SPLITTINGS OF MANIFOLDS WITH BOUNDARY AND RELATED INVARIANTS (*)

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SUMMARY. - We construct special handle decompositions for a compact connected PL manifold with non empty boundary and study the associated topological invariants. As a consequence, we characterize the unknot in $\mathbb{S}^{n+2}$ ($n \leq 2$) as the unique n-knot whose complement has genus one. Then we obtain a simple geometric proof of the non cancellation theorem for tame n-knots in $\mathbb{S}^{n+2}$, $n \leq 2$.

1. Introduction.

Let $M^n$ be a compact connected (orientable) triangulated n-manifold with non empty boundary $\partial M$. We construct special handle decompositions of $M$ and define the concept of regular splitting of $M$. Then we describe regular Heegaard diagrams of $M$ ($n = 3$) and relate them to another known 3-manifold representation, named $P$-graph theory (see [19] and [23]). As a consequence, we obtain nice properties about (geometric)

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finite presentations of \( \Pi_1(M) \) which arise from regular Heegaard diagrams of \( M \). Then we extend the notion of Heegaard genus of a closed 3-manifold to the boundary case, also considering higher dimensions. The concept of genus yields a nice characterization of cubes with handles among bordered 3-manifolds. As a consequence, we also characterize the unknot in \( \mathbb{S}^3 \) as the unique 1-knot whose complement has genus one. This gives a simple alternative proof of the classical non-cancellation theorem for 1-knots in \( \mathbb{S}^3 \) (see for example [20]). Then we extend these results to dimension four. More precisely, we characterize the unknot in \( \mathbb{S}^4 \) as the unique 2-knot whose complement has genus one and obtain a geometric proof of the non-cancellation theorem for tame 2-knots in \( \mathbb{S}^4 \). Some examples complete the paper.

2. Handle Decompositions.

Throughout the paper we work in the piecewise linear category in the sense of [12] and [21]. For convenience, we assume that any considered (pseudo) manifold is orientable. Recall that a cube with \( n \) handles is a 3-manifold \( V \) which contains \( n \) pairwise disjoint properly embedded 2-cells such that the result of cutting \( V \) along them is a 3-cell. The integer \( n \) is called the \textit{genus} of \( V \), written \( g(V) \).

Let \( M^3 \) be a connected compact (orientable) 3-manifold with non-empty boundary components \( \partial_1M, \partial_2M, \ldots, \partial_hM \). We define the concept of “regular” splitting of \( M \) as follows. A pair \((V_1, V_2)\) of cubes with handles is said to be a (regular) \textit{Heegaard splitting} of \( M \) if it satisfies the following properties:

1. \( V_1 \cup V_2 = M \)
2. \( V_2 \cap \partial_iM \) is a closed 2-cell \( D_i \) for \( i = 1, 2, \ldots, h \)
3. \( V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = \partial V_2 \setminus \bigcup_{i=1}^{h} D_i \)
4. \( \partial V_1 = \partial V_2 \# \partial_1M \# \ldots \# \partial_hM \).

As a consequence, we have the relation \( g(V_1) = g(V_2) + \sum_{i=1}^{h} g(\partial_iM) \), where \( g \) also denotes the genus of a closed surface. The \textit{genus} of the regular splitting \((V_1, V_2)\) is defined to be \( g(V_1) \). The (regular) \textit{Heegaard genus} of \( M \) is the minimum \( n \) for which \( M \) admits regular splittings of genus \( n \). Obviously this concept extends to the boundary case the usual Heegaard
genus of a closed (orientable) 3-manifold as \( g(V_1) = g(V_2) \) whenever \( \partial M \) is a 2-sphere.

The following existence theorem was first proved in [4], Proposition 4, for manifolds with connected boundary and successively extended to the general case in [11], Proposition 12; we shall present an alternative proof of the result, which follows closely a construction contained in [24] (section 8.3.6, pp. 260-261).

**Theorem 1.** Let \( M^3 \) be a compact connected (orientable) 3-manifold with non empty boundary components \( \partial_1 M, \partial_2 M, \ldots, \partial_h M \). Then \( M \) admits a regular Heegaard splitting.

**Proof.** Let \( K \) be a simplicial triangulation of \( M \) and \( Sd^r K \) the \( r \)-th barycentric subdivision of \( K \). Let us denote by \( \Gamma_1 \) and \( \Gamma_2 \) the 1-skeleton and the dual 1-skeleton of \( K \) respectively. Recall that \( \Gamma_2 \) is the maximal 1-simplex of \( Sd^r K \) disjoint from \( \Gamma_1 \). We consider a derived simplicial neighbourhood \( H_i \) of \( \Gamma_i \) in \( Sd^r K \). Then the polyhedron underlying \( H_i \), also named \( H_i \), is a tubular neighbourhood of \( \Gamma_i \) in \( M \). Obviously we have that \( M = H_1 \cup H_2 \) and \( H_1 \cap H_2 = \partial H_1 \cap \partial H_2 \). Furthermore, \( H_1 \) and \( H_2 \) are not identified along their whole boundaries as the points where \( \partial H_1 \) and \( \partial H_2 \) are not identified constitute \( \partial M \). The pieces of \( \partial_i M \) on \( \partial H_2 \) are 2-cells \( e_j \), \( j = 1, 2, \ldots, \alpha(i) \), arising from the middle of the faces in the triangulation of \( \partial_i M \). By doing isotopies inside a collar of \( \partial M \) in \( M \), we push the 2-cells \( e_j \) into the interior of a 2-cell \( f_i \) of \( \partial_i M \) for any \( i = 1, 2, \ldots, h \). Let \( C_i \) be a 3-cell such that \( C_i \cap M = \partial C_i \cap \partial M = f_i \) and \( \partial C_i \setminus f_i \) is the 2-cell \( D_i \).

