THE HYPERSPACE OF AN ANALYTIC METRIZABLE SPACE IS ANALYTIC (*)

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SOMMARIO. - *Se (X,τ) è uno spazio analitico metrizzabile e d una metrica compatibile con la topologia τ allora l'iperspazio $CL(X)$ dotato della topologia d-Wijsman è anch'esso analitico e metrizzabile. Il risultato è ottenuto indirettamente mediante argomenti di misurabilità.*

SUMMARY. - *If (X,τ) is an analytic metrizable space and d any metric compatible with the topology τ then the hyperspace $CL(X)$ endowed with the d-Wijsman topology is again an analytic and metrizable space. The result is achieved indirectly by means of a measurability argument.*

1. Introduction.

Let (X,τ) be a T_1 space. The **hyperspace** $\mathrm{CL}(X)$ is the family of all closed and non-empty subsets of X; X will be called the **base space** of the hyperspace. As every singleton set is closed, the base space can be embedded naturally in its hyperspace. A **hypertopology** is a topology on $\mathrm{CL}(X)$ which makes such an embedding a homeomorphism, so that the hyperspace is a topological "extension" of its base space. One of the first questions that arise is the following: what kind of properties the hyperspace inherits from the base space? Of course, as there are many possible ways to define a hypertopology, some of them will reproduce properties better than others. In this framework, the Wijsman hypertopology for a metrizable X (which will be described later) seems to be well suited. As an example, $\mathrm{CL}(X)$ with the Wijsman topology is separable or analytic or polish when the base space has the same property. For separability

(*) Pervenuto in Redazione il 28 dicembre 1993.

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the result is well-known [11], while the analytic case is the scope of this paper. The result about polishness has been established recently by other authors (references are in last section).

This work is part of the author's dissertation [2] (in Italian), where more extensive proofs and references on the subject can be found.

2. Analytic spaces and σ-algebras.

A topological space is said to be **polish** iff it is separable and completely metrizable; **analytic** iff it is T_2 and a continous image of a polish space. As an example of an analytic space which is not polish one can consider an infinite-dimensional separable Banach space with the weak topology.

A **measurable space** is a pair $(X, \mathcal{A})$ where X is a set and $\mathcal{A}$ is a σ-algebra on X. Given a topological space (X, τ) the **Borel σ-algebra** $\mathcal{B}(\tau)$ is the smallest σ-algebra on X that contains every τ-open subset.

A map $f: (X, \mathcal{A}) \to (Y, \mathcal{B})$ is **measurable** iff $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$. The map f is called a (measurable) **isomorphism** iff it is measurable, bijective and f^{-1} is also measurable as a mapping from $(Y, \mathcal{B})$ to $(X, \mathcal{A})$.

A σ-algebra $\mathcal{A}$ on X is **analytic** iff the measurable space $(X, \mathcal{A})$ is isomorphic to a subset of $[0, 1]$ which is the projection onto one factor of a Borel subset of $[0, 1] \times [0, 1]$ (here $[0, 1]$ is endowed with the Borel σ-algebra of the euclidean topology) [5].

Let $\mathcal{F}$ be a given family of subsets of X. The **Souslin operation** is a set-theoretic operation that generates new subsets starting from the element of $\mathcal{F}$. First, define an **$\mathcal{F}$-Souslin scheme** as a mapping $A: P \to \mathcal{F}$ where P is the set of all multi-indexes (i.e. n-tuples of integers for any n). Then consider the set

$$S(A) = \bigcup_{n \in \mathbb{N}^\infty} \bigcap_{k=1}^{\infty} A(n_1, \ldots, n_k)$$

where $\mathbb{N}^\infty$ is the set of all sequences $\{n_1, n_2, \ldots\}$ of integers. Any set $S(A)$ where A is an $\mathcal{F}$-Souslin scheme is said to be an **$\mathcal{F}$-Souslin set** or Souslin set with respect to $\mathcal{F}$. Any $F \in \mathcal{F}$ is clearly $\mathcal{F}$-Souslin but there may be many more Souslin sets, even if $\mathcal{F}$ is a σ-algebra (notice that the union is uncountable!). For further information the reader is referred to [10, 5], or any text about descriptive set theory.

