COMPLEXITY OF COMPACTIFICATIONS OF N (*)

by G. D. FAULKNER (in Raleigh)  
and M.C. Vipera (in Perugia)**

SOMMARIO. - A causa della sua complessità, la compattificazione di Stone-Čech  
dei numeri naturali è tra gli spazi topologici più studiati. Altre compattificazioni di N possono essere altrettanto complesse. In questo lavoro si esamina il reticolo delle compattificazioni di N rispetto a due misure di complessità.

SUMMARY. - Because of its complexity, the Stone-Čech compactification of the natural numbers is among the most studied of topological spaces. Other compactifications of N share in this complexity. This paper begins an examination of the lattice of compactifications of N with respect to two measures of complexity that compactifications may share with βN.

1. Introduction.

The Stone-Čech compactification of the natural numbers, βN, has long been a fascination of topologist. This is almost certainly due to the fact that something of such tantalizing complexity could arise out of such a topologically trivial object as the natural numbers. This complexity has made βN into something of an example machine and one of the principle grounds of interaction between set theory and topology.

Of course the lattice of compactifications of N is huge and there are very many compactifications of N which are “near” βN in the lattice. Perhaps these compactification share in the complexity of βN. It is this that we begin to address in this paper.

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(**) Indirizzi degli Autori: G.D. Faulkner: Department of Mathematics, North Carolina State University, Raleigh NC 27695-825, (USA); M.C. Vipera: Dipartimento di Matematica dell’Università, Via Vanvitelli 1, 06123 Perugia (Italia).
What do we mean when we say that $\beta\mathbb{N}$ is complex? We might of course mean any one of an almost inexhaustible list of properties. We will consider two of these:

1. First $\beta\mathbb{N}$ contains no nontrivial convergent sequences.

Vaughan and Dow call spaces with this property “contrasquential” [3]. We will usually use the notation $\omega + 1 \not\subset X$. If $\omega + 1 \not\subset \alpha\mathbb{N}$, then this alone forces restrictions on how small the remainder can be. In particular $|\alpha\mathbb{N} \setminus \mathbb{N}| \geq 2^t$, where $t$ is the smallest cardinality of a tower in $\mathcal{P}(\mathbb{N})$. In fact it is known that a compact space of cardinality less than $2^t$ is sequentially compact [14, Thm 6.3], [16, Thm. 5.9]. Of course, within the lattice of compactifications, the presence of nontrivial convergent sequences in some cases yields as much information as their absence.

2. $\beta\mathbb{N}$ contains a wealth of copies of $\beta\mathbb{N}$.

In particular if $F$ is any infinite closed subset of $\beta\mathbb{N}$ then $\beta\mathbb{N} \rightarrow F$. Clearly, all compactifications of $\mathbb{N}$ which share this property with $\beta\mathbb{N}$ are contrasquential. However, it is known, at least consistently, that there are compactifications of $\mathbb{N}$ which contain neither $\omega + 1$ or $\beta\mathbb{N}$ [6].

The notation substantially follows [2, 7]. Information on cardinal functions can be found in [8], and information on small cardinals in [14, 16].

We will denote by $\mathbb{N}$ the set and the discrete space of the natural numbers and by $\omega$ the cardinality of $\mathbb{N}$. Also, as is customary, $A^* = Cl_{\beta\mathbb{N}}(A) \setminus A$ for each infinite $A \subseteq \mathbb{N}$. In particular $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$.

If $X$ is a Tychonoff space, the canonical quotient map from a larger compactification $\gamma X$ to a smaller compactification $\alpha X$ will be denoted by $\pi_{\gamma\alpha}$. As usual, $C^*(X)$ denotes the algebra of all bounded real valued continuous functions on $X$. $C^*(X)$ is a Banach space with the supremum norm. In particular, $C^*(\mathbb{N})$ coincides with the space $l_\infty$ of all bounded real valued sequences. As usual, $c_0$ is the subspace of consisting of those sequences which converge to zero.

The closed subalgebra of $C^*(X)$ generated by a subcollection $\mathcal{F}$ is denoted by $\overline{\langle \mathcal{F} \rangle}$. $C_\alpha(X)$ is the (closed) subalgebra of $C^*(X)$ consisting of those functions which have continuous extensions to the compactification $\alpha X$. Usually, the unique extension of $f \in C_\alpha(X)$ to $\alpha X$ will be denoted by $f^\alpha$. The map $f \mapsto f^\alpha$ is an isomorphism from $C_\alpha(\mathbb{N})$ onto $C(\alpha X)$ and one has $\|f^\alpha\| = \|f\|$. The extension of $f \in C^*(\mathbb{N})$ to $\beta\mathbb{N}$ will be denoted, as is customary, by $f^*$. 
If $Y$ is a Banach space and $M$ is a subspace of $Y$, then we say that $M$
complemented in $Y$ provided there is another closed subspace $N$ of $Y$
for which $Y = M \oplus N$. A subspace $M$ is complemented if and only if $M$
is the nullspace of a continuous projection defined on $Y$.