The manifold \( \widetilde{M} = M \cup \cup_{i=1}^h C_i \) is homeomorphic to \( M \). Now \( \widetilde{M} \) splits into two cubes with handles \( V_2 = H_2 \cup \cup_{i=1}^h C_i \) and \( V_1 = \text{cl}(\widetilde{M} \setminus V_2) \cong \text{cl}(M \setminus H_2) \cong H_1 \). Here we have also denoted by the same symbol the image of \( H_i \) under the above mentioned isotopies. Finally the pair \((V_1, V_2)\) satisfies the statement. \( \diamond \)

By Theorem 1 we can analyze the bordered 3-manifolds in terms of the manner in which the pieces are attached and thus we reduce the study of these 3-manifolds to problems about 2-manifolds.

Suppose we have a (regular) Heegaard splitting \((V_1, V_2)\) of a 3-manifold \( M \) with non empty boundary components \( \partial_1 M, \partial_2 M, \ldots, \partial_h M \). Render \( V_2 \) simply connected by removing suitable meridian plates \( P_k \), \( k = 1, 2, \ldots, m \). More precisely, let \( \{B_1, B_2, \ldots, B_m\} \) be any collection of pairwise disjoint properly embedded 2-cells in \( V_2 \) which cut \( V_2 \) into
a 3-cell. The pairwise disjoint 1-spheres \( \{ J_1, J_2, \ldots, J_m \} \), \( J_k = \partial B_k \), cut \( \partial V_2 \) into a 2-sphere with \( 2m \) holes. The plates \( P_k \) are precisely \( B_k \times I \subset V_2 \), where \( I = [0,1] \). Since the pieces of \( \partial M \) on \( \partial V_2 \) are the 2-cells \( D_i \), \( i = 1, 2, \ldots, h \), we can place the plates \( P_k \) so that they do not meet \( \partial M \) by pushing their rims \( \partial B_k \times I = J_k \times I \) away from the discs \( D_i \) where necessary.

Let \( V_2' \) be the result of cutting \( V_2 \) along \( \bigcup_{k=1}^m B_k \). Then \( V_2' \) is a 3-cell as \( g(V_2) = m \). Furthermore \( V_2' \) meets \( \partial_i M \) along the 2-cell \( D_i \). For any \( i = 1, 2, \ldots, h - 1 \) cut a plate \( P_i' = B_i' \times I \) from \( V_2' \) which has \( D_i \) as its top face and its rim \( \partial B_i' \times I = J_i' \times I \) is an annulus common to \( \partial V_1 \) and \( \partial V_2 \).

We call the system \( (V_1; J_1, J_2, \ldots, J_m, J_1', J_2', \ldots, J_h') \) a (regular) \textit{Heegaard diagram} of \( M \). We can recover \( M \) from a (regular) Heegaard diagram of it. Conversely, every set of disjoint simple closed curves on a cube \( V_1 \) with \( n \) handles determines a bordered 3-manifold \( M \). Indeed, \( M \) is obtained by gluing plates to annular neighbourhoods of the curves.

Given a (regular) Heegaard diagram \( (V_1; J_1, J_2, \ldots, J_m, J_1', J_2', \ldots, J_h') \) as above we can construct a presentation for \( \Pi_1(M) \) as follows. Choose a free basis \( \{ x_1, x_2, \ldots, x_n \} \) for the free group \( \Pi_1(V_1) \cong \ast_{n \geq 2} \mathbb{Z} \), where \( n = g(V_1) \). For \( k = 1, 2, \ldots, m \) and \( i = 1, 2, \ldots, h - 1 \), let \( r_k \) and \( r_i' \) be words in \( x_1, x_2, \ldots, x_n \) representing the elements of \( \Pi_1(V_1) \) determined by \( J_k \) and \( J_i' \) respectively. These words are unique up to inversion and conjugation. By Van Kampen’s theorem we have that

\[
< x_1, x_2, \ldots, x_n; r_1, r_2, \ldots, r_m, r_1', r_2', \ldots, r_{h-1}' >
\]

is a presentation for \( \Pi_1(M) \).

In particular, we obtain the following result:

**Theorem 2.** Let \( M^3 \) be a compact connected (orientable) 3-manifold with non empty boundary components \( \partial_1 M, \partial_2 M, \ldots, \partial_h M \). Then the fundamental group \( \Pi_1(M) \) has a finite presentation of deficiency

\[
\sum_{i=1}^h g(\partial_i M) - h + 1 = 1 - \chi(M).
\]

**Proof.** By Theorem 1, we have

\[
\Pi_1(M) \cong < x_1, x_2, \ldots, x_n; r_1, r_2, \ldots, r_m, r_1', r_2', \ldots, r_{h-1}' >
\]
where \( n = g(V_1) = g(V_2) + \sum_{i=1}^{h} g(\partial_i M) \) and \( m = g(V_2) \). Thus the deficiency \( d \) of the presentation is

\[
  d = n - m - (h - 1) = \sum_{i=1}^{h} g(\partial_i M) - h + 1.
\]

Now let \( D(M) \) be the closed 3-manifold which is the double of \( M \). Then we have \( \chi(D(M)) = 2\chi(M) - \chi(\partial M) = 0 \), i.e.

\[
  2\chi(M) = \sum_{i=1}^{h} \chi(\partial_i M) = 2h - 2 \sum_{i=1}^{h} g(\partial_i M).
\]

This implies that \( \chi(M) = h - \sum_{i=1}^{h} g(\partial_i M) \), hence \( d = -\chi(M) + 1 \) as requested.

