ONE-PARAMETER GROUPS OF VOLUME-PRESERVING AUTOMORPHISMS OF $C^2$ (*)

by Chiara De Fabritiis (in Trieste)(**) 

SOMMARIO. - Si esaminano i gruppi ad un parametro nel gruppo degli automorfismi polinomiali di $C^2$ e nel gruppo degli shears, provando che sono coniugati a gruppi a un parametro nel gruppo degli automorfismi affini di $C^2$ o nel gruppo degli automorfismi elementari; da ciò si deducono risultati sui comportamento asintotico del gruppo ad un parametro, sui suoi punti periodici e sui suoi punti fissi.

SUMMARY. - In this work we study the one-parameter groups in the group of all polynomial automorphisms of $C^2$ and in the group of all shears. We prove that any such one-parameter group is conjugate to a one-parameter group contained either in the group of all affine automorphisms of $C^2$ or in the group of elementary automorphisms. This implies some results on the asymptotic behaviour of the one-parameter group, on its periodic points and on its fixed points.

0. Introduction.

In this work we investigate the structure of the one-parameter groups in $Aut_P C^2$, the group of all polynomial automorphisms of $C^2$, and in the group of all shears, $G_1$, introduced by J.-P. Rosay and W. Rudin in [12]. A one-parameter group in $Aut_P C^2$ ($G_1$) is a continuous homomorphism from $R$ to $Aut_P C^2$ ($G_1$), where these last two groups are both endowed with the compact-open topology.

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(**) Indirizzo dell’Autore: Scuola Internazionale Superiore di Studi Avanzati, Via Beirut 2-4, 34013 Trieste (Italia).
In particular we will prove that, both in the case of $\text{Aut}_p \mathbb{C}^2$ and in the case of $G_1$, each one-parameter group is conjugated to a one-parameter group in two suitable subgroups: the subgroup of affine automorphisms of $\mathbb{C}^2$ and the subgroup of "elementary transformations" (to be defined in §1). This gives us the possibility to study the asymptotic behaviour of the one-parameter group, its set of common fixed points and other qualitative results on the behavior of its orbits. We will show, in particular, the lack of any chaotic phenomena, in contrast with the discrete case of iterates (see [4]).

In the first section we give the definition of the shear group, $G_1$, as well as the definition of the subgroup of all elementary automorphisms of the group $\text{Aut}_p \mathbb{C}^2$ and $G_1$, and present the main results.

In the second section we prove a structure theorem both for $\text{Aut}_p \mathbb{C}^2$ and $G_1$, i.e., we prove the fact that there exist two subgroups $A, E \subseteq \text{Aut}_p \mathbb{C}^2$ such that $\text{Aut}_p \mathbb{C}^2$ is the free product of $A$ and $E$ amalgamated over their intersection. Similarly for $G_1$, we prove that there exist two subgroups $A_1, E_1 \subseteq G_1$ such that $G_1$ is the free product of $A_1$ and $E_1$, amalgamated over their intersection.

In the third section we prove that any one-parameter group in $\text{Aut}_p \mathbb{C}^2$ ($G_1$) is conjugated to a one-parameter group in $A$ or $E$ ($A_1$ or $E_1$, respectively).

1. Definitions and Main Results.

Let us first consider the group of polynomial automorphisms of $\mathbb{C}^2$, which we denote by $\text{Aut}_p \mathbb{C}^2$. An elementary automorphism of $\text{Aut}_p \mathbb{C}^2$ is a transformation of the form

$$
\begin{pmatrix}
  x \\
  y
\end{pmatrix} \mapsto \begin{pmatrix}
  \alpha x + p(y) \\
  \beta y + \gamma
\end{pmatrix},
$$

where $\alpha, \beta \in \mathbb{C}^*$, $\gamma \in \mathbb{C}$ and $p$ is a polynomial with coefficients in $\mathbb{C}$. Let $A$ be the group of affine automorphisms of $\mathbb{C}^2$ and $E$ be the group consisting of all elementary automorphisms of $\mathbb{C}^2$. Obviously $A$ and $E$ are subgroups of $\text{Aut}_p \mathbb{C}^2$. $E$ is said to be the subgroup of elementary automorphisms in $\text{Aut}_p \mathbb{C}^2$.

The structure of the group $\text{Aut}_p \mathbb{C}^2$ is given by the following

**Theorem 1.1.** $\text{Aut}_p \mathbb{C}^2$ is the free product of $A$ and $E$ amalgama-
mated over $A \cap E$.

The structure theorem (whose proof is postponed, together with the proof of Theorem 1.3, to §2) will be useful in the following to understand the behaviour of one-parameter groups in $Aut_P \mathbb{C}^2$.

Together with the group of polynomial automorphisms of $\mathbb{C}^2$, we want to consider also the group of all shears, i.e., the group generated by the automorphisms of $\mathbb{C}^2$ of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + f \left( \Lambda \begin{pmatrix} x \\ y \end{pmatrix} \right) e,$$

where $f \in \text{Hol}(\mathbb{C}, \mathbb{C})$, $e \in \mathbb{C}^2$ and $\Lambda$ is a linear form on $\mathbb{C}^2$ with $\Lambda e = 0$.

These automorphisms—which were introduced by J-P. Rosay and W. Rudin in [12] (see also [2])—are called shears and the group generated by them will be denoted by $G_1$ (by a slight modification of the notation introduced in [2]).

Let $Aut_1 \mathbb{C}^2$ be the group of all holomorphic automorphisms of $\mathbb{C}^2$ whose Jacobian is equal to 1. E. Andersen proved in [2] that $G_1$ is a proper subgroup of $Aut_1 \mathbb{C}^2$ and that $G_1$ is dense in $Aut_1 \mathbb{C}^2$ for the topology of uniform convergence on compact sets. Since any polynomial automorphism of $\mathbb{C}^2$ has constant Jacobian, the group $G_1$ can be seen as a generalization of $Aut_P \mathbb{C}^2$. This is also confirmed by the following Proposition, which is proved by Andersen in [2].

**Proposition 1.2.** The special linear group on $\mathbb{C}^2$, $SL(2, \mathbb{C})$, is contained in $G_1$.

Let $A_1$ be the subgroup of all affine automorphisms of $\mathbb{C}^2$ with Jacobian equal to 1. $A_1$ is the semidirect product between $SL(2, \mathbb{C})$ and the group $\mathbb{C}^2$ of translations with the left action of $SL(2, \mathbb{C})$ on $\mathbb{C}^2$. In particular it is contained in $G_1$. Let $E_1$ be the subgroup of $G_1$ given by

$$E_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} ax + f(y) \\ \alpha^{-1}y + \beta \end{pmatrix}, \quad \alpha \in \mathbb{C}^*, \beta \in \mathbb{C}, \quad f \in \text{Hol}(\mathbb{C}, \mathbb{C}) \right\}.$$ 

Since each element of $E_1$ is the composition of a shear such that $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, with an element in $SL(2, \mathbb{C})$ and a translation, it follows that $E_1 \subseteq G_1$. The group $E_1$ is said to be the group of elementary automorphisms of $G_1$. 
The intersection $A_1 \cap E_1$ is given by

$$A_1 \cap E_1 = \left\{ \left( \begin{array}{c} x \\ y \end{array} \right) \mapsto \left( \begin{array}{c} ax + ay + b \\ \alpha^{-1}y + \beta \end{array} \right), \quad \alpha \in \mathbb{C}^*, \ \beta, a, b \in \mathbb{C} \right\}.$$  

**Theorem 1.3.** $G_1$ is the free product of $A_1$ and $E_1$ amalgamated over $A_1 \cap E_1$.

As we already said, we postpone the proof of Theorem 1.3 to §2. We endow $\text{Aut}_P \mathbb{C}^2$ and $G_1$ with the compact-open topology.

**Definition 1.1.** A one-parameter polynomial group or a one-parameter shear group is a continuous homomorphism of $\mathbb{R}$ into $\text{Aut}_P \mathbb{C}^2$ or $G_1$ respectively.

Using Theorem 1.1 and Theorem 1.3 we shall prove the following

**Theorem 1.4.** A one-parameter polynomial group $\Phi$ (or a one-parameter shear group) is conjugated in $\text{Aut}_P \mathbb{C}^2$ (or $G_1$) to a one-parameter group in $E$ or $A$ (or $E_1$ or $A_1$), i.e. there exists $X \in \text{Aut}_P \mathbb{C}^2$ ($G_1$) such that for all $t \in \mathbb{R}$, $X \circ \Phi_t \circ X^{-1} \in A$ or $E$ ($A_1$ or $E_1$).

The proof of Theorem 1.4 is postponed to §3. In order to describe the possible conjugates, i.e. the one-parameter groups in $A$, $A_1$, $E$ and $E_1$, we start by considering the one-parameter groups in $A$ and $A_1$, looking for fixed points, periodic points and the behavior of the one-parameter groups as $t \to +\infty$ or $t \to -\infty$.

**Definition 1.2.** A fixed point for a one-parameter group $\Phi$ is a fixed point for $\Phi_t$ for all $t \in \mathbb{R}$.

**Definition 1.3.** A periodic point $x$ for a one-parameter group $\Phi$ is a point such there exists $t_0 \in \mathbb{R}^*$ so that $x$ is a fixed point for $\Phi_{t_0}$ (hence for $\Phi_{nt_0}$ for all $n \in \mathbb{Z}$).

**Definition 1.4.** A limit point $x$ for a one-parameter group $\Phi$ is a point $x$ such there exists $\lim_{t \to +\infty} \Phi(x)$.

The following proposition collects the results of our investigation for the case of the group $A$. It can be also found in [3], together with
Proposition 1.7.