1. First we will consider several nontrivial examples of compactifications
of $\mathbb{N}$ which are consequential.

If $F$ is a closed subset of $\mathbb{N}^*$, we will denote by $\beta\mathbb{N}/F$
the compactification resulting from identifying $F$ to a single point.

**Theorem 1.1.** Suppose that $\alpha\mathbb{N} = \beta\mathbb{N}/F$, where $F$
is a retract of $\beta\mathbb{N}$. Then $\omega + 1 \not\to \alpha\mathbb{N}$.

*Proof.* Let $r : \beta\mathbb{N} \to F$ be a retraction and let $r_1 = r|\mathbb{N}$. Let $I_F$
be the ideal of functions in $C^*(\mathbb{N}) = l_\infty$ whose extensions to $\beta\mathbb{N}$
vanish on $F$. Define a continuous linear projection $P : C^*(\mathbb{N}) \to C^*(\mathbb{N})$
by $Pf = f^* \circ r_1$. Clearly the nullspace of $P$ is exactly $I_F$. Thus $I_F$
is complemented in $C^*(\mathbb{N})$, and since $I_F$ is of codimension one in $C_\alpha(\mathbb{N})$, $C_\alpha(\mathbb{N})$
is complemented as well. Thus $C_\alpha(\mathbb{N})$ is isomorphic to $C^*(\mathbb{N})$ [10]. Now suppose
$\omega + 1 \to \alpha\mathbb{N}$. In particular, let $\{p_n \mid n < \omega\} \cup \{p\} \subseteq \alpha\mathbb{N}$, with $p_n \to p$.
Define a mapping $Q : C_\alpha(\mathbb{N}) \to c_0$ by $Qf = \{f^*(p_n) - f^*(p)\}_{n=1}^\infty$. Clearly
$||Q|| \leq 2$, so that $Q$ is a continuous linear operator. Since any copy of
$\omega + 1$ must be $C^*$-embedded in $\alpha\mathbb{N}$, $Q$ is surjective. By the open mapping theorem, $Q$
must be open. However, $Q$ must also be weakly compact [8]. This would imply that $c_0$
is reflexive. Hence $\omega + 1 \not\to \alpha\mathbb{N}$. ◊

The hypothesis that $F$ is a retract of $\beta\mathbb{N}$ is satisfied, for example, if
$F$ is of countable $\pi$-weight [15, Thm. 1.8.2]. In particular, if $F \subset \mathbb{N}^*$
homeomorphic to $\beta\mathbb{N}$, then $\omega + 1 \not\to \beta\mathbb{N}/F$.

It is known that there are closed separable subsets of $\beta\mathbb{N}$ which are not
retracts of $\beta\mathbb{N}$ [13]. This leads to:

**Question.** Suppose $\alpha\mathbb{N} = \beta\mathbb{N}/F$, where $F$ is separable. Can $\alpha\mathbb{N}$
contain nontrivial convergent sequences?

In any case it is clear that, if you identify a large enough set, then
convergent sequences must arise.
Lemma 1.2. Let \( \alpha N \) be a compactification of \( N \). If, for some \( p \in \alpha N \setminus N \), \( \pi_{\alpha}(p) \) has nonempty interior in \( N^* \), then \( \omega + 1 \Rightarrow \alpha N \).

Proof. Suppose \( \pi_{\alpha}(p) \) has nonempty interior in \( N^* \). Then \( \pi_{\alpha}(p) \) contains a set of the form \( A^* \) for some \( A \subseteq N \). This set \( A \), considered as a sequence in \( \alpha N \), converges to \( p \). \( \diamond \)

Remark 1.3. We may observe that, if there is a sequence in \( N \) which converges to a point \( p \) in \( \alpha N \setminus N \), then \( \pi_{\alpha}(p) \) has nonempty interior.

Now, how does one typically obtain convergent sequences in a compactification of \( N \) when viewed as a quotient of \( \beta N \)? One can proceed as follows. Let \( D \) be any countable discrete subset of \( \beta N \). Since \( D \) is \( C^* \)-embedded in \( \beta N \), \( Cl_{\beta N}(D) \cong \beta N \). If in this copy of \( \beta N \) we identify the remainder to a point, we obtain a convergent sequence. We might in fact collapse any closed set which contains this remainder and misses the points of \( D \). This is in fact what always happens. Suppose \( \{ p_n \mid n < \omega \} \) is a nontrivial sequence in \( \alpha N \) converging to a point \( p \). Let \( q_n \in \pi_{\alpha}(p_n) \) and \( D = \{ q_n \mid n < \omega \} \). As before \( D \) is \( C^* \)-embedded in \( \beta N \) so that \( Cl_{\beta N}(D) \setminus D \cong N^* \). In the quotient \( \alpha N \), this set, and perhaps more, must be collapsed to \( p \). From this it is easy to observe that:

Proposition 1.4. If \( \omega + 1 \Rightarrow \alpha N \), then there is a compactification \( \gamma N \), strictly larger than \( \alpha N \), for which \( \omega + 1 \Rightarrow \gamma N \).

