RIEMANNIAN MANIFOLDS WITH SPECIAL CURVATURE TENSOR (*)

by Fabio Podestà (in Parma)
and Franco Tricerri † (in Firenze)(**) 

SOMMARIO. - Lo scopo di questo lavoro è studiare varietà Riemanniane a curvatura omogenea il cui tensor di curvatura è della forma $aR_S^s+bK$, $a, b \in \mathbb{R}$, dove $K$ è semisimmetrico, i.e. $K \cdot K = 0$, e di Einstein. Quando $a > 0$, si prova che la varietà deve avere curvatura sezionale costante, mentre il caso $a < 0$ rimane aperto.

SUMMARY. - This paper is dealing with the problem of characterizing those curvature homogeneous Riemannian manifolds whose Riemannian curvature tensor is of the form $aR_S^s+bK$, $a, b \in \mathbb{R}$, where $K$ is semisymmetric, i.e. $K \cdot K = 0$, and Einsteinian. When $a > 0$, it is shown that the manifold must be of constant sectional curvature, while the case $a < 0$ still remains open.

0. Introduction.

The main purpose of this note is to continue the study of Riemannian manifolds having a special assigned curvature tensor (see, for example, [KP], [KTV], [TV1], [TV2]).

The authors heartily thank Professor L. Vanhecke for stimulating conversations on this subject. This work was partially supported by M.U.R.S.T. 40% and by G.N.S.A.G.A. of C.N.R. Italy.

(*) Pervenuto in Redazione il 30 maggio 1994.
(**) Indirizzi degli Autori: F. Podestà: Università di Parma, Dipartimento di Matematica, Via M. D’Azeglio, 43100 Parma (Italia), E-mail address: podesta@vm.idg.fcnr.it; F. Tricerri: è docente presso l’Università di Firenze, Dipartimento di Matematica “U. Dini”. È scomparso nel mese di giugno del 1995.
An $n$-dimensional Riemannian manifold $(M, g)$ is said to be curvature homogenous (see [TV1]) if, for each $p, q \in M$, there exists a linear isometry $F : T_p M \to T_q M$ such that

$$F^* R_q = R_p$$

where $R$ denotes the curvature tensor of $(M, g)$. This is also equivalent to saying that $R_p$ and $R_q$ have the same components with respect to suitable orthonormal bases in $T_p M$ and $T_q M$. A homogeneous Riemannian manifold is clearly curvature homogenous, but there are many examples of curvature homogenous Riemannian manifolds which are not even locally homogenous (see [KTV] and [TV1] for more details and further references). To give an alternative description of curvature homogeneity, we consider the principal fibre bundle $O(M)$ of orthonormal frames on $M$; the curvature tensor $R$ may be viewed as a $C^\infty$ map

$$R : O(M) \to \mathfrak{A}(V)$$

where $\mathfrak{A}(V)$ denotes the space of algebraic curvature tensors on an $n$-dimensional vector space $V$. The map $R$ is $O(n)$-equivariant in the sense that $R(au) = a^{-1}R(u)$ for all $u \in O(M)$ and all $a \in O(n)$, the orthogonal group acting on $O(M)$ and $\mathfrak{A}(V)$ in a standard way. It turns out that curvature homogeneity is actually equivalent to saying that $R(O(M))$ is contained in a single $O(n)$-orbit in $\mathfrak{A}(V)$. Moreover, given any algebraic curvature tensor $R_\alpha$, we will say that $(M, g)$ has a curvature tensor of type $R_\alpha$ if $R(O(M)) \subset O(n) \cdot R_\alpha$, that is, if the image of the map $R$ is contained in the $O(n)$-orbit through $R_\alpha$. Recently, Kowalski and Prüfer ([KP]) have constructed a family of algebraic curvature tensors in dimension four which do not belong to any curvature homogenous space.

One starting point for the problem we will be dealing with in this paper is the following result proved in [TV2]:

**Theorem.** Let $(M, g, J)$ be an almost Hermitian manifold with real dimension $2n \geq 6$ and Riemannian curvature tensor of the form

$$R = f R_{S^{2n}} + h R_{C^{2n}}$$

where $f$ and $h$ are $C^\infty$ functions with $h$ not identically zero. Then $M$ is a complex space form.
Remark. If \( 2n = 4 \) the previous theorem also holds when \( h \) is supposed to be constant (see [TV2]). But if not, counterexamples are given in [Ol].