Define:

1) \( rk(M) \) the minimum rank of \( \Pi_1(M) \);
2) \( d(M) \) the minimum deficiency over all presentations of \( \Pi_1(M) \).

The following facts are straightforward:

**Proposition 3.** Let \( M^3 \) be a compact connected (orientable) 3-manifold with non empty boundary components \( \partial_1 M, \partial_2 M, \ldots, \partial_h M \). Then we have:

1) \( g(M) \geq g(\partial M) \).
2) \( g(M) \geq rk(M) \).
3) \( 0 \leq d(M) \leq g(\partial M) - h + 1 = 1 - \chi(M) \).
4) \( d(M) + \beta_2(M) \leq \beta_i(M) \) where \( \beta_i(M) \) is the \( i \)-th Betti number of \( M \).
   In particular, if \( d(M) > 0 \), then \( H_1(M) \) (and hence \( \Pi_1(M) \)) is an infinite group.
5) \( g(M) = 0 \) if and only if \( M \) is a punctured 3-cell, i.e. a manifold which becomes a 3-sphere by capping off each 2-sphere component of \( \partial M \) with a 3-cell.

Now we prove a nice characterization of cubes with handles among 3-manifolds with non empty connected boundary.
THEOREM 4. Let $M^3$ be a compact connected (orientable) 3-manifold with non-empty connected boundary $\partial M$. Then $M$ is a cube with $n$ handles if and only if $g(M) = g(\partial M) = n$.

Proof. The necessity is clear. For sufficiency, let $(V_1, V_2)$ be a (regular) Heegaard splitting of $M$ such that $g(M) = g(V_1) = g(V_2) + g(\partial M) = g(\partial M)$. By hypothesis, it follows that $g(V_2) = 0$, hence $V_2$ is a 3-cell. Furthermore $V_2$ meets $V_1$ in a 2-cell in the boundary of each as $\partial M \cap V_2 = D$, a 2-cell, and $V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = \partial V_2 \setminus D$ is a 2-cell. Hence $M$ is the 3-manifold obtained from the cube with handles $V_1$ by attaching a 3-cell along a 2-cell in their boundaries. Thus $M \cong_{PL} V_1$ as required.

Note that Theorem 4 gives a simple (non-combinatorial) proof of the main theorem of [3]. Indeed the regular genus $\tilde{g}(M)$ of a 3-manifold $M$ with boundary, used in [3], satisfies the relation $\tilde{g}(M) \geq g(M) \geq g(\partial M) = \tilde{g}(\partial M)$ as one can easily verify.

COROLLARY 5. Let $K$ be a tame knot in $S^3$ and $M$ the knot manifold of $K$, i.e. $M$ is the closed complement of a regular neighbourhood of $K$ in $S^3$. Then $K$ is the trivial knot if and only if $g(M) = g(\partial M) = 1$.

PROPOSITION 6. Let $K_i$ be a tame knot in $S^3$, $M_i$ the knot manifold of $K_i$, $i = 1, 2$, and $K$ the composite knot $K_1 \# K_2$. If $M$ is the knot manifold of $K$, then we have $g(M) = g(M_1) + g(M_2) - 1$.

Proof. For composite knots it is convenient to use a new view of the knot manifold as described in [1] and [2], chp. 15, part B. One looks at the complement $M_1$ of a regular neighbourhood of the knot $K_1 \subset S^3$ from the centre of a ball in the regular neighbourhood. Now $M_1$ looks like a cube with a knotted hole (for details see the quoted papers). Suppose $W_2$ is a regular neighbourhood of $K_2$ in $S^3$ such that $M_1 \subset W_2$ and $M_1 \cap M_2 = \partial M_1 \cap \partial M_2$ is an annulus $A$, where $M_2 = S^3 \setminus W_2$. Then $M_1 \cup_A M_2$ is just the knot complement of the composite knot $K = K_1 \# K_2$ if the annulus $A$ is meridional with respect to $K_1$ and $K_2$. Let $(V_1^{(i)}, V_2^{(i)})$, $i = 1, 2$, be a minimal regular Heegaard splitting of $M_i$, i.e. $g(M_i) = g(V_2^{(i)}) + 1$. By isotopy there exist closed 2-cells $D_2^{(i)}$, $C_2^{(i)}$, $B_2$ which satisfy the following properties:

1) $V_2^{(i)} \cap \partial M_i = D_2^{(i)}$
2) \( V_2^{(i)} \cap A = B_2 \subset D_2^{(i)} \)

3) \( V_2^{(i)} \cap (\partial M_i \setminus \tilde{A}) = C_2^{(i)} \subset D_2^{(i)} \)

4) \( B_2 \cup C_2^{(i)} = D_2^{(i)} \)

5) \( B_2 \cap C_2^{(i)} = \partial B_2 \cap \partial C_2^{(i)} \) is an 1-arc properly embedded in \( D_2^{(i)} \).