**Proposition 1.5.** All the one-parameter groups in $A$ are given, up to conjugation in $A$, by the following expressions

\[
a) \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} e^{t c_1} x \\ e^{t c_2} y \end{array} \right), \ b) \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + ts_1 \\ e^{t c_2} y \end{array} \right), \\
c) \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} e^{t c_1} x \\ e^{t c_1} (tx + y) \end{array} \right), \\
d) \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + ts_1 \\ tx + y + ts_2 + t^2 s_1/2 \end{array} \right), \ c) \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + ts_1 \\ y + ts_2 \end{array} \right),
\]

where $c_1, c_2, s_1, s_2 \in \mathbb{C}$.

**Proof.** Set $\Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = R_t \left( \begin{array}{c} x \\ y \end{array} \right) + S_t$, where $S_t \in \mathbb{C}^2$ and $R_t \in GL(2, \mathbb{C})$ if $\Phi_t \in A$ and $R_t \in SL(2, \mathbb{C})$ if $\Phi_t \in A_1$. The fact that $\Phi_{t+\tau} = \Phi_t \circ \Phi_{\tau}$ is equivalent to

\[
i) \ R_{t+\tau} = R_t R_\tau \quad \text{and} \quad ii) \ S_{t+\tau} = R_t S_\tau + S_t.
\]

The first equation implies that $R$ is a one-parameter group in $GL(2, \mathbb{C})$ or in $SL(2, \mathbb{C})$. Hence $R_t = \exp tW$, where $W$ is an element of $M(2, \mathbb{C})$ which, up to conjugation by a suitable element $M \in SL(2, \mathbb{C})$, may be assumed to be

\[
W = \left( \begin{array}{cc} c_1 & 0 \\ 0 & c_2 \end{array} \right) \text{ or } W = \left( \begin{array}{cc} c_1 & 0 \\ 1 & c_1 \end{array} \right),
\]

where $\Phi_t \in A_1$ iff $\text{tr} W = 0$, Equation ii) implies that $R_t S_\tau + S_t = R_\tau S_t + S_\tau$.

If there is a $t_0$ such that $R_{t_0} - I$ is invertible, then we obtain $S_t = (R_t - I)S$, where $S = (R_{t_0} - I)^{-1} S_{t_0} \in \mathbb{C}^2$. Otherwise $c_1 c_2 = 0$, if $\Phi_t \in A$ and $c_1 = 0$, if $\Phi_t \in A_1$. If $c_1 = 0$ and $c_2 \neq 0$, then a simple computation in ii) yields $S_t = \left( \begin{array}{c} ts_1 \\ (e^{t c_2} - 1)s_2 \end{array} \right)$. If $c_1 = 0$ and $c_2 = 0$, then $S_t = \left( \begin{array}{c} ts_1 \\ ts_2 \end{array} \right)$, in the case in which we have $R_t = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$, or $S_t = \left( \begin{array}{c} ts_1 \\ ts_2 + t^2 s_1/2 \end{array} \right)$, if we have $R_t = \left( \begin{array}{cc} 1 & 0 \\ t & 1 \end{array} \right)$. 


We begin by examining the case in which
\[
\Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1} (x + s_1) - s_1 \\ e^{tc_2} (y + s_2) - s_2 \end{pmatrix},
\]
where \( c_1 c_2 \neq 0 \); conjugating by a suitable translation we obtain \( \Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1} x \\ e^{tc_2} y \end{pmatrix} \), which is case a).

Now we examine the case \( \Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ts_1 \\ e^{tc_2} (y + s_2) - s_2 \end{pmatrix} \).

Again by conjugating by a suitable translation, we obtain \( \Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ts_1 \\ e^{tc_2} y \end{pmatrix} \), which is case b).

We turn to consider \( \Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1} (x + s_1) - s_1 \\ e^{tc_1} (tx + y + s_2) - s_2 \end{pmatrix} \) by conjugating by a suitable translation we obtain
\[
\Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1} x \\ e^{tc_1} (tx + y) \end{pmatrix}
\]
that is case c).

We are left with
\[
\Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ts_1 \\ tx + y + ts_2 + t^2 s_1 / 2 \end{pmatrix}
\]
or
\[
\Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ts_1 \\ y + ts_2 \end{pmatrix},
\]
i.e. cases d) and e).

\( \diamond \)

Now we investigate fixed points, periodic points and limit points in the different cases listed in Proposition 1.5. We begin with case a): if \( x \neq 0 \) and \( y \neq 0 \), then \( \Phi_t \begin{pmatrix} x \\ y \end{pmatrix} \) converges for \( t \to +\infty \) iff \( \text{Rec}_1 < 0 \) and \( \text{Rec}_2 < 0 \). If \( x \neq 0 \) and \( y = 0 \) (or \( x = 0 \) and \( y \neq 0 \)), then \( \Phi_t \begin{pmatrix} x \\ y \end{pmatrix} \) has limit for \( t \to +\infty \) iff \( \text{Rec}_1 < 0 \) (or \( \text{Rec}_2 < 0 \)); moreover \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is a fixed point for all \( \Phi_t \), hence the only case in which the limit of \( \Phi_t(z) \) exists both for \( t \to \pm \infty \) is given by \( z = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), which is a fixed point for the one-parameter group \( \Phi \).
If $x \neq 0$ and $y \neq 0$, the periodic points of $\Phi$ are given by $\text{Rec}_1 = 0$ and $\text{Rec}_2 = 0$, with $c_1$ and $c_2$ linearly dependent over $\mathbb{Q}$, if $x = 0$ (or $y = 0$), the periodic points are given by $\text{Rec}_2 = 0$ (or $\text{Rec}_1 = 0$) respectively.

Gathering all the results established so far, we obtain that for the one-parameter group $\Phi$ of case a) the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is always a fixed point, $\Phi$ has periodic points iff $\text{Rec}_1 = 0$ or $\text{Rec}_2 = 0$; moreover $\Phi_t$ converges for $t \to +\infty$ iff $\text{Rec}_1 < 0$ and $\text{Rec}_2 < 0$.

For case b) a simple computation shows that, if $s_1 \neq 0$, there are neither limit points nor periodic or fixed points; if $s_1 = 0$, $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is always a fixed point and the condition for $\Phi_t \begin{pmatrix} x \\ y \end{pmatrix}$ to converge, as $t \to +\infty$, for all $\begin{pmatrix} x \\ y \end{pmatrix}$, is $\text{Rec}_2 < 0$. The condition for the existence of periodic points (in this case every point in $\mathbb{C}^2$ becomes a periodic one) is $\text{Rec}_2 = 0$.

In that same way we obtain that, for case c), $\Phi_t \begin{pmatrix} x \\ y \end{pmatrix}$ has limit for $t \to +\infty$ iff $\text{Rec}_1 < 0$. Periodic points are given by $x = 0$ and $\text{Rec}_1 = 0$; the unique fixed point for the whole group is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

In case d) there are limit points iff $s_1 = 0$ and $x = -s_2$ and these are the only fixed points. Periodic points never exist. In case e) it is easily seen that, if $s_1 \neq 0$ or $s_2 \neq 0$, there are neither limit nor periodic or fixed points.

**Corollary 1.6.** Let $\Phi$ be a one-parameter group in $A$ or $A_1$ such that $\Phi$ has a fixed point. Then $\Phi$ can be expressed, up to conjugation in $A$, in one of the following ways:

$$
\Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1}x \\ e^{tc_2}y \end{pmatrix}, \quad \Phi_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^{tc_1}x \\ e^{tc_1}(tx + y) \end{pmatrix},
$$

where $c_1, c_2 \in \mathbb{C}$. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is always a fixed points in these three cases.

**Proof.** In fact the proof comes down to showing that, if there are fixed points in case b), then we can pass to case a), and, if there are fixed points in case d), then we can pass to case c). In fact, as the
presence of fixed points in cases b) and d) is equivalent to $s_1 = 0$, if this happens, then case b) reduces to case a) (with $c_1 = 0$), and, by conjugating with a translation $x \mapsto x + s_2$ on the first component we obtain that case d) reduces to case c). \hfill \Box

Turning our attention to the one-parameter groups in $E$ and $E_1$, we consider separately the two subgroups.

**Proposition 1.7.** All the one-parameter groups in $E$ are expressed, up to conjugation in $A \cap E$, by

\[
a) \quad \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} e^{tc_1} x + p_t(y) \\ e^{tc_2} y \end{array} \right),
\]

where $p_t$ satisfies $p_{t+\tau}(y) = e^{tc_1} p_t(y) + p_\tau(e^{tc_2} y)$

\[
b) \quad \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} e^{tc_1} x + p_t(y) \\ y + ts_2 \end{array} \right),
\]

where $p_t$ satisfies $p_{t+\tau}(y) = e^{tc_1} p_t(y) + p_\tau(y + ts_2)$.

**Proof.** Let $\Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} \alpha_t x + p_t(y) \\ \beta_t y + \gamma_t \end{array} \right)$ be a one-parameter group in $E$. Then the condition $\Phi_{t+\tau} = \Phi_t \circ \Phi_\tau$ yields, up to a conjugation with a translation on the second variable, $\alpha_t = e^{tc_1}$, $\beta_t = e^{tc_2}$ and $\gamma_t = (e^{tc_2} - 1)c$, if $c_2 \neq 0$ (in which case by a suitable translation we obtain case a) of Proposition 1.7). Otherwise $\gamma_t = ts_2$ if $c_2 = 0$ (which yields b) of Proposition 1.7). \hfill \Box

A direct inspection shows that $\Phi_t$ has a limit for $t \to +\infty$ in case a) iff $\text{Re} c_1 < 0$, $\text{Re} c_2 < 0$ and there exists $\lim_{t \to +\infty} p_t(y)$; in case b) iff $\text{Re} c_1 < 0$ and there exists $\lim_{t \to +\infty} p_t(y)$.