The three definitions of analytic space, analytic σ-algebra and Souslin set, although formally different one from the other, are intimatetly related. The following deep result can be found in [5, theorem 2.6]:

THEOREM 1. *Let* (X, τ) *be an analytic topological space and* S *a subset of* X*. Then the following statements are equivalent:*

i) S with the induced topology is an analytic topological space;

ii) the σ-algebra $\mathcal{B}(\tau)$ restricted to S is analytic;

iii) S is a $\mathcal{B}(\tau)$-Souslin subset.

COROLLARY 2. If (X, τ) is an analytic space then $\mathcal{B}(\tau)$ is an analytic σ-algebra.

The converse of corollary 2 is not true in general, as the Sorgenfrey line, that is the real line endowed with the topology generated by the half-open intervals $(a, b]$ is not topologically analytic [8, example 2], while its Borel σ-algebra coincides with the Borel σ-algebra of the euclidean topology which is analytic.

Thus, to obtain the converse of corollary 2 we need some additional hypotheses. The key result is due to Frolík [8, theorem 4] and reads as follows:

THEOREM 3. [Frolík] *If* (X, τ) *is a metrizable space and* $\mathcal{B}(\tau)$ *is analytic then* (X, τ) *is an analytic topological space.*

It has to be noticed that the original result was stated in terms of the Baire σ-algebra instead of Borel's, the former being the smallest σ-algebra with respect to which every τ-continous real-valued function is measurable. In the metrizable case the two σ-algebras coincide. Theorem 3 is not in contrast with the example given above as the Sorgenfrey line is not metrizable (it is separable but not second countable).

3. The Wijsman hypertopology.

For the rest of the paper X will be a metrizable space and $\mathcal{M}$ will be the set of all metrics on X compatible with its topology. The **Wijsman** topology induced by a metric $d \in \mathcal{M}$ is the weakest topology on $\mathrm{CL}(X)$ such that the real functions

$$\begin{array}{rcl} d_x : \mathrm{CL}(X) & \rightarrow & \mathbb{R} \\ A & \mapsto & d(x, A) \end{array}$$

are continous for every $x \in X$ (the distance functional $d(x, A)$ is defined in the usual way as $\inf_{a \in A} d(x, a)$). This topology will be denoted by τ_{W_d} and depends on the metric d chosen on X; it is known that two distinct metrics $d, \rho \in \mathcal{M}$ may generate distinct Wijsman hypertopologies.

The **Effros σ-algebra** $\mathcal{E}$ is the σ-algebra on $\mathrm{CL}(X)$ generated by the collection of all

$$G^- = \{A \in \mathrm{CL}(X) : A \cap G \neq \emptyset\}$$

where G varies in the family of open subsets. The definition of $\mathcal{E}$ is purely topological and does not depend on a specific metric chosen. The Effros σ-algebra has been deeply studied as it is related with the concept of weak measurability of multifunctions (see [9, 1, 2]). An interesting fact, outlined by Hess in [9, theorem 3.1.1], is that the Effros σ-algebra is the Borel σ-algebra of every Wijsman topology on the hyperspace, at least in the separable case:

THEOREM 4. [Hess] *If X is separable and $d \in \mathcal{M}$ then $\mathcal{B}(\tau_{W_d}) = \mathcal{E}$.*

If X is separable, many other important hypertopologies have their Borel σ-algebra equal to $\mathcal{E}$, for example the Fell topology [2] (even if the base space is not locally compact), many weak topologies described in term of gap and excess functionals and the slice topology when the base space is a normed space with separable dual [4].

We now investigate the hyperspace of an analytic space. The first result is about the Effros σ-algebra:

THEOREM 5. [Christensen] *If X is analytic then $\mathcal{E}$ is an analytic* σ-algebra.

To give the reader an idea of the instruments involved in the proof, we include here a sketch of it. For the complete proof [5], an italian version with up-to-date notation is contained in [2].