In this note we give some first results about characterizing Riemannian curvature homogeneous manifolds whose curvature tensor is of type

\[ R_o = a R_{S^n} + b K_o \]  

\( (*) \)

where \( a, b \) are real numbers and \( K_o \) is a symmetric curvature tensor, that is, satisfies \( K_o \cdot K_o = 0 \). We recall here that, by a classical theorem due to E. Cartan (see [He]), each symmetric algebraic curvature tensor is the curvature tensor of a Riemannian symmetric space.

As we will see, the problem of giving a full classification of curvature homogeneous spaces having curvature of type \( (*) \) seems to be not trivial, at least when the curvature \( K_o \) corresponds to a reducible Riemannian symmetric space. Indeed, we have the following contrasting examples:

**Example 1.** In [Ts], Tsukada gave an explicit example of a four-dimensional curvature homogeneous manifold \((M, g)\) whose curvature tensor is of type

\[ R_o = -R_{S^1} + R_{\mathbb{R}^2 	imes S^2}. \]

The manifold \((M, g)\) turns out to be not locally homogeneous. Moreover, it can be proved (see [KTV]) that there is no homogeneous model space, that is, there does not exist any homogeneous space \((\tilde{M}, \tilde{g})\) with \( R(O(M)) \subset \tilde{R}(O(\tilde{M})) \), where \( \tilde{R} \) is the curvature map for \((M, \tilde{g})\).

**Example 2.** Cartan isoparametric hypersurfaces \((M, g)\) with three distinct principal curvatures in the space form \( S^4(c) \) of constant curvature \( c > 0 \) provide homogeneous examples whose curvature is of type \( (*) \). Indeed, it is easy to see that, if \(-\lambda, 0, \lambda\) denote the three different principal curvatures, with \( \lambda = \sqrt{3c} \), the curvature tensor of \((M, g)\) is of type

\[ R_o = 4 R_{S^1} + 4 \lambda^2 R_{\mathbb{R}^2 	imes S^2}. \]
Our main result deals with the case when the Ricci tensor of $(M, g)$ is parallel and gives a partial classification in this case.

**Theorem.** Let $(M, g)$ be a Ricci parallel, curvature homogeneous Riemannian manifold whose curvature is of type $(\ast)$. Then the constant $a$ is nonpositive or $(M, g)$ has constant sectional curvature.

**Remark.** Note that $(M, g)$ is Ricci parallel if and only if it is locally isometric to the product of Einstein spaces. In particular, this hypothesis of the Theorem is satisfied if $K_o$ is the curvature tensor of an irreducible or, more generally, an Einstein symmetric space. In fact, in such a case $(M, g)$ itself is Einstein.

Up to now, the authors were not able to find any example of a manifold satisfying the conditions of the Theorem with a negative $a$ and not of constant sectional curvature.

1. **Proof of the Theorem.**

   We may suppose that $R = aR^c + bK_o$ with $b \neq 0$, otherwise our claim follows immediately. In such a case, since $(M, g)$ is curvature homogeneous, $M$ has a $G$-structure $P$, where $G$ is the Lie subgroup of $O(n)$ defined by

   \[ G = \{ a \in O(n) | aK_o = K_o \} \]

   (see [KTV]). Then $K_o$ induces a tensor field $K$ on $M$, defined by

   \[ K_p(x, y, z, w) = K_o(u^{-1}x, u^{-1}y, u^{-1}z, u^{-1}w) \]

   where $u$ is an element of the reduced bundle $P$ belonging to the fibre over the point $p \in M$. Hence the Riemann curvature tensor $R$ of $(M, g)$ can be written as

   \[ R = aR^c + bK, \]

   where $R^c$ is the tensor field on $M$ given by

   \[ R^c(x, y, z, w) = g(x, z)g(y, w) - g(x, w)g(y, z). \]
Note that
\[
\nabla R = b \nabla K
\]
where $\nabla$ denotes the Levi Civita connection of $(M, g)$. It follows that $\nabla K = \frac{1}{2} \nabla R$ satisfies the second Bianchi identity.