This latter limit may exist or may not exist: for instance take $\gamma_t \equiv 0$, $c_1 \neq c_2$, and $p_t(y) = (e^{tc_1} - e^{tc_2})y$ : it is easily seen that this gives a one-parameter group in $E$ and that, if $\text{Re} c_1 < 0$ and $\text{Re} c_2 < 0$, then $p_t(y)$ has always limit for $t \to +\infty$. If $\text{Re} c_1 > 0$ and $\text{Re} c_2 < 0$, then $p_t(y)$ has no limit for $t \to +\infty$ or $t \to -\infty$. Notice that in certain cases there are values of $t$ different from 0 such that $p_t$ is identically 0 : in the above example, if $c_1 = c_2 + 2\pi i$ and $t \in \mathbb{Z}$, then $p_t \equiv 0$. Hence it is not true that $p_t$ has always the same degree.
Going back to the general case, periodic points do exist only if \( \text{Re}c_2 = 0 \), in which case we must solve \( e^{tc_1}x + p_t(y) = x \). If we want every point in \( \mathbb{C}^2 \) to be a periodic point, then we must require \( \text{Re}c_1 = 0 \) and \( c_1 \) and \( c_2 \) must be linearly dependent over \( \mathbb{Q} \). Moreover \( p_t \) must be zero for suitable values of \( t \). If we want periodic points to exist, then it is enough that \( (e^{tc_1} - 1)x + p_t(y) = 0 \). For fixed \( t \), the last equation defines hypersurfaces in \( \mathbb{C}^2 \).

We look for the solutions of equations

i) \( p_{t+\tau}(y) = e^{tc_1}p_t(y) + p_\tau(e^{tc_2}y) \) and

ii) \( p_{t+\tau}(y) = e^{tc_1}p_t(y) + p_\tau(y + ts_2) \),

where \( p_t \) is a polynomial and \( t \mapsto p_t \) is a \( C^1 \) map.

Notice that by integrating i) and ii) in \( \tau \) between 0 and 1 we can prove that the flow depends smoothly on \( t \).

In case i) write \( p_t(y) = \sum_{n \in \mathbb{N}} a_n(t)y^n \), where, for any fixed \( t \), \( a_n(t) \) vanishes when \( n \gg 0 \). Then \( p \) satisfies i) iff

\[
\sum_{n \in \mathbb{N}} a_n(t + \tau)y^n = \sum_{n \in \mathbb{N}} (e^{tc_1}a_n(t) + e^{ntc_2}a_n(\tau))y^n \quad \forall y \in \mathbb{C},
\]

hence \( a_n(t + \tau) = e^{tc_1}a_n(t) + e^{ntc_2}a_n(\tau) \) \( \forall n \in \mathbb{N} \).

We subtract \( a_n(\tau) \) from both members, divide by \( t \) and let \( t \) go to 0; then, by recalling that \( a_n(0) = 0 \) \( \forall n \in \mathbb{N} \) because \( \Phi_0 = \text{id} \mathbb{C} \), we obtain that

\[
\frac{da_n}{d\tau}(\tau) = e^{tc_1} \frac{da_n}{d\tau}(0) + nc_2a_n(\tau),
\]

which gives, together with the condition \( a_n(0) = 0 \),

\[
a_n(t) = \begin{cases} 
\alpha_n(e^{tc_1} - e^{nc_2t}) & \text{if } c_1 \neq nc_2 \\
\alpha_n t e^{tc_1} & \text{if } c_1 = nc_2,
\end{cases}
\]

where \( \alpha_n \in \mathbb{C} \). It is easily seen that these functions give the solutions for i). This proves that there is an upper bound, independent on \( t \), for the degree of \( p_t \).

In case ii) write again \( p_t(y) = \sum_{n \in \mathbb{N}} a_n(t)y^n \), where, for any fixed \( t \), \( a_n(t) \) vanishes when \( n \gg 0 \). Then \( p \) satisfies ii) iff

\[
\sum_{n \in \mathbb{N}} a_n(t + \tau)y^n =
\]
\[
\sum_{n \in \mathbb{N}} \left( e^{\tau c_1} a_n(t) + \sum_{j \geq n} a_j(\tau) \left( \frac{j}{n} \right) (ts_2)^{j-n} \right) y^n \quad \forall y \in \mathbb{C},
\]
whence
\[
a_n(t + \tau) = e^{\tau c_1} a_n(t) + \sum_{j \geq n} a_j(\tau) \left( \frac{j}{n} \right) (ts_2)^{j-n} \quad \forall n \in \mathbb{N}.
\]
By subtracting \(a_n(\tau)\) from both members, dividing by \(t\), letting \(t\) go to 0 and recalling that \(a_n(0) = 0 \forall n \in \mathbb{N}\), we obtain
\[
\frac{da_n(\tau)}{d\tau} = e^{\tau c_1} \frac{da_n(0)}{d\tau} + (n + 1)s_2 a_{n+1}(\tau).
\]
Choosing \(a_0\), then we can find \(a_n\) by a recursive step obtaining that \(a_k \in C^\infty\) for all \(k \in \mathbb{N}\).

Now we consider one-parameter groups in \(E_1\):

**Proposition 1.8.** All the one-parameter groups in \(E_1\) are expressed (up to conjugation in \(E_1 \cap A_1\)) by

i) \( \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} e^{ia} x + f_t(y) \\ e^{-ia} y \end{array} \right) \)

ii) \( \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + f_t(y) \\ y + tb \end{array} \right) \),

where \(a \in \mathbb{C}\) and in case i) \(f_t\) satisfies \(f_{t+\tau}(y) = e^{\tau a}f_t(y) + f_\tau(e^{-ia}y)\), while in case ii) it satisfies \(f_{t+\tau}(y) = f_t(y + \tau b) + f_\tau(y)\).

**Proof.** The fact that \( \Phi_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} \alpha_t x + f_t(y) \\ \alpha_t^{-1} y + \beta_t \end{array} \right) \) satisfies the composition rule is equivalent to the fact that \(\alpha_t^{-1} y + \beta_t\) is a one parameter group of affine transformations of \(\mathbb{C}\), hence it can be conjugated with a translation to obtain \(y \mapsto e^{ia}y\) or \(y \mapsto y + tb\); the relation on \(f\) follows immediately. \(\diamondsuit\)

In case i), write \(f_t(y) = \sum_{n \in \mathbb{N}} a_n(t) y^n\), where \(a_n(0) = 0\) for all \(n \in \mathbb{N}\). The relation \(f_{t+\tau}(y) = e^{\tau a}f_t(y) + f_\tau(e^{-ia}y)\) implies
\[
a_n(t + \tau) = e^{\tau a} a_n(t) + e^{-nta} a_n(\tau).
\]
If \(a = 0\), then there exists an entire function \(g\) such that \(f_t(y) = tg(y)\) for all \(t \in \mathbb{R}\) and \(y \in \mathbb{C}\). It is easily seen that the point \(\left( \begin{array}{c} x \\ y \end{array} \right) \) is fixed
point for \( \Phi \) iff \( g(y) = 0 \) and that it is a limit point iff it is a fixed point.

If \( a \neq 0 \), then \( a_n(t + \tau) = e^{\tau a}a_n(t) + e^{-n\tau a}a_n(\tau) \) must be equal to \( e^{\tau a}a_n(\tau) + e^{-n\tau a}a_n(t) \), therefore we obtain that

\[
(e^{-n\tau a} - e^{\tau a})a_n(\tau) = (e^{-n\tau a} - e^{\tau a})a_n(t).
\]

Since \( a \neq 0 \), then there exists \( \tau_0 \) such that \( e^{-n\tau_0 a} - e^{\tau_0 a} \neq 0 \); hence we obtain that

\[
a_n(t) = (e^{-n\tau_a} - e^{\tau a})a_n(\tau_0) \left( e^{-n\tau_0 a} - e^{\tau_0 a} \right)^{-1} = c_n \left( e^{-n\tau a} - e^{\tau a} \right),
\]

where \( c_n \in \mathbb{C} \), and in this way we can recover the function \( f_t \).

If we look for a limit point \( \left( x_0, y_0 \right) \) with \( y_0 \neq 0 \), then we must have \( \text{Re} a > 0 \) and \( x_0 + \sum c_n y_0^n = 0 \). If we look for a limit point \( \left( x_0, y_0 \right) \) with \( y_0 = 0 \), then we must have \( x_0 + c_0 = 0 \).

To study periodic points we have to split up our investigation in two cases: the first case, in which \( \text{Re} a \neq 0 \), and the second case, in which \( \text{Re} a = 0 \). If \( \text{Re} a \neq 0 \) and \( \left( x_0, y_0 \right) \) is a periodic point of period \( t_0 \), then \( y_0 = 0 \) and \( e^{-t_0 a}x_0 + f_{t_0}(0) = x_0 \), that is \( e^{-t_0 a}x_0 + c_0(1 - e^{-t_0 a}) = x_0 \) and therefore \( x_0 = c_0 \).

If \( \text{Re} a = 0 \) and \( t_0 a \) is an integer multiple of \( 2\pi i \), then it is easily seen that \( f_{t_0} \equiv 0 \), and therefore \( \Phi_{t_0} = 1d\mathbb{C}^2 \).

When we look for a fixed point \( \left( x_0, y_0 \right) \), it is easily seen that we must have \( x_0 = y_0 = 0 \) and \( f_t(0) = 0 \) for all \( t \in \mathbb{R} \), that is \( c_0 = 0 \).

In case \( ii \), we write again \( f_t(x) = \sum_{n \geq 0} a_n(t)x^n \) and we obtain

\[
a_n(t + \tau) = a_n(\tau) + \sum_{k \geq n} a_k(t)(\tau b)^{k-n} \binom{k}{n}.
\]

We subtract \( a_n(\tau) \) from both members, we divide by \( t \) and let \( t \) go to \( 0 \), obtaining

\[
\frac{da_n}{d\tau}(\tau) = \sum_{k \geq n} \frac{d a_k}{d\tau}(0)(\tau b)^{k-n} \binom{k}{n}.
\]

In this way we can recover the form of \( f \). It is easily seen that there is no possibility of having fixed points, periodic points or limit points.
The above considerations indicate that, in the continuous case, there is no chaotic behaviour (such as having many different periods), in sharp contrast with the discrete case of iterates (see [4] and [5]). The behaviour of all one-parameter groups in Aut\(p\mathbb{C}^2\) or in \(G_1\) is clarified by the above models.