It follows that the pair \((V_1^{(1)} \cup A \setminus B_2, V_1^{(2)} \cup B_2, V_2^{(1)} \cup B_2, V_2^{(2)})\) is a regular Heegaard splitting of \( M \). Then we have

\[
g(M) = g(V_2^{(1)} \cup B_2, V_2^{(2)}) + 1 = g(V_2^{(1)}) + g(V_2^{(2)}) + 1 = g(M_1) + g(M_2) - 1.
\]

Conversely, let \((V_1, V_2)\) be a minimal regular Heegaard splitting of \( M \), i.e. \( g(M) = g(V_2) + 1 \). By the general position theorem we can always assume that \( V_2 \) transversely intersects the annulus \( A \) in a finite number of disjoint closed 2-cells \( e_j \). Then \( V_2^{(i)} = V_2 \cap M_i \) and \( V_1^{(i)} = V_1 \cap M_i \) are cubes with handles. Now we cut a plate \( B_j^{(i)} \times I \) from \( V_2^{(i)} \) which has the 2-cell \( e_j \) as its top face and its rim \( \partial B_j^{(i)} \times I \) is an annulus in \( \partial V_2^{(i)} \). Repeating this process yields a cube with handles, \( \overline{V}_2^{(i)} \), say, which has the same genus of \( V_2^{(i)} \). Moreover, attaching the plates \( B_j^{(i)} \times I \) to \( V_1^{(i)} \) gives a homeomorphic cube with handles, \( \overline{V}_1^{(i)} \), say. By construction the pair \((\overline{V}_1^{(i)}, \overline{V}_2^{(i)})\) is a regular Heegaard splitting of \( M_i \) such that \( g(\overline{V}_2^{(i)}) = g(V_2^{(i)}) \). Finally we have

\[
g(M) = g(V_2) + 1 = g(V_2^{(1)}) + g(V_2^{(2)}) + 1 = g(\overline{V}_2^{(1)}) + g(\overline{V}_2^{(2)}) \geq g(M_1) + g(M_2) - 1.
\]

This proves the statement. \( \Diamond \)

Corollary 5 and Proposition 6 yield a simple alternative proof of the classical non-cancellation theorem for 1-knots in \( S^3 \) (see for example [2] and [20]).

**Corollary 7.** (The non-cancellation theorem for 1-knots in \( S^3 \)). The composite knot \( K_1 \# K_2 \) is trivial if and only if \( K_1 \) and \( K_2 \) are trivial.

**Proof.** If \( K_1 \# K_2 \) is unknotted, then \( g(M) = 1 \). Because \( g(M) = g(M_1) + g(M_2) - 1 \) and \( g(M_i) \geq g(\partial M_i) = 1 \), it follows that \( g(M_i) = 1 \), \( i = 1, 2 \), and hence \( K_i \) is trivial by Corollary 5. \( \Diamond \)

Now we shall apply Theorem 5.2 of [15] and the additivity of the genus ([13] and [14]) to obtain the following result:
Proposition 8. Let $M^3$ be a compact connected orientable 3-manifold with nontrivial free fundamental group. If $g(M) = \text{rk}(M)$, then $M$ is homeomorphic to a connected sum whose factors are cubes with handles and copies of $S^1 \times S^2$.

Proof. Let $\partial_1 M$, $\partial_2 M$, \ldots, $\partial_h M$ be the boundary components of $M$ and let us denote the genus of $\partial_i M$ by $g_i$, $i = 1, 2, \ldots, h$. By Theorem 5.2 and Corollary 5.3 of [15] the manifold $M$ is a connected sum of type $\Sigma \# H_1 \# \cdots \# H_h \# \Lambda_1 \# \cdots \# \Lambda_s$, where $H_i$ is a cube with $g_i$ handles, $\Lambda_j$ is a copy of $S^1 \times S^2$ and $\Sigma$ is a homotopy 3-sphere. Furthermore, the following relation

$$s = \text{rk}(M) - \sum_{i=1}^h g_i = \text{rk}(M) - g(\partial M)$$

is verified. To prove the result we have to show that $\Sigma$ is really a 3-sphere. Let $(V_1, V_2)$ be a minimal regular Heegaard splitting of $M$, i.e. $g(M) = g(V_1) = g(V_2) + g(\partial M) = g(V_2) + \text{rk}(M) - s$. Then the hypothesis of the statement implies that $g(V_2) = s$. Let $H_i' \subset \partial H_i$ so that the union $H_i \cup H_i'$ is a connected sum of $g_i$ factors of type $S^1 \times S^2$. Let $M'$ be the closed orientable 3-manifold obtained from $M$ by capping off each boundary component $\partial_i M = \partial H_i$ with $H'_i$. Then $M'$ is homeomorphic to a connected sum $\Sigma \# p(S^1 \times S^2)$, where $p = s + g(\partial M)$. Haken’s theorem on the additivity of the Heegaard genus in the closed case (see [13] and [14]) implies that

$$g(M') = g(\Sigma) + s + g(\partial M) = g(\Sigma) + g(V_2) + g(\partial M) = g(\Sigma) + g(M).$$

Because $V_2$ meets each boundary component $\partial_i M = \partial H_i$ in a 2-cell, the union $V_2' = V_2 \cup \bigcup_{i=1}^h H'_i$ is a cube with handles whose genus is

$$g(V_2') = g(V_2) + \sum_{i=1}^h g_i = g(V_2) + g(\partial M) = g(V_1) = g(M).$$

Thus the closed 3-manifold $M'$ admits the Heegaard splitting $(V_1, V_2')$, in the usual sense, of genus $g(M)$. This implies that $g(M') \leq g(M)$ and hence $g(\Sigma)$ vanishes as $g(M') = g(\Sigma) + g(M)$. Thus $\Sigma$ must be a genuine 3-sphere and the proof is complete. \hfill \Box

Corollary 9. $g(M) = 1$ if and only if $M^3$ is either a punctured lens space (including $S^1 \times S^2$) or $M = S^1 \times D^2$ (cube with 1-handle).
Examples of genus two 3-manifolds with toroidal boundary components are given by the closed complements of small regular neighbourhoods of certain knots and links in $S^3$ (see the next section).
3. $P$-graphs.