Moreover, by the curvature homogeneity, the length of the tensor field $K$ is constant on the whole of $M$, so that, if $\Delta$ denotes the Laplace operator on $(M, g)$, we have
\[
\frac{1}{2} \Delta(||K||^2) = ||\nabla K||^2 + (\nabla^2_{\text{mn}} K_{ijkl}) K_{ijkl} = 0 \tag{2.1}
\]
where repeated indices mean summation.

Now, using the second Bianchi identity and the Ricci commutation formulas, we have
\[
(\nabla^2_{\text{mn}} K_{ijkl}) K_{ijkl} = - (\nabla^2_{\text{mj}} K_{mikl} + \nabla^2_{\text{mi}} K_{jmkl}) K_{ijkl} = [\nabla^2_{\text{jm}} K_{mikl} - \nabla^2_{\text{im}} K_{jmkl} + (R_{mj} \cdot K)_{mikl} + (R_{mi} \cdot K)_{jmkl}] K_{ijkl}. \tag{2.2}
\]
Since $(M, g)$ is Ricci parallel, also $\nabla S = 0$, where $S$ is the Ricci tensor field of type $(0,2)$ corresponding to $K$. Hence, we have
\[
\nabla^2_{\text{jm}} K_{mikl} = \nabla^2_{\text{im}} K_{jmkl} = 0.
\]
Further, $K \cdot K = 0$ yields that (2.2) may be rewritten in the following way:
\[
(\nabla^2_{\text{mn}} K_{ijkl}) K_{ijkl} = a[(R^m_{mj} \cdot K)_{mikl} + (R^m_{mi} \cdot K)_{jmkl}] K_{ijkl} = 2a(R^m_{mj} \cdot K)_{mikl} K_{ijkl}. \tag{2.3}
\]
We now write (2.3) as follows:
\[
(\nabla^2_{\text{mn}} K_{ijkl}) K_{ijkl} = 2a[R^p_{pnmj} K_{pikl} + R^p_{pimj} K_{mpkl} + R^p_{pkmj} K_{mpl} + R^p_{pimj} K_{mkpl}] K_{ijkl} = 2a(n - 1) ||K||^2 + 2a[\delta_{pj} \delta_{im} K_{mpkl} + (\delta_{pm} \delta_{kj} - \delta_{km} \delta_{pj}) K_{mpl} + (\delta_{pm} \delta_{ij} - \delta_{im} \delta_{pj}) K_{mkpl}] K_{ijkl} = 2a[(n - 2)||K||^2 - 2||S||^2 - 2K_{mpl} K_{ipml}], \tag{2.4}
\]
The first Bianchi identity yields
\[ K_{mipl}K_{ipml} = -||K||^2 - K_{impl}K_{ipml} = -||K||^2 - K_{mipl}K_{ipml}, \]
so that (2.4) becomes
\[ (\nabla^2_{mn} K_{ijkl})K_{ijkl} = 2a[(n - 1)||K||^2 - 2||S||^2]. \]

Hence, (2.1) can be written as
\[ \frac{1}{2} \Delta(||K||^2) = ||\nabla K||^2 + 2a[(n - 1)||K||^2 - 2||S||^2] = 0. \quad (2.5) \]

On the other hand, for an arbitrary algebraic curvature tensor \( K_\circ \) with Ricci tensor \( S_\circ \), we have
\[ ||K_\circ||^2 \geq \frac{2}{n - 1}||S_\circ||^2, \quad (2.6) \]
with equality if and only if \( K_\circ \) has constant sectional curvature (see, for example, [Be]).

It follows from (2.5) and (2.6) that \( a \) has to be nonpositive unless
\[ ||K||^2 = \frac{2}{n - 1}||S||^2 \]
everywhere on \( M \), that is, \( (M, g) \) has constant sectional curvature.

Note that, when \( a = 0 \), \( (M, g) \) is a curvature homogeneous semi-symmetric space. These spaces have been treated thoroughly in [BKV].

References


Riemannian Manifolds with Special etc. 101