In particular the regularity in \(t\) is a consequence of the well-known theorem for continuous one-parameter groups on a complex manifold.

**Theorem 1.9.** Let \(X\) be a complex domain in \(\mathbb{C}^n\) and \(\Phi\) a one-parameter semigroup (i.e. a continuous homomorphism from \(\mathbb{R}^+\) to \(\text{Hol}(X, X)\)). The map \(t \mapsto \Phi_t(x)\) is analytic on \(\mathbb{R}^+\) for all \(x \in X\); moreover there exists a holomorphic map \(F\) from \(X\) to \(\mathbb{C}^n\) such that

\[
\frac{\partial \Phi}{\partial t} = F \circ \Phi.
\]

The proof of this theorem can be found, e.g., in [1] (see, p.296).

Fix \(x \in \mathbb{C}^2\), let \(K\) be a compact neighborhood of \(x\), then, by a corollary to Cauchy’s theorem, we can solve the Cauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t}(t, z) = F(u(t, z)), \\
u(0, z) = z \quad \text{on } U
\end{cases}
\]

on \((-a, a) \times U\) (where \(a > 0\) and \(U\) is a neighborhood of \(K\)) with \(u\) analytic.

For the uniqueness of the solution we have \(\Phi(t, z) = u(t, z)\) if \(t \geq 0\) and \(z \in K\); moreover, if \(t, s, t + s \in (-a, a)\) we have

\[
u(s, u(t, z)) = u(t + s, z)
\]

on \(K\).

Let \(t \in [0, a)\) and \(s = -t\), then last equality implies that,

\[
u(-t, u(t, z)) = z \quad \text{if } z \in K:\) then \(u(-t, \cdot)\) is a local inverse of \(\Phi(t, \cdot)\), hence \(u(-t, z) = \Phi(-t, z)\) if \(z \in K_1\) and \(t \in [0, a)\) (where \(K_1\) is a suitable neighborhood of \(x\)).

Therefore \(\Phi(t, x)\) is analytic on \((-a, a)\), adding the fact that it is analytic on the two half-lines we obtain that the dependence is analytic on the whole line and by analytic continuation

\[
\frac{\partial \Phi}{\partial t} = F \circ \Phi.
\]
2. Group Structures.

In this section we prove Theorem 1.1 and Theorem 1.3, i.e., we prove that $\operatorname{Aut}_P \mathbb{C}^2$ is the free product of $A$ and $E$ amalgamated over their intersection $A \cap E$ and that $G_1$ is the free product of $A_1$ and $E_1$ amalgamated over their intersection $A_1 \cap E_1$.

To prove Theorem 1.1 and Theorem 1.3 we first prove that $A$ and $E$ (in the case of Theorem 1.1), and $A_1$ and $E_1$ (in the case of Theorem 1.3) generate $\operatorname{Aut}_P \mathbb{C}^2$ and $G_1$, respectively.

**Theorem 2.1.** (Jung) The group $\operatorname{Aut}_P \mathbb{C}^2$ of polynomial automorphisms of $\mathbb{C}^2$ is generated by $A$ and $E$.

The proof of Theorem 2.1 can be found in [10].

**Remark 2.1.** If $g \in E \cap A$, then

$$g \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} ax + ay + b \\ \beta y + \gamma \end{array} \right),$$

where $\alpha, \beta \in \mathbb{C}^*$, $a, b, \gamma \in \mathbb{C}$.

**Lemma 2.2.** $A_1$ and $E_1$ generate $G_1$.

**Proof.** In order to prove that $E_1$ and $A_1$ generate $G_1$, it is enough to prove that $E_1$ and $A_1$ generate all shears. First of all notice that, if $e \in \mathbb{C}^2 \setminus \{0\}$, then there exists $T \in \text{SL}(2, \mathbb{C}) \subset A_1$ such that $T \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = e$; moreover, if

$$S \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x + f(y) \\ y \end{array} \right),$$

then

$$T \circ S \circ T^{-1} \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} x \\ y \end{array} \right) + f \left( \Lambda \left( \begin{array}{c} x \\ y \end{array} \right) \right) e,$$

where $\Lambda$ is a linear form on $\mathbb{C}^2$ with $\Lambda e = 0$. Hence, conjugating $S \in E_1$ by a suitable element $T \in A_1$, we obtain every shear. That proves that $G_1$ is generated by $A_1$ and $E_1$. 

These two lemmas give us the possibility to prove a first, very simple result on the structure of $\operatorname{Aut}_P \mathbb{C}^2$ and $G_1$. 

Lemma 2.3. Both $\text{Aut}_p \mathbb{C}^2$ and $G_1$ are arcwise connected.

Proof. In fact it is easily seen that both $A$ and $A_1$ are arcwise connected. Moreover for an element $g \in E$ as in (1.1) it is enough to consider two continuous paths $\alpha, \beta : [0, 1] \to \mathbb{C}^*$ such that $\alpha(1) = \beta(1) = 1$ and $\alpha(0) = \alpha, \beta(0) = \beta$; then

$$g_t \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} \alpha(t)x + (1 - t)p(y) \\ \beta(t)y + (1 - t)\gamma \end{array} \right)$$

is a continuous path in $E$ such that $g_0 = g$ and $g_1 = \text{id}_{\mathbb{C}^2}$.

In the same way we can connect any element $g$ in $E_1$ to the identity map with a continuous path in $E_1$.

Now, for $h \in \text{Aut}_p \mathbb{C}^2$, (or $g \in G_1$) we choose a representation of $h$ as $h = h_n \circ \ldots \circ h_1$ with $h_i \in A \cup E$ (respectively a representation of $g \in G_1$ as $g = g_n \circ \ldots \circ g_1$ with $g_j \in A_1 \cup E_1$) and we take $n$ continuous paths $h_j(t)$ in $A$ or $E$ (or $n$ continuous paths $g_j(t)$ in $A_1$ or $E_1$) such that $h_j(0) = h_j$ and $h_j(1) = \text{id}_{\mathbb{C}^2}$. In this way we obtain that $h(t) = h_n(t) \circ \ldots \circ h_1(t)$ is a continuous path in $\text{Aut}_p \mathbb{C}^2$ which connects $h$ with the identity (and the same for $G_1$).

We start by proving Theorem 1.1, whose proof is much simpler than the proof of Theorem 1.3, due to the fact that the degree induces a partial ordering on polynomials.

Definition 2.1. A sequence $(g_n, \ldots, g_1)$ of length $n \geq 1$ is called a reduced word with respect to the subgroups $A$ and $E$ if, for each $i = 1, \ldots, n$, $g_i \in (A \cup E)/(A \cap E)$ and $g_i, g_{i+1}$ do not belong both to the same of the two subgroups.

Let $g = g_n \circ \ldots \circ g_2 \circ g_1 \in \text{Aut}_p \mathbb{C}^2$, where $(g_n, \ldots, g_1)$ is a reduced word; we shall prove that this representation is “unique” up to products in $A \cap E$. For this we need the following

Theorem 2.4. If $g = g_n \circ \ldots \circ g_2 \circ g_1$ and $(g_n, \ldots, g_1)$ is a reduced word, then $g \neq \text{id}_{\mathbb{C}^2}$.

Definition 2.2. For $h \in \text{Aut}_p \mathbb{C}^2$, we define the degree of $h$ to be the maximum between the degrees of its two scalar components.

The following theorem implies Theorem 2.4 and therefore Theorem 1.1.
THEOREM 2.5. If \( g = g_n \circ \ldots \circ g_2 \circ g_1 \) and \((g_n, \ldots, g_1)\) is a reduced word, then the degree of \( g \) is the product of the degrees of its factors.

Proof. Since for \( n = 1 \) the statement is trivial, we proceed by induction on \( n \).

Let \( w_1 \) and \( w_2 \) be the two scalar components of \( g_k \circ \ldots \circ g_2 \circ g_1 \), where \( g_k \) is an element in \( E \): by the induction step we can suppose that \( \text{deg} g_k \circ \ldots \circ \text{deg} g_1 = \text{deg} w_1 > \text{deg} w_2 \) (the relation \( \text{deg} w_1 > \text{deg} w_2 \) is a part of the inductive step).

As \( g_{k+1} \in A \setminus E \), then \( g_{k+1} \circ g_k \circ \ldots \circ g_2 \circ g_1 \) has the same degree as \( g_k \circ \ldots \circ g_2 \circ g_1 \). Moreover if \( u_1 \) and \( u_2 \) are the two scalar components of \( g_{k+1} \circ g_k \circ \ldots \circ g_2 \circ g_1 \), then \( \text{deg} u_2 \geq \text{deg} u_1 \).

Now we consider \( g_{k+2} \in E \setminus A \). If \( v_1 \) and \( v_2 \) are the scalar components of \( g_{k+2} \circ g_{k+1} \circ g_k \circ \ldots \circ g_2 \circ g_1 \), then \( \text{deg} v_1 = \text{deg} (\alpha u_1 + p(u_2)) \), where \( \alpha \in C^* \) and \( p \) is a polynomial of degree \( > 1 \). Hence

\[
\text{deg} v_1 = \text{deg} p \cdot \text{deg} u_2 = \text{deg} g_{k+2} \cdot \text{deg} g_{k+1} \ldots \text{deg} g_1 > \text{deg} v_2 = \text{deg} g_{k+1} \ldots \text{deg} g_1,
\]

and that completes the proof. \( \diamond \)

COROLLARY 2.6. If \((g_n, \ldots, g_1)\) is a reduced word, then its length is an invariant of the element \( g = g_n \circ \ldots \circ g_1 \in \text{Aut}_P C^2 \). Moreover the representation \( g = g_n \circ \ldots \circ g_1 \) is unique up to replacing \( g_k \) by \( h g_k h^{-1} \) by \( g_{k+1} \).