Let $M$ be a connected compact (orientable) 3-manifold with non-empty boundary $\partial M$. In this section we relate the concept of (regular) Heegaard diagram of $M$ to another known 3-manifold representation, named $P$-graph theory (see for example [19] and [23]). As a consequence, we obtain a nice property about the finite presentations of $\Pi_1(M)$, which arise from (regular) Heegaard diagrams of $M$. In order to do this, we recall some definitions and results about $P$-graphs, listed in the quoted papers. Let $\varphi$ be a group presentation with $n$ generators and $m$ relations, $n \geq m$, i.e.

$\varphi = \langle x_1, x_2, \ldots, x_n : r_1, r_2, \ldots, r_m \rangle$. By $K_\varphi$ we denote the canonical 2-complex associated to $\varphi$. Then $K_\varphi$ is a 2-dimensional CW-complex with one vertex $v$ and $n$ 1-cells (resp. $m$ 2-cells) corresponding to generators (resp. relations) of $\varphi$. Each 1-cell of $K_\varphi$ will be labelled by the associated generator $x_i$ of $\varphi$, for $i = 1, 2, \ldots, n$. Every presentation $\varphi$ determines a unique $P$-graph $P_\varphi$ obtained as the boundary of a regular neighbourhood of the vertex $v$ in $K_\varphi$. If $x_i \cap P_\varphi = \{ e^+_i, e^-_i \}$, then the points (vertices) on the boundary of regular neighbourhoods of $e^+_i$, $e^-_i$ in $P_\varphi$ will be denoted by $e^+_{ij}$, $e^-_{ij}$ respectively ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, k_i$). Then we set $E^+_i = \{ e^+_{ij} : j = 1, 2, \ldots, k_i \}$ and $E^-_i = \bigcup_{i=1}^{k_i} E^+_{ij}$ for $\varepsilon = +$ or $-$. Now let $B = B(\varphi)$ be the involutory permutation of $E$, defined by $B(e^+_{ij}) = e^-_{ij}$. If $P_\varphi$ is embedded into the 2-sphere $S^2$, then walking clockwise around each vertex of $E^+_{ij}$ induces a permutation $C = C(\varphi)$ of $E$, whose orbits are the sets $E^+_{ij}$. An embedding $f : P_\varphi \rightarrow S^2$ is said to be faithful if $B = CBC$. In this case, we say that $\varphi$ fits.

A basic result of $P$-graph theory is the following representation theorem (see [18], [19] and [23]).

**Theorem 10.** Let $M$ be a connected compact orientable 3-manifold (with or without boundary). Suppose $\varphi$ is a finite presentation of $\Pi_1(M)$. Then $\varphi$ fits if and only if $K_\varphi$ is a spine of $M$, i.e. there exists an embedding $K_\varphi \subset M$ such that $M \setminus K_\varphi$ is homeomorphic to $\partial M \times [0, 1]$. Moreover, the manifold $M$ is uniquely determined by the faithful embedding of $P_\varphi$ in $S^2$.

Now we are going to construct a Heegaard diagram of $M$ from a faithfully embedded $P$-graph $(P_\varphi, f)$. We consider the disc $B^2_i \subset S^2$ with center $e_i$ and such that $E^+_i \subset \partial B^2_i$. Since $B = CBC$, there exists an orientation reversing homeomorphism $\psi_i : \partial B^2_i \rightarrow \partial B^2_i$ such that $\psi_i(e^+_i) = e^-_i$ for
$i = 1, 2, \ldots, n$. Let $\Sigma$ denote the closed complement of $\bigcup_{i \in \mathbb{Z}} B_i$ in $S^2$. Then the quotient space obtained from $\Sigma$ by identifying each $\partial B_i^+$ with $\partial B_i^-$ via $\psi_i$ is the closed orientable surface $S$ of genus $n$, standardly embedded in the euclidean 3-space $\mathbb{R}^3$. Let $H = H(\varphi, f)$ denote the orientable cube with $n$ handles, in $\mathbb{R}^3$, such that $\partial H = S$. Let $\gamma = \gamma(\varphi, f)$ be the set of simple disjoint closed curves in $\partial H$ obtained from $f(\varphi) \cap \Sigma$ via the natural projection $\pi : \Sigma \to S$. Now the pair $(H, \gamma)$ is a Heegaard diagram of $M$, called the diagram induced by $(P\varphi, f)$. This construction can be reversed as follows. Let $(H, \gamma)$ be a (regular) Heegaard diagram of $M$ and let $\varphi$ denote the group presentation of $\Pi_1(M)$ arising from $(H, \gamma)$. We construct a faithfully embedded $P$-graph $(P\varphi, f)$ such that the induced diagram $(H(\varphi, f), \gamma(\varphi, f))$ coincides with $(H, \gamma)$. For this, it is convenient to take the usual representation of the diagram in the euclidean plane as shown in [22]. Let $S^2$ be the 2-sphere, represented as the $(x, y)$-plane plus a point at infinity. For $i = 1, 2, \ldots, n$, let $e_i^+ \equiv (i, +1)$, $e_i^- \equiv (i, -1)$ and $B_i^\varepsilon$ the 2-cell of radius 1/4 and center at $e_i^\varepsilon$, where $\varepsilon = +$ or $-$. As usual, $\Sigma$ denotes the bordered surface $S^2 \setminus \bigcup_{i \in \mathbb{Z}} \partial B_i^\varepsilon$. Let $\pi : \Sigma \to \partial H$ be a map, one-to-one everywhere except that each point of $\pi(\partial \Sigma)$, has two points, one of $\partial B_i^+$ and one of $\partial B_i^-$, as inverse image. Let $\Sigma^+$ (resp. $\Sigma^-$) be the subset of $\Sigma$ consisting of all the points with non negative (resp. non positive) ordinate, plus the point at infinity. By isotoping, if necessary, the curves of $\gamma \subset \partial H$, we can suppose that the following conditions are satisfied:

1) for each $j = 1, 2, \ldots, m$, $\pi^{-1}(\gamma_j)$ is the disjoint union of a finite set of arcs $\{\alpha_jr\}$, each meeting a circle only at its endpoints;

2) $\alpha_{jr} \cap \Sigma^\varepsilon$ is either empty or the disjoint union of a finite set of arcs $\{\beta_{jr}^\varepsilon\}$, none of which meets the $x$-axis (plus $\infty$) at an inner point.

Each curve $\partial B_i^\varepsilon$ is split by the endpoints of the arcs $\beta_{jr}^\varepsilon$ into the union of a finite set of arcs with ends $e_i^\varepsilon$. We can consider the pseudo-graph $G = (V, E)$ (multiple edges and loops may occur) where:

1) $V = \{e_i^+, e_i^-\}_{\varepsilon \in \mathbb{Z}}$ is the vertex-set;

2) two vertices $v, w \in V$ are joined by an edge in $E$ if either they are the endpoints of the same arc $\alpha_{jr}$ or $\{v, w\} = \{e_i^+, e_i^-\}$.

The pseudo-graph $G$ is the desired $P$-graph $P\varphi$ associated to $\varphi$. Moreover, $G$ is faithfully embedded in $S^2$ and the induced diagram coincides with $(H, \gamma)$. 
Thus Theorem 4.1 of [19] applies to obtain the following result:

**Theorem 11.** Let $M$ be a connected compact orientable 3-manifold (with or without boundary), $(H, \gamma)$ a (regular) Heegaard diagram of $M$ and $\varphi$ the finite presentation of $\pi_1(M)$ arising from the diagram. Suppose that $x$ is an arbitrary generator of $\varphi$ and that $\{x^{m_1}, x^{m_2}, \ldots, x^{m_s}\}$ is the set of $x$-syllables in the relators of $\varphi$. Then there exist relatively prime integers $m_x, p_x$ such that the absolute value $|m_t|$ of $m_t$, $t = 1, 2, \ldots, s$, belongs to the set $\{m_x, p_x, m_x + p_x\}$.

Now we illustrate our constructions showing Heegaard diagrams and faithfully embedded $P$-graphs of certain classical knot and link complements.

Let us consider the figure-eight knot (see for example [20]) in $S^3$, shown in figure 1.

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**Fig. 1 - The figure-eight knot $K$.**
We prove the following result:

**Proposition 12.** Let \( \varphi \) be the finite presentation

\[
< x, y : x y x y^{-1} x^{-1} y x y^{-1} y^{-1} > .
\]

Then the complement of the figure-eight knot is the unique orientable prime 3-manifold with connected boundary which has the canonical 2-complex \( K \varphi \) as spine.

**Proof.** Let us denote the oriented 1-cells of \( K \varphi \) by \( x, y \) and the unique 2-cell of \( K \varphi \) by \( c \). Then there exists an attaching map \( \partial B^2 \to x \lor y \) (one point union) given by the relator of \( \varphi \). The set \( E \) consists of exactly 20 elements, two for each occurrence of a generator in the relator of \( \varphi \). Suppose we denote these elements by \( e_{1,1}^+, e_{1,2}^+, \ldots, e_{1,5}^+, e_{2,6}^+, e_{2,7}^+, \ldots, e_{2,10}^+, e_{1,1}^- \\
 e_{1,2}^-, \ldots, e_{1,5}^-, e_{2,6}^-, e_{2,7}^-, \ldots, e_{2,10}^- \\
 e_{1,10}^+ \) which is more convenient to identify with \( 1, 2, \ldots, 5, 6, 7, \ldots, 10, 1, 2, \ldots, 5, 6, 7, \ldots, 10 \). Assume this numbering chosen so that an appropriate closed curve parallel to and near \( \partial c \) intersects \( 1, 1, 6, 5, 2, 2, 7, 7, 3, 3, 8, 8, 4, 4, 9, 9, 5, 5, 10, 10 \) in this order. Then we have the involutory permutation

\[
B = B(\varphi) = (1 \ 1)(2 \ 2)(3 \ 3)(4 \ 4)(5 \ 5)(6 \ 6)(7 \ 7)(8 \ 8)(9 \ 9)(10 \ 10).
\]

Now the \( P \)-graph \( P \varphi \), determined by \( \varphi \), is embedded in the 2-sphere \( S^2 \), as shown in figure 2.

Then walking clockwise around each vertex of \( E_i^+, i = 1, 2 \), induces the permutation

\[
C = C(\varphi) = (1 \ 3 \ 4 \ 5 \ 2)(6 \ 7 \ 9 \ 8 \ 10)(2 \ 5 \ 4 \ 3 \ 1)(1 \ 0 \ 8 \ 9 \ 7 \ 6).
\]

Obviously the presentation \( \varphi \) fits, i.e. the embedding of \( P \varphi \) in \( S^2 \) satisfies the relation \( B = C BC \) as one can easily verify. Now we apply Theorem 10. The unicity of the manifold follows from the Whitten rigidity theorem, (see [9], [25] and [26]).
Fig. 2 - A $P$-graph of the complement of the figure-eight knot.