Proof. [(sketch)] as X is separable, there exists in $\mathcal{M}$ a totally bounded metric ρ. It is known [11] that total boundedness of ρ implies that the ρ-Wijsman topology is induced by the Hausdorff metric on $\mathrm{CL}(X)$ defined by

$$\mathrm{H}_\rho(A,B) = \sup_{x\in X} |\rho(x,A) - \rho(x,B)|$$

We can consider the completion $\widetilde{X}$ of X with the metric $\tilde{\rho}$ induced by ρ, and its hyperspace $\mathrm{CL}(\widetilde{X})$ with the Hausdorff metric $\mathrm{H}_{\tilde{\rho}}$. $\widetilde{X}$ and $\mathrm{CL}(\widetilde{X})$ are compact metric space, and there is a natural embedding

$$\begin{array}{rcl} i : \mathrm{CL}(X) & \to & \mathrm{CL}(\widetilde{X}) \\ A & \mapsto & \tilde{A} \end{array}$$

where $\widetilde{A}$ is the closure of A viewed as a subset of $\widetilde{X}$. A simple calculation shows that i is an isometry, so $\mathrm{CL}(X)$ is homeomorphic to a subset Σ of $\mathrm{CL}(\widetilde{X})$ ($\mathrm{CL}(\widetilde{X})$ is polish!). Then we *just* need to show that Σ is Souslin with respect to the Borel sets (that's the nasty part!), and the theorem 1 will lead us to the result. $\diamondsuit$

We now have all the ingredients to prove our main result.

THEOREM 6. *If X is analytic and $d \in \mathcal{M}$ then $(\mathrm{CL}(X), \tau_{W_d})$ is an analytic (metrizable) space.*

Proof. We have already noted that X analytic implies that X is separable. So, from Christensen's theorem, we know that $\mathcal{E}$ is an analytic σ-algebra and from Hess' result that $\mathcal{E} = \mathcal{B}(\tau_{W_d})$. The result will follow from Frolík's theorem provided we show that the Wijsman topology is metrizable. But this fact is well known [11], a metric is given by

$$\mathrm{W}(A,B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|d(x_n,A) - d(x_n,B)|}{1 + |d(x_n,A) - d(x_n,B)|}$$

where $\{x_n\}_{n=1}^{\infty}$ is a countable dense subset of X. ◊

4. Final remarks.

The same arguments can be applied as well to every metrizable hypertopology on $\mathrm{CL}(X)$ that has $\mathcal{E}$ as its Borel σ-algebra. That is the case of the *countably determined* topologies defined in [4]. But we do not have to restrict ourselves to the whole $\mathrm{CL}(X)$. For example, if X is normed with separable dual, the slice topology τ_S we mentioned earlier has $\mathcal{B}(\tau_S) = \mathcal{E}$; but τ_S is defined on $\mathrm{C}(X)$, that is the hyperspace of all closed, non-empty and convex subsets of X. Can we say that $(\mathrm{C}(X), \tau_S)$ is analytic if X is, say, Banach? The problem is that we don't know, a priori, whether or not $\mathcal{E}$ restricted on $\mathrm{C}(X)$ is an analytic σ-algebra. The answer is yes, but we have to use theorem 1 in its full strength. The key is that $\mathrm{C}(X)$, as a subset of $\mathrm{CL}(X)$, is $\mathcal{E}$-Souslin (in fact it is a member of $\mathcal{E}$ itself [2]).

The ideas contained in this paper can be applied to other situations, there are in fact cases in which these techniques quickly lead to some criteria for the weak measurability of convex-valued multifunctions [1] (a more extensive paper is being prepared by C. Hess and the author).

The first possible generalization of theorem 6 is certainly the following: let X be polish and $d \in \mathcal{M}$, is $(\mathrm{CL}(X), \tau_{W_d})$ polish? Of course we know from theorem 6 that it is analytic; Effros and Beer proved polishness when d is either a totally bounded or complete metric [7, 3]. Very recently, C. Costantini proved that the answer is affirmative in the general case [6]. Yet, theorem 6 is independent from Costantini's result as both the hypothesis and the thesis are weaker. We remark here that the arguments in [6] are of the same nature as the ones in theorem 5, as there is no way to use a Frolík-type theorem. In any case these two results show how faithfully the Wijsman topology reproduces on the hyperspace the properties of the base space.

The author would like to thank professor S.Levi for his help during and after the preparation of the author's dissertation and also for kindly reviewing this article.

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