In the same way, to clarify the structure of \( G_1 \), we prove now that \( G_1 \) is the free product of \( A_1 \) and \( E_1 \) amalgamated over the intersection \( E_1 \cap A_1 \).

First of all we introduce the notion of rosary, due to Andersen (see [2]).

For any \( r \geq 1 \) set \( \mathcal{H}_0(C^r) = \{ f \in \text{Hol}(C^r, C) : f(0) = 0 \} \).

DEFINITION 2.3. A rosary is a sequence \( L = \{L_1, \ldots, L_n\} \) of linear subspaces of \( \mathcal{H}_0(C^2) \) such that

i) \( \dim L_i = 2 \),

ii) \( \dim L_i \cap L_{i+1} \geq 1 \),
iii) there are \( u, v \in L_i \) such that \( (u, v) \) has Jacobian equal to 1 on \( \mathbb{C}^2 \).

**Definition 2.4.** A rosary is said to be non-tautological if \( L_i \cap L_{i+2} = \{0\} \) for \( i = 1, \ldots, n-2 \) and \( L_i \neq L_{i+1} \) for \( i = 1, \ldots, n-1 \).

Notice that the second condition is necessary only if \( n = 2 \), because, if \( n > 2 \) and \( L_i = L_{i+1} \) for some \( i = 1, \ldots, n-1 \), then \( \dim L_i \cap L_{i+2} \geq 1 \) if \( i < n-1 \) or \( \dim L_{i-1} \cap L_{i+1} \geq 1 \) if \( i = n-1 \).

We denote by \( \langle v_1, \ldots, v_j \rangle \) the complex vector space spanned by the vectors \( v_1, \ldots, v_j \).

**Definition 2.5.** A sequence \( \mathcal{U} = \{u_0, u_1, \ldots, u_n\} \subset \mathcal{H}_0(\mathbb{C}^2) \) is called a basis for the rosary \( L \) if

1) \( L_i = \langle u_{i-1}, u_i \rangle \) and
2) \( u_{i+1} = f_{i+1}(u_i) + u_{i-1} \), where \( f_{i+1} \in \mathcal{H}_0(\mathbb{C}) \).

Notice that, if \( f_{i+1} \) is linear, then \( L_i = L_{i+1} \), and therefore the rosary \( L \) is tautological.

**Lemma 2.7.** Let \( g_1, \ldots, g_n \) be a sequence of shears such that \( g_j \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) for all \( j = 1, \ldots, n \), and set

\[
L_1 = \langle x, y \rangle, \quad L_2 = \langle g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_1, g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_2 \rangle, \ldots,
\]

\[
L_i = \langle g_i \circ \cdots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_1, g_i \circ \cdots \circ g_1 \begin{pmatrix} x \\ y \end{pmatrix} \cdot e_2 \rangle,
\]

where the dot indicates the canonical hermitian product in \( \mathbb{C}^2 \) and \( \{e_1, e_2\} \) is the canonical basis in \( \mathbb{C}^2 \). Then \( L = \{L_1, \ldots, L_n\} \) is a rosary.

**Proof.** Conditions i) and iii) of Definition 2.3 are trivial.

If, with the same notation as before,

\[
g_{j+1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + f \left( A \begin{pmatrix} x \\ y \end{pmatrix} \right) e,
\]

then taking \( e \) and another suitable vector \( \varepsilon \) as a basis of \( \mathbb{C}^2 \) we can find a basis \( u, v \) of \( L_j \) such that \( g_{j+1}(u, v) = u \varepsilon + (v + f(u))e \). Thus \( L_{j+1} \) is spanned by \( u \) and \( v + f(u) \), which yields ii). \( \diamond \)
The proof of the following proposition essentially follows an argument given in [2] for a more restrictive case.

**Proposition 2.8.** A non tautological rosary has a basis.

**Proof.** We proceed by induction on the length \( n \) of the rosary, the case \( n = 1 \) being trivial.

We first consider \( n = 2 \), to clarify notations. Let \( L = \{L_1, L_2\} \) be a rosary. We have, by definition, \( L_1 = \langle u_0, u_1 \rangle \) and \( L_2 = \langle u_1, v \rangle \), for some \( u_0, u_1, v \). By property iii) of Definition 2.3, \( J(u_0, u_1) = cJ(v, u_1) \), for some \( c \in \mathbb{C}^* \), where \( J \) denotes the Jacobian. Hence \( J(u_0 - cu_1, u_1) = 0 \), which gives \( cv = f(u_1) + u_0 \), where \( f \) is an entire function. The choice \( u_2 = cv \) completes the proof for \( n = 2 \).

By induction we can suppose we have found \( u_1, \ldots, u_k \) with \( L_k = \langle u_k, u_{k-1} \rangle \), \( L_{k-1} = \langle u_{k-1}, u_{k-2} \rangle \) and \( u_k = f_k(u_{k-1}) + u_{k-2} \). By definition of rosary there is \( 0 \neq v \in L_{k+1} \cap L_k \), \( v = au_k + bu_{k-1} \). As \( L \) is non-tautological, \( a \neq 0 \). Hence we can replace \( u_k \) by \( u'_k = u_k + a^{-1}bu_{k-1} = a^{-1}v \in L_k \cap L_{k+1} \) and then we get \( u_{k+1} \) with the same procedure as above. \( \diamond \)

**Remark 2.2.** In the choice of a basis of a non-tautological rosary \( u_{k+1} = f_{k+1}(u_k) + u_{k-1} \), hence \( f_{k+1} \) cannot be linear.

We introduce an ordering on \( \mathcal{H}_0(\mathbb{C}) \) which will be useful in the following. This ordering is provided by Nevanlinna's value distribution theory; as our use of this tool is almost incidental we refer to [2] for a more exhaustive treatment of the subject and further references.

Let \( f \in \mathcal{H}_0(\mathbb{C}) \) and set

\[
m(f, r) = \frac{1}{2\pi} \int_{S^1} \ln^+ \left| f(r\xi) \right| |d\xi|,
\]

where \( S^1 \) is the unit circle in \( \mathbb{C} \) oriented counterclockwise, and \( |d\xi| \) is the standard Lebesgue measure on \( S^1 \).

The following proposition is obtained gathering Lemma 3.2.1 and Corollary 3.2.3 in [2], together with the trivial remark that, if \( f = id_{\mathbb{C}} \), then \( m(f, r) = \ln^+ r \). If \( f = i\overline{d_{\mathbb{C}}} \) we denote \( m(f, r) \) by \( m(z, r) \).

**Proposition 2.9.** Let \( f \in \mathcal{H}_0(\mathbb{C}) \).
If \( f \) is a polynomial of degree \( d \geq 1 \), then \( \limsup_{r \to +\infty} \frac{m(f,r)}{m(x,r)} = d \);

if \( f \) is a transcendental function, then \( \limsup_{r \to +\infty} \frac{m(f,r)}{m(x,r)} = +\infty \).

**Definition 2.5.** If \( u \) and \( v \) are entire functions on \( \mathbb{C} \) we say that

\[
u \succ v \text{ if } \limsup_{r \to +\infty} \frac{m(u,r)}{m(v,r)} > 1.
\]

Then we have the following proposition which has been established in [2]; see Theorem 3.3.1 of [2], where it was stated in a slightly more restrictive form than in Proposition 2.10. The proof given by Andersen extends almost verbatim to our more general setting.

**Proposition 2.10.** Let \( p, q \in \mathcal{H}_0(\mathbb{C}) \), let \( u_0, \ldots, u_n \) be a basis of a non tautological rosary and let \( u_j(s) = u_j(p(s), q(s)) \). Then \( u_j(s) \in \mathcal{H}_0(\mathbb{C}) \), and, if \( u_2(s) \) is non-zero and \( u_2(s) \succ u_1(s) \), then \( u_{k+1}(s) \succ u_k(s) \) for all \( k \geq 1 \).

Our proof of Theorem 1.3 makes use of the ordering \( \succ \) on \( \mathcal{H}_0(\mathbb{C}) \) introduced above to show that, if \( (g_n, \ldots, g_1) \) is a reduced word in \( G_1 \), then \( g_n \circ \ldots \circ g_1 \neq id_{\mathbb{C}^2} \). In fact, given a rosary \( L \), we show that we can find a suitable basis \( u_1, \ldots, u_k \) which is naturally ordered by \( \succ \). This will imply that, given any reduced word \( (g_n, \ldots, g_1) \) in \( G_1 \), then \( g_n \circ \ldots \circ g_1 \neq id_{\mathbb{C}^2} \).

Take \( g \notin A_1 \cap E_1 \), and let \( (g_n, \ldots, g_1) \) be a reduced word of length \( n \) with respect to the subgroups \( A_1 \) and \( E_1 \) such that \( g = g_n \circ \ldots \circ g_1 \). We prove that the length \( n \) is an invariant of the element \( g \in G_1 \). As in the case of \( g \in \text{Aut}_P \mathbb{C}^2 \), all we need follows from

**Theorem 2.11.** If \( g = g_n \circ \ldots \circ g_1 \), where \( (g_n, \ldots, g_1) \) is a reduced word, then \( g \neq id_{\mathbb{C}^2} \).

If \( g = g_n \circ \ldots \circ g_1 = id_{\mathbb{C}^2} \), then the two scalar components of \( \begin{pmatrix} x \\ y \end{pmatrix} \mapsto g \begin{pmatrix} x \\ y \end{pmatrix} \) are both linear in \( x \) and \( y \); moreover they both belong to \( \mathcal{H}_0(\mathbb{C}^2) \).