The Heegaard diagram (full outside) of the complement of the figure-eight knot, induced from the above-mentioned faithfully embedded $P$-graph, is shown in figure 3.

Fig. 3 - A Heegaard diagram of the knot complement of the figure-eight knot.
Let us consider the link $L \subseteq S^3$ with two components shown in figure 4.

Fig. 4 - A link $L$ with two components $J, K$.

As before, one can prove the following result:

**Proposition 13.** Let $\varphi$ be the finite presentation

$$< x, y : xyz^{-1}yz^{-1}y^{-1}xy^{-1} > .$$

Then the complement of the link $L$ is the unique orientable prime 3-manifold with two toroidal boundary components, which has the canonical 2-complex $K \varphi$ as spine.

The faithfully embedded $P$-graph $P\varphi$, induced by $\varphi$, is shown in figure 5.

Walking clockwise around each vertex of $E_i^1$, $i = 1, 2$, yields the permutation

$$C = C(\varphi) = (1 \ 3 \ 4 \ 2)(5 \ 8 \ 6 \ 7)(2 \ 4 \ 3 \ 1)(7 \ 5 \ 3 \ 8).$$

Because the permutation $B = B(\varphi)$ is given by

$$B = (1 \ 1)(2 \ 2)(3 \ 3)(4 \ 4)(5 \ 5)(6 \ 6)(7 \ 7)(8 \ 8),$$

one can easily verify that the relation $B = CBC$ holds. The unicity of the manifold follows from the fact that the above $C = C(\varphi)$ is the unique permutation for which $\varphi$ fits. Finally the Heegaard diagram, induced by the $P$-graph of figure 5, is shown in figure 6.
Fig. 5 - A $P$-graph of the knot space of $L$.

Fig. 6 - A Heegaard diagram of the knot space of $L$.

4. Results in Higher Dimension.

In this section we partially extend some results, proved for bordered 3-manifolds, to higher dimension. As a consequence, we obtain a simple geometric proof of the non cancellation theorem for tame 2-knots embedded into the 4-sphere $S^4$.

Let $M^n$ be a compact connected (PL) $n$-manifold with $h$ boundary components $\partial_1 M, \partial_2 M, \ldots, \partial_h M$. A handle of dimension $n$ and index $p$ (briefly a $p$-handle) $H^p$ is a homeomorph of $D^p \times D^{n-p}$ ($0 \leq p \leq n$), $D^i$ being a closed $i$-cell.

Given a $p$-handle $H = D^p \times D^{n-p}$, let us consider a (PL) homeomorphism $\psi : \partial D^p \times D^{n-p} \rightarrow \partial M$. Then $M \cup_{\psi} H$ is the manifold obtained from $M$ by attaching a $p$-handle $H$ via $\psi$. Attaching disjoint 1-handles to a closed $n$-cell yields an $n$-cube with handles (compare section 2 for $n = 3$), also named $n$-handlebody.

A handle decomposition of $M$ is a presentation

$$M = H_0 \cup H_1 \cup \ldots \cup H_t,$$

where $H_0$ is a closed $n$-cell and $H_i$ is a handle attached to $M_{i-1} = \bigcup \{H_j : j \leq i - 1\}$. It is well-known that any (PL) $n$-manifold with non void
boundary admits a handle decomposition with one 0-handle and no n-handles ([21]).

Let \( K \) be a simplicial triangulation of \( M \). Let us denote by \( \Gamma_1 \) and \( \Gamma_2 \) the \((n-2)\)-skeleton and the dual 1-skeleton of \( K \) respectively. Now one can directly repeat the arguments developed in the proof of Theorem 1 to obtain the following natural extension.

**Proposition 14.** Let \( M^n \) be a compact connected (orientable) \( n \)-manifold with \( h \) boundary components \( \partial_1 M, \partial_2 M, \ldots, \partial_h M \). Then there exists a pair \((V_1, V_2)\) of bordered connected \( n \)-manifolds satisfying the following properties:

1) \( V_1 \cup V_2 = M \),

2) \( V_2 \cap \partial_i M \) is a closed \((n-1)\)-cell \( D_i \) for \( i = 1, 2, \ldots, h \);

3) \( V_1 \cap V_2 = \partial V_1 \cap \partial V_2 = \partial V_2 \setminus \bigcup_{i=1}^{h} \partial D_i \);

4) \( V_1 \) admits a handle decomposition with handles of index \( \leq n - 2 \);

5) \( V_2 \) is an \( n \)-dimensional handlebody;

6) \( \partial V_1 = \partial V_2 \# \partial_1 M \# \ldots \# \partial_h M \).

According to section 2, any pair \((V_1, V_2)\) with the properties of Proposition 14 is called a \( (regular) \) splitting of \( M \). From now on, we suppose that \( M \) is a compact connected orientable 4-manifold with \( h \) boundary components. The \textit{genus} of a splitting \((V_1, V_2)\) of \( M \) is defined to be the Heegaard genus of the closed orientable 3-manifold \( \partial V_1 \). As usual, the \textit{genus} of \( M^4 \) is the minimum \( m \) for which \( M \) admits splittings of genus \( m \). By [11] it follows that \( g(M^4) \geq g(\partial M) \) since \( g(M) = g(\partial V_1) = g(\partial V_2) + g(\partial M) \) for any splitting \((V_1, V_2)\) of minimal genus. We also observe that the genus \( g(M^4) \) equals the following expression

\[
\alpha_1(M^4) - h + 1 + \sum_{i=1}^{h} g(\partial_i M^4)
\]

where \( \alpha_1(M^4) \) is the minimum number of 1-handles in \( V_2 \) among all regular splittings \((V_1, V_2)\) of \( M^4 \) and \( g(\partial_i M^4) \) is the Heegaard genus of \( \partial_i M^4 \). For instance, suppose that \( M^4 \) is a compact connected orientable 4-manifold with non-empty connected boundary \( \partial M \). Then \( g(M) = g(\partial M) \) if and
only if $\alpha_1(M^4) = 0$, i.e. $V_2$ is a 4-cell and $M^4$ is homeomorphic to $V_1$. In particular, if $M^4$ is a cube with $n$ handles, then $g(M) = g(\partial M) = n$.

Given a regular splitting $(V_1, V_2)$ of a compact connected orientable 4-manifold $M^4$, let $V_1 = H^0 \cup \lambda H^1 \cup \mu H^2$ and let $\psi_j : (\partial D^2 \times D^2)_j \rightarrow (H^0 \cup \lambda H^1) \cong \#J S^1 \times S^2$ be the attaching map of the $j$-th handle of index 2. We consider the set $\gamma$ of simple closed curves $\gamma_j = \psi_j(\partial D^2 \times \{0\})$. Then the pair $(\#J S^1 \times S^2, \gamma)$ is a Heegaard diagram of the bordered orientable 4-manifold $M$ in the sense of [17]. This extends the results of the quoted paper to the boundary case. Now we are going to study some application about knot theory.

**Proposition 15.** Let $K$ be a tame (PL or smooth) 2-knot in the 4-sphere $S^4$. Let $M \subset S^4$ be the knot manifold of $K$. Then $K$ is unknotted if and only if $g(M) = 1$.

**Proof.** If $K$ is trivial, then $M \cong_{PL} D^3 \times S^1$, hence $g(M) = 1$. Conversely, let $(V_1, V_2)$ be a regular splitting of $M$ of minimal genus. By [13] and [14] it follows that $g(M) = g(\partial V_1) = g(\partial V_2) + g(\partial M)$. Because $\partial M \cong \partial(S^2 \times D^2) \cong S^2 \times S^1$ and $g(S^2 \times S^1) = 1$, we have $g(M) = g(\partial V_2) + 1$. Hence $g(M) = 1$ implies that $g(\partial V_2) = 0$, i.e. $V_2$ is a 4-cell as $V_2$ is a handlebody. Thus $M \cong_{PL} V_1$. Because $H_1(M) \cong \mathbb{Z}$ and $H_2(M) \cong 0$, the Mayer-Vietoris sequence of the pair $(H^0 \cup \lambda H^1, \mu H^2)$, where $V_1 = H^0 \cup \lambda H^1 \cup \mu H^2$, yields $\lambda = 1$, hence $\pi_1(V_1) \cong \pi_1(M) \cong \mathbb{Z}$. By [8] the manifold $M$ is homotopy equivalent to $S^1 \times D^3$. Thus the results of [6], [7], [10] and [16] get that $M$ is (TOP) homeomorphic to $S^1 \times D^3$. Hence $K$ is trivial. \hfill \Box

**Proposition 16.** Let $K_i$ be a tame 2-knot in the 4-sphere $S^4$, $i = 1, 2$, and $M_i$ the knot manifold of $K_i$. If $M$ is the knot manifold of the connected sum $K_1 \# K_2$, then we have $g(M) = g(M_1) + g(M_2) - 1$.

**Proof.** By definition of connected sum there exists a tame 3-sphere $\Sigma \subset S^4$ which divides $S^4$ into two 4-balls $B_1, B_2$ containing $K_1, K_2$ respectively. Furthermore, $K_1 \cap K_2$ is a closed 2-cell $C$, tamely embedded in $\Sigma$, and $K = K_1 \# K_2$ is just the union of $K_1, K_2$ minus the interior of $C$. Let $W$ be a regular neighbourhood of the unknotted 1-sphere $\partial C$ in $\Sigma$ and let $W'$ denote the closed complement of $W$ in $\Sigma$. Then the pair $(W, W')$ of solid tori represents the standard genus one splitting of $\Sigma$. If we set $K_i' = K_i \setminus \overset{\circ}{C}$, $i = 1, 2$, then the composite knot $K$ is $K_1' \cup K_2'$ and its knot manifold $M$ is $M_1' \cup M_2'$, where $M_i'$ denotes the closed complement of a small regular neighbourhood of $K_i'$ in $B_i$, $i = 1, 2$. Moreover, the intersection of $M_i'$ with
$M'_2$ is just the solid torus $W'$. Thus, according to notation of Proposition 6, there exists a 3-dimensional annulus $A = S^1 \times D^2 \cong_{PL} W'$ such that $M = M_1 \cup_A M_2$. Furthermore, $A$ is properly embedded essential annulus in $\partial M$, i.e. the inclusion induced homomorphism $\Pi_1(A) \rightarrow \Pi_1(\partial M)$ is monic. Now we can repeat the arguments discussed in the proof of Proposition 6 to obtain the result. 

The next result gives a partial solution to a problem stated in [5].

**Corollary 17.** (The non cancellation theorem for 2-knots in $S^4$.) *Suppose a connected sum $K = K_1 \# K_2$ of two tame 2-knots is unknotted in $S^4$. Then both $K_1$ and $K_2$ are themselves unknotted.*
References

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References
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