The first step in the proof of Theorem 2.11 consists in showing that we can replace \( g = g_n \circ \ldots \circ g_1 \) by \( \tilde{g} = \tilde{g}_n \circ \ldots \circ \tilde{g}_1 \), where
\[ \tilde{g}_k \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]
\[ g_k \text{ and } \tilde{g}_k \text{ are in the same subgroup, and } \tilde{g}_k \notin A \cap E, \]
for \( k = 1, \ldots, n. \)

**Proposition 2.12.** Suppose \((g_n, \ldots, g_1)\) is a reduced word in \( G_1 \)
with—letting as before \( g = g_n \circ \ldots \circ g_1 - g \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). We can find a
representation of \( g \) as a reduced word in \( G_1 \), \( g = \tilde{g}_n \circ \ldots \circ \tilde{g}_1 \), where
\[ \tilde{g}_k \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]
for all \( k = 1, \ldots, n. \)

**Proof.** Set \( B_1 = g_1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and \( \tilde{g}_1(z) = g_1(z) - B_1 \), that is \( \tilde{g}_1 = r_1 \circ g_1 \), where \( r_1 \) is the translation of vector \(-B_1\).

Set again \( B_{k+1} = g_{k+1}(B_k) \) and \( \tilde{g}_{k+1}(z) = g_{k+1}(z + B_k) - B_{k+1} \).
Hence \( \tilde{g}_{k+1} = r_{k+1} \circ g_{k+1} \circ r_k^{-1} \), where \( r_k \) is the translation of vector
\(-B_k\). Then \( \tilde{g}_k \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), and, as the translations are contained in
\( A_1 \cap E_1 \), \( \tilde{g}_k \) is contained in the same subgroup as \( g_k \) (i.e. \( g_k \in A_1 \) iff \( \tilde{g}_k \in A_1 \) and similarly \( g_k \in E_1 \) iff \( \tilde{g}_k \in E_1 \))
moreover \( \tilde{g}_k \notin A_1 \cap E_1 \).

Since \( B_n = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), then \( g_n \circ \ldots \circ g_1 = \tilde{g}_n \circ \ldots \circ \tilde{g}_1 \), and we are
done.

By Proposition 2.12 we can suppose that, if \( g = g_n \circ \ldots \circ g_1 = id_{C^\varepsilon} \), then \( g_k \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), for \( k = 1, \ldots, n. \) Now we can come to
the proof of Theorem 2.11.

**Proof.** (of Theorem 2.11) By Proposition 2.12 there is no restriction
in assuming that each \( g_k \) maps \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) to \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). Since the case \( n = 1 \) is obvious, we can suppose \( n \geq 2 \) and proceed by induction on \( n. \)

If \( g_n \) and \( g_1 \) are both in \( A_1 \) or both in \( E_1 \), then we can replace
\( g_n \circ \ldots \circ g_2 \circ g_1 \) by \( g_1 \circ g_n \circ \ldots \circ g_2 \); this is still equal to the identity,
but, as a reduced word, has length \( n - 1 \). So we can suppose that \( n \)
is even, in fact we have seen that we can suppose that \( g_n \) and \( g_1 \) do
not belong to the same subgroup; hence, as each \( g_j \) does not belong
to the same subgroup of \( g_{j+1} \), we can suppose that \( n \) is even.
If \((g_n, \ldots, g_1)\) is a reduced word such that \(g_n \circ \cdots \circ g_1 = id_{C^2}\), we can suppose that \(g_1 \in E_1\). In fact if \(g_1 \in A_1\) we have \(g_2 \in E_1\) and \(g_1 \circ g_n \circ \cdots \circ g_2\) is still equal to the identity map. If \(n = 2\) we get \(g_1 = g_2^{-1}\), whence \(g_1 \in A_1 \cap E_1\), which is a contradiction. Thus we can suppose \(n \geq 4\).

Now we prove that \(L_1, \ldots, L_m\) given by

\[
L_1 = \langle x, y \rangle, \quad L_2 = \langle g_1 \left( \begin{array}{c} x \\ y \end{array} \right) \cdot e_1, g_1 \left( \begin{array}{c} x \\ y \end{array} \right) \cdot e_2 \rangle, \\
L_3 = \langle g_3 \circ g_2 \circ g_1 \left( \begin{array}{c} x \\ y \end{array} \right) \cdot e_1, g_3 \circ g_2 \circ g_1 \left( \begin{array}{c} x \\ y \end{array} \right) \cdot e_2 \rangle, \ldots, \\
L_m = \langle g_{n-1} \circ \cdots \circ g_1 \left( \begin{array}{c} x \\ y \end{array} \right) \cdot e_1, g_{n-1} \circ \cdots \circ g_1 \left( \begin{array}{c} x \\ y \end{array} \right) \cdot e_2 \rangle,
\]

where \(m = n/2 + 1\), is a non tautological rosary.

First of all \(L = \{L_1, L_2, \ldots, L_m\}\) is a rosary because, if we set

\[
M_1 = \langle x, y \rangle, \\
M_{j+1} = \langle g_j \circ \cdots \circ g_2 \circ g_1 \left( \begin{array}{c} x \\ y \end{array} \right) \cdot e_1, g_j \circ \cdots \circ g_2 \circ g_1 \left( \begin{array}{c} x \\ y \end{array} \right) \cdot e_2 \rangle, \ldots,
\]

then \(M = \{M_1, M_2, \ldots, M_n\}\) is a rosary by Lemma 2.7 and so, \(L_1 = M_1, L_2 = M_2, L_3 = M_3, L_4 = M_4, \ldots\), is a rosary.

We now prove that \(L\) is a non tautological rosary. Note first that \(L_1 \neq L_2\) because \(L_2 = \langle x + f_1(y), y \rangle\), where \(f_1\) is non-linear (because \(g_1 \in E_1 \setminus A_1\) and \(f_1(0) = 0\).

Suppose that \(L_1 \cap L_3 \neq \{0\}\), and write

\[
g_1 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} \alpha_1 x + f_1(y) \\ \alpha_1^{-1} y \end{array} \right), \quad g_2 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} a_1 x + b_1 y \\ e_1 x + d_1 y \end{array} \right) \quad \text{and}
\]

\[
g_3 \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} \alpha_3 x + f_3(y) \\ \alpha_3^{-1} y \end{array} \right),
\]

where \(f_1\) and \(f_3\) are on-linear elements of \(H_0(C)\) and \(e_1 \neq 0\). Then

\[
g_3 \circ g_2 \circ g_1 \left( \begin{array}{c} x \\ y \end{array} \right) =
\left( \frac{\alpha_3 (a_1 (\alpha_1 x + f_1(y)) + b_1 \alpha_1^{-1} y) + f_3 (\alpha_1 (\alpha_1 x + f_1(y)) + d_1 \alpha_1^{-1} y)}{\alpha_3^{-1} (\alpha_1 (\alpha_1 x + f_1(y)) + d_1 \alpha_1^{-1} y)} \right).
\]
If \( \langle x, y \rangle \cap \langle g_3 \circ g_2 \circ g_1 \left( \frac{x}{y} \right), e_1, g_3 \circ g_2 \circ g_1 \left( \frac{x}{y} \right), e_2 \rangle \neq \{0\} \) there exist \( \gamma, \delta \in \mathbb{C} \) such that \( |\gamma| + |\delta| > 0 \) and

\[
\gamma [a_3(a_1 x + f_1(y)) + b a_1^{-1} y] + f_3(c_1(a_1 x + f_1(y)) + d a_1^{-1} y)] + \\
\delta [a_3^{-1}(c_1(a_1 x + f_1(y)) + d a_1^{-1} y)]
\]
is linear in \( x \) and \( y \). Then \( \gamma (a_3 f_1(y) + f_3(c_1(a_1 x + f_1(y)) + d a_1^{-1} y) + \\
d c_1 a_3^{-1} f_1(y) \) is linear in \( x \) and \( y \), and therefore taking the derivative with respect of \( x \) we find that \( a_1 \gamma c_1 f_1^1(c_1(a_1 x + f_1(y)) + d a_1^{-1} y) \) is constant. As \( f_3 \) is non-linear and \( c_1(a_1 x + f_1(y)) + d a_1^{-1} y \) is non-constant, then \( a_1 \gamma c_1 = 0 \), and since \( a_1 c_1 \neq 0 \), we obtain that \( \gamma = 0 \). Thus \( \delta c_1 f_1(y) \) is linear in \( x \) and \( y \) in contrast with the fact that \( \delta c_1 \neq 0 \) and \( f_1 \) is non-linear.

Suppose now that \( L_k \cap L_{k+2} \neq \{0\} \), with \( k > 1 \) and set \( w_j = g_{2k-2} \circ \cdots \circ g_1 \left( \frac{x}{y} \right), e_j, j = 1, 2 \). Then

\[
L_k = \langle g_{2k-3} \circ \cdots \circ g_1 \left( \frac{x}{y} \right), e_1, g_{2k-3} \circ \cdots \circ g_1 \left( \frac{x}{y} \right), e_2 \rangle = \langle g_{2k-2} \circ \cdots \circ g_1 \left( \frac{x}{y} \right), e_1, g_{2k-2} \circ \cdots \circ g_1 \left( \frac{x}{y} \right), e_2 \rangle = \langle w_1, w_2 \rangle,
\]
because \( g_{2k-2} \) is linear. Moreover

\[
L_{k+2} = \langle g_{2k+1} \circ \cdots \circ g_1 \left( \frac{x}{y} \right), e_1, g_{2k+1} \circ \cdots \circ g_1 \left( \frac{x}{y} \right), e_2 \rangle = \langle g_{2k+1} \circ g_2 \circ g_{k-1} \left( \frac{w_1}{w_2} \right), e_1, g_{2k+1} \circ g_2 \circ g_{k-1} \left( \frac{w_1}{w_2} \right), e_2 \rangle.
\]

If \( L_k \cap L_{k+2} \neq \{0\} \), then there are \( \gamma, \delta \in \mathbb{C} \) such that \( |\gamma| + |\delta| > 0 \) and the holomorphic function

\[
\gamma g_{2k+1} g_2 g_{k-1} \left( \frac{w_1}{w_2} \right) + e_1 + \delta g_{2k+1} g_2 g_{k-1} \left( \frac{w_1}{w_2} \right) = e_2
\]
is linear in \( w_1 \) and \( w_2 \). Hence, arguing as in the case \( k = 1 \), we get a contradiction, proving that \( L \) is a non tautological rosary.

We now choose a basis \( \{ u_0, u_1, \ldots, u_m \} \) of the rosary \( L \). As \( L_1 = \langle x, y \rangle, L_2 = \langle x + f_1(y), y \rangle, \) and so on, then we have \( u_0 = x, u_1 = y \) and \( u_2 = \mu(x + f_1(y)) + \nu y \), where \( \mu \in \mathbb{C}^* \). Choosing \( p(s) = q(s) = s \),
and, using the notations of Proposition 2.10, we obtain that $u_0(s) = u_1(s) = s$, $u_2(s) = \mu f_1(s) + (\mu + \nu)s$, where $\mu \neq 0$. It is easily seen that $u_2(s)$ is non-zero. To apply Proposition 2.10 we only need to prove that $u_2(s) \succ u_1(s)$.

For this goal note that, if $f_1$ is a polynomial of degree $d > 1$ and $\mu \neq 0$, then $\mu f_1(s) + (\mu + \nu)s$ is still a polynomial of degree $d$, hence, by Proposition 2.9,

$$
\lim \sup_{r \to +\infty} \frac{m(\mu f_1(s) + (\mu + \nu)s, r)}{m(s, r)} = d > 1;
$$

whereas, if $f_1$ is transcendental, we have

$$
\lim \sup_{r \to +\infty} \frac{m(\mu f_1(s) + (\mu + \nu)s, r)}{m(s, r)} = +\infty,
$$

again by Proposition 2.9. Hence we can apply Proposition 2.10, obtaining $u_m(s) \succ u_{m-1}(s) \succ \ldots u_2(s) \succ u_1(s) = s$. As $m = n/2 + 1 \geq 3$, neither $u_m(s)$ nor $u_{m-1}(s)$ are linear in $s$.

If $g = g_n \circ \ldots \circ g_1$ were the identity map, then the components of

$$
\begin{pmatrix} x \\ y \end{pmatrix} \mapsto g \begin{pmatrix} x \\ y \end{pmatrix}
$$

were $x$ and $y$. Thus $L_m$ would be equal to $(u_m, u_{m-1})$, with $u_m(s)$ and $u_{m-1}(s)$ non-linear in $s$, whereas, replacing $x$ by $p(s) = s$ and $y$ by $q(s) = s$, we obtain two functions which are both linear in $s$. This contradiction shows that $g_n \circ \ldots \circ g_1$ is not the identity map and completes the proof of Theorem 2.11.

Then we obtain, as a trivial consequence, the following

**Corollary 2.13.** If $(g_n, \ldots, g_1)$ is a reduced word, then its length is an invariant of the element $g = g_n \ldots g_1 \in G_1$. Moreover the representation $g = g_n \circ \ldots \circ g_1$ is unique up to replacing $g_k$ by $h g_k$ and $g_{k+1}$ by $g_{k+1} h^{-1}$, for some $h \in A_1 \cap E_1$.

Theorem 1.1 and Theorem 1.3 are the keys of the forthcoming section, which contains the proof of the conjugacy theorem for the one-parameter groups in Aut$\mathbb{P}\mathbb{C}^2$ and $G_1$. 
3. Proof of the Conjugacy Theorem.

This section contains the proof of Theorem 1.4. As the proof is equal in the case of the two groups $Aut_p C^2$ and $G_1$, we introduce the following notations:

$$\mathcal{G} = Aut_p C^2 \text{ (respectively } G_1)$$

$$\mathcal{E} = E \text{ (respectively } E_1)$$

$$A = A \text{ (respectively } A_1)$$

Let $g \in \mathcal{G} \setminus (A \cap \mathcal{E})$ and let $(g_1, \ldots, g_k)$ be its representation as a reduced word in $\mathcal{G}$ (this representation is almost “unique”, in the sense specified in Corollary 2.6 and Corollary 2.13).

Obviously $g_k \circ g_{k-1} \circ \ldots \circ g_1 \circ g_0 \circ \ldots \circ g_{k+1}$ is conjugated to $g$ in $\mathcal{G}$. Hence, if $g_n$ and $g_l$ both belong to either $A$ or $\mathcal{E}$, in the conjugacy class of $g$ there is an element which has a representation as a reduced word whose length is strictly less than the length of $g$.

At this point only two cases can occur: either the element $\hat{g}$ of minimal length in the conjugacy class of $g$ is of length 1, i.e., $g = X \circ h_1 \circ X^{-1}$, where $X \in \mathcal{G}$ and $h_1 \in (A \cup \mathcal{E}) \setminus (A \cap \mathcal{E})$, or $\hat{g}$ has a representation as a reduced word of even length and the first and last elements of this word do not both belong to the same of the two groups $A$ or $\mathcal{E}$.

Our proof of Theorem 1.4 relies on the following estimate of the length of the powers of a word.

**Proposition 3.1.** If $g \in \mathcal{G} \setminus (A \cap \mathcal{E})$ is conjugated in $\mathcal{G}$ to an element of minimal length $2r$, then the length of $g^m$ is at least $2rm$.

**Proof.** As $g$ is conjugated in $\mathcal{G}$ to an element of minimal length $2r$, then $g = X \circ h_{2r} \circ \ldots \circ h_1 \circ X^{-1}$, where $\hat{g} = h_{2r} \circ \ldots \circ h_1$, $(h_{2r}, \ldots, h_1)$ is a reduced word, $X \in \mathcal{G}$ and $h_1$ and $h_{2r}$ do not both belong to the same of the two groups $A$ or $\mathcal{E}$. Hence

$$g^m = X \circ (h_{2r} \circ \ldots \circ h_1)^m \circ X^{-1} \ (3.1)$$

We remark that, if $X \in A \cap \mathcal{E}$, then (3.1) implies that the length of $g^m$ is equal to $2rm$. If $X \notin A \cap \mathcal{E}$, let $X = \xi_1 \circ \ldots \circ \xi_j$, where $(\xi_1, \ldots, \xi_j)$ is a reduced word. We proceed by induction on $j$.\[\]
If \( j = 1 \), then \( g = \xi_1 \circ h_2 \circ \ldots \circ h_1 \circ \xi_1^{-1} \) and, as \( h_2r \) and \( h_1 \) do not belong to the same of the two subgroup \( \mathcal{A} \) and \( \mathcal{E} \), this is not a representation as a reduced word. Suppose that \( h_2r \) and \( \xi_1 \) belong to the same subgroup (the case in which \( h_2r \) is replaced by \( h_1 \) can be dealt with exactly in the same way). If \( \xi_1 \circ h_2r \notin \mathcal{A} \cap \mathcal{E} \), then \((\xi_1 \circ h_2r), h_{2r-1}, \ldots, h_1, \xi_1^{-1})\) is a reduced word and thus
\[
g^m = (\xi_1 \circ h_2r) \circ h_{2r-1} \circ \ldots \circ h_1 \circ h_2r \circ \ldots \circ h_1 \circ \xi_1^{-1}.
\]
Hence a representation of \( g^m \) as a reduced word is obviously given by
\[
((\xi_1 \circ h_2r), h_{2r-1}, \ldots, h_1, h_2r, \ldots, h_1, \xi_1^{-1})
\]
and the length of \( g^m \) is bigger than \( 2rm \). If \( \xi_1 \circ h_2r \in \mathcal{A} \cap \mathcal{E} \), we set \( e = \xi_1 \circ h_2r \), \( \tilde{h}_{2r-1} = e \circ h_{2r-1} \) and \( \tilde{h}_{2r} = h_{2r} \circ e^{-1} \). Then \( \tilde{h}_{2r-1} \) and \( \tilde{h}_{2r} \) are not in the same of the two subgroups \( \mathcal{A} \) and \( \mathcal{E} \) (because \( h_{2r-1} \) and \( h_2r \) are not). Then we have \( g = \tilde{h}_{2r-1} \circ h_{2r-2} \circ \ldots \circ h_1 \circ h_2r \), where \((\tilde{h}_{2r-1}, h_{2r-2}, \ldots, h_1, \tilde{h}_{2r})\) is a reduced word; then, using the above remark, we obtain that the length of \( g^m \) is \( 2rm \).

Proceeding by induction on \( j \), we can suppose that the statement is true for \( j - 1 \) and consider \( X = \xi_j \circ \ldots \circ \xi_1, \) where \( (\xi_j, \ldots, \xi_1) \) is a reduced word.

Then \( g = \xi_1 \circ \ldots \circ \xi_j \circ h_2r \circ \ldots \circ h_1 \circ \xi_j^{-1} \circ \ldots \circ \xi_1^{-1} \) and, as \( h_2r \) and \( h_1 \) do not belong to the same of the two subgroups \( \mathcal{A} \) and \( \mathcal{E} \), this is not a representation as a reduced word. Suppose that \( h_2r \) and \( \xi_j \) belong to the same subgroup (the case in which \( h_2r \) is replaced by \( h_1 \) can be dealt with exactly in the same way). If \( \xi_j \circ h_2r \notin \mathcal{A} \cap \mathcal{E} \), then
\[
(\xi_1, \ldots, \xi_{j-1}, (\xi_j \circ h_2r), h_{2r-1}, \ldots, h_1, \xi_j^{-1}, \ldots, \xi_1^{-1})
\]
is a reduced word and we obtain
\[
g^m = \xi_1 \circ \ldots \circ (\xi_j \circ h_2r) \circ h_{2r-1} \circ \ldots \circ h_1 \circ \ldots \circ h_2r \circ \ldots \circ h_1 \circ \xi_j^{-1} \circ \ldots \circ \xi_1^{-1}.
\]
Thus a representation of \( g^m \) as a reduced word is given by
\[
(\xi_1, \ldots, \xi_{j-1}, (\xi_j \circ h_2r), h_{2r-1}, \ldots, h_1, \ldots, h_2r, \ldots, h_1, \xi_j^{-1}, \ldots, \xi_1^{-1}),
\]
showing that the length of \( g^m \) is \( 2rm + 2j - 1 \geq 2mr \).
If $\xi_j \circ h_{2r} \in \mathcal{A} \cap \mathcal{E}$, setting $c = \xi_j \circ h_{2r}$, we obtain $g = \xi_1 \circ \ldots \circ \xi_{j-1} \circ c \circ h_{2r-1} \circ \ldots \circ h_1 \circ h_{2r} \circ c^{-1} \circ \xi_{j-1} \circ \ldots \circ \xi_1^{-1}$. If we define $\tilde{h}_{2r-1} = c \circ h_{2r-1}$ and $\tilde{h}_{2r} = h_{2r} \circ c^{-1}$, then $\tilde{h}_{2r-1}$ and $\tilde{h}_{2r}$ are still in different subgroups and they both lie outside $\mathcal{A} \cap \mathcal{E}$. Hence $(\tilde{h}_{2r-1}, \tilde{h}_{2r-2}, \ldots, h_1, h_{2r})$ is a reduced word and we have found a representation of $g$

$$g = Y \circ \tilde{h}_{2r-1} \circ h_{2r-2} \circ \ldots \circ h_1 \circ \tilde{h}_{2r} \circ Y^{-1},$$

with $Y = \xi_1 \circ \ldots \circ \xi_{j-1}$. Then we can proceed by induction on $j$ to obtain that $g^m$ has length greater or equal than $2rm$.

At last we come to the proof of Theorem 1.4, which will be given in the case of a one-parameter group $\Phi : \mathbb{R} \rightarrow \mathcal{G}$.

First of all we prove that $\mathcal{A}$ and $\mathcal{E}$ are closed in $\mathcal{G}$: in fact, let $\varphi_n \rightarrow \varphi$ in $G_1(\text{Aut}_p \mathbb{C}^2)$ and suppose that $\varphi_n \in E_1(E)$, then the first component of $\varphi_n$ is affine in $x$ and the second component of $\varphi_n$ does not depend on $x$ and is affine in $y$, and therefore also the first component of $\varphi$ is affine in $x$ (it depends on $x$ because $\varphi$ is a biholomorphism of $\mathbb{C}^2$) and the second component of $\varphi$ does not depend on $x$ and is affine in $y$ (it depends on $y$ because $\varphi$ is a biholomorphism of $\mathbb{C}^2$). Moreover the Jacobian of $\varphi$ is equal to the limit of the Jacobian of $\varphi_n$, therefore also $E_1$ is closed in $G_1$. If $\varphi_n \rightarrow \varphi$ in $G_1(\text{Aut}_p \mathbb{C}^2)$ and $\varphi_n$ is affine, then $\varphi$ is affine as well, and the same reasoning on Jacobians implies that $\mathcal{A}$ is closed in $\text{Aut}_p \mathbb{C}^2$ and $\mathcal{A}_1$ is closed in $G_1$. Therefore $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$ is closed in $\mathcal{G}$.

Next we prove that, for any $t \in \mathbb{R}$, $\Phi_t$ is conjugated to an element in $\mathcal{A}$ or $\mathcal{E}$.

If $\Phi_t \in \mathcal{A} \cap \mathcal{E}$, this is obvious. If $\Phi_t \notin \mathcal{A} \cap \mathcal{E}$, let $l$ be the length of $\Phi_t$ and choose $m_0 \in \mathbb{N}$ such that $l < 2m_0$. Consider $\Phi_{t/m_0}$: if this were not conjugated to an element in $\mathcal{A}$ or $\mathcal{E}$, then it should be conjugated to an element of minimal length $2r$, therefore Proposition 2.2 implies that the length of $\Phi_t = (\Phi_{t/m_0})^{m_0}$ is greater than or equal to $2rm_0 \geq 2m_0 > l$, that is a contradiction. Then $\Phi_{t/m_0}$ is conjugated to an element of $\mathcal{A}$ or $\mathcal{E}$ and hence $\Phi_t = (\Phi_{t/m_0})^{m_0}$ is conjugated to an element of $\mathcal{A}$ or $\mathcal{E}$.

We recall that we proved that $\mathcal{G}$ is the free product of $\mathcal{A}$ and $\mathcal{E}$, amalgamated over $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$. We call in the following theorem, due to Moklavanski (see [13], Theorem 0.3),

**Theorem 3.2.** Let $H$ be an abelian subgroup of $\mathcal{G}$ where $\mathcal{G}$ is
the free product of $\mathcal{A}$ and $\mathcal{E}$ amalgamated over their intersection $\mathcal{B}$. Precisely one of the following situations holds:

1) $H$ is conjugated in $\mathcal{G}$ to a subgroup of $\mathcal{A}$ or $\mathcal{E}$, 

2) $H$ is not conjugated in $\mathcal{G}$ to any subgroup of $\mathcal{A}$ or $\mathcal{E}$. There exists a nested chain of subgroups $H_0 \subset H_1 \subset \ldots \subset H_{i-1} \subset H_i \subset \ldots$ such that $H = \bigcup_{i=0}^{\infty} H_i$ and each $H_i$ is conjugated in $\mathcal{G}$ to a subgroup of $\mathcal{B}$, 

3) $H = F \times < g >$, where $< g >$ is the subgroup of $\mathcal{G}$ generated by $g$, $F$ is conjugated to a subgroup of $\mathcal{B}$, $g$ is not conjugated to any element of $\mathcal{A}$ or $\mathcal{E}$ (where $\times$ denotes the map $\mathcal{B} \times W \to \mathcal{G}$ given by multiplication and $W$ denotes the set of reduced words in $\mathcal{G}$, see [13]).

The subgroups of $\mathcal{G}$ are called of type 1), 2) or 3), according to the fact that they satisfy 1), 2) or 3).

**Remark 3.1.** If $H$ is of type 3), in particular it contains the element $g$, which is not conjugated to any element in $\mathcal{A}$ or $\mathcal{E}$.

Now, let $\Phi$ be a one-parameter group in $\mathcal{G}$: we already proved that for all $t \in \mathbb{R}$, $\Phi_t$ is conjugated to an element of $\mathcal{A}$ or $\mathcal{E}$, hence $H = \{ \Phi_t, t \in \mathbb{R} \}$ is an abelian subgroup of $\mathcal{G}$ which cannot be of type 3).

Let us prove that $H$ cannot be of type 2).

If $H$ is of type 2), we denote by $C_i$ the set $\{ t \in \mathbb{R} : \Phi_t \in H_i \}$, where the $H_i$'s are the subgroups in the definition of subgroup of type 2).

Let $X_t \in \mathcal{G}$ be such that $X_t \circ h \circ X_t^{-1} \in \mathcal{B}$ for all $h \in H_i$, then $X_t \circ \Phi_t \circ X_t^{-1} \in \mathcal{B}$ for all $t \in C_i$. As $\mathcal{B}$ is closed and both conjugation and $\Phi$ are continuous mappings, then $X_t \circ \Phi_t \circ X_t^{-1} \in \mathcal{B}$ for all $t \in C_i$.

Since $H = \bigcup_{i=0}^{\infty} H_i$, then $\bigcup_{i=0}^{\infty} C_i = \mathbb{R}$ and therefore $\bigcup_{i=0}^{\infty} C_i = \mathbb{R}$.

Baire’s category theorem implies that there exists $i_0 \in \mathbb{N}$ such that the inner part of $C_{i_0}$ is not empty, therefore there exist $\tau \in \mathbb{R}$ and a positive number $\varepsilon$ such that $(\tau - \varepsilon, \tau + \varepsilon) \subset C_{i_0}$. Therefore $X_t \circ \Phi_t \circ X_t^{-1} \in \mathcal{B}$ for all $t \in (\tau - \varepsilon, \tau + \varepsilon)$.

Let us consider the one-parameter group $\Psi_t = X_{i_0} \circ \Phi_t \circ X_{i_0}^{-1}$, we already proved that $\Psi_t \in \mathcal{B}$ for all $t \in (\tau - \varepsilon, \tau + \varepsilon)$.

Let $\rho \in (-\varepsilon, \varepsilon)$, then $\Psi_{\rho} = \Psi_{\tau+\rho-\tau} = \Psi_{\tau+\rho} \circ \Psi_{-\tau} = \Psi_{\tau+\rho} \circ (\Psi_{\tau})^{-1}$, as $\tau + \rho$ and $\tau$ belong to $(\tau - \varepsilon, \tau + \varepsilon)$ and $\mathcal{B}$ is a subgroup
of \( \mathcal{G} \), then \( \Psi_{\rho} \in \mathcal{B} \) for all \( \rho \in (-\varepsilon, \varepsilon) \). Taking all integer multiples of the interval \((-\varepsilon, \varepsilon) \), we obtain the \( \Psi_t \in \mathcal{B} \) for all \( t \in \mathbb{R} \).

Then \( H \) is conjugated to a subgroup of \( \mathcal{B} \), as \( \mathcal{B} \) is in particular a subgroup of \( \mathcal{A} \), then \( H \) is conjugated to a subgroup of \( \mathcal{A} \); this contradicts the fact that \( H \) is of type 2], therefore, by Theorem 3.2, \( H \) is of type 1), i.e., \( H \) is conjugated to a subgroup of \( \mathcal{A} \) or \( \mathcal{E} \) and this proves Theorem 1.4. \( \diamond \)

References


[13] Wright D., *Abelian subgroups of \( \text{Aut}_k(k[X, Y]) \) and applications to actions on the affine plane*, Illinois Jour. of Math. 23 (1979), 579-634.