Proof. Let \( \{ p_n \mid n < \omega \} \) be a nontrivial convergent sequence in \( \alpha N \), and let \( p \) be its limit point. Then there is a copy \( F \) of \( N^* \) in \( \beta N \) such that \( \pi_{\beta}(F) = \{ p \} \). Take a copy of \( \beta N \) in \( F \). The compactification of \( N \) resulting from the collapse of the remainder in this copy of \( \beta N \) to a point is the desired \( \gamma N \). \( \diamond \)

If \( X \) is a locally compact space, \( K \) is compact, and \( f : X \to K \) is continuous, then, clearly, there exists a minimum compactification \( \alpha X \) of \( X \) to which \( f \) extends. Such a compactification can actually be constructed by endowing the disjoint union \( X \cup K \) with a suitable topology, which makes it compact, and putting \( \alpha X = Cl_{X \cup K}(X) \) [1,5,11]. The remainder of \( \alpha X \) is the singular set of \( f \), \( S(f) = \{ y \in K \mid \text{for each neighborhood} \ V \text{ of} y, \overline{f^{-1}(V)} \text{is not compact} \} \). The extension \( f \) of \( f \) to \( \alpha X \) is the identity on
$\alpha X \setminus X$. Moreover, if $\gamma X$ is a compactification to which $f$ extends, then $\pi_{\gamma X} \mid \gamma X \setminus X = f \mid \gamma X \setminus X$, where $\tilde{f}$ is the extension of $f$ to $\gamma X$.

If $f \in C^*(X)$, then the minimum compactification to which $f$ extends is denoted by $\omega_f X$. Clearly $\omega_f X \setminus X$ is homeomorphic to a compact subspace of $\mathbb{R}$.

If $\mathcal{F} \subseteq C^*(X)$, then we denote by $\omega_{\mathcal{F}} X$ the minimum compactification to which each member of $\mathcal{F}$ extends. Clearly $\omega_{\mathcal{F}} X = \sup \{\omega_f X \mid f \in \mathcal{F}\}$. Therefore $\omega_{\mathcal{F}} X \setminus X$ is homeomorphic to a subspace of the Tychonoff cube $\mathbb{I}^{\mathcal{F}}$. Clearly the family of the extensions of the elements of $\mathcal{F}$ to $\omega_{\mathcal{F}} X$ must separate points of $\omega_{\mathcal{F}} X \setminus X$. One has $w(\omega_{\mathcal{F}} X \setminus X) \leq |\mathcal{F}|$ and, if $X$ is second countable, $w(\omega_{\mathcal{F}} X) \leq |\mathcal{F}|$.

**Theorem 1.5.** Let $g$ be a map from $\alpha N$ into $\alpha N$ and let $f : \alpha N \to \beta N$ be the composition of $g$ and the inclusion map. Let $\alpha N$ be the minimum of the compactifications of $N$ to which $f$ extends. Then $\omega + 1 \not\rightarrow \alpha N$ if and only if $g$ is finite-to-one.

**Proof.** We can put $\alpha N = N \cup S(f) \subseteq N \cup \beta N$. Let $\tilde{f}$ denote the extension of $f$ to $\beta N$. First suppose that there is $n \in N$ such that $A = g^{-1}(n) = f^{-1}(n)$ is infinite. Then $n \in S(f) = \alpha N \setminus N$ and one has $\tilde{f}(A^*) = \{n\}$. Then $\pi_{\beta N}(A^*) = \{n\}$, hence, by lemma 1.2, $\omega + 1 \not\rightarrow \alpha N$.

Now, let $g$ be finite-to-one. Then $g(N)$ is infinite. We can replace $g(N)$ by $N$ (and $Cl_{\beta N}(g(N))$ by $\beta N$), hence, without loss of generality, we can suppose $g$ surjective. Then, obviously, $\alpha N \setminus N = S(f) = N^*$, so $\omega + 1 \not\rightarrow \alpha N \setminus N$. Now, let $B$ be an infinite subset of $N$. Then $f(B) = g(B)$ is infinite. One has $\tilde{f}(B^*) = \pi_{\beta N}(B^*) \subseteq \alpha N \setminus N = N^*$. Clearly one has $\tilde{f}(B^*) = \tilde{f}(Cl_{\beta N}(B)) \setminus N$. Moreover, since $\tilde{f}$ is closed, one has $\tilde{f}(Cl_{\beta N}(B)) = Cl_{\beta N}(\tilde{f}(B)) = Cl_{\beta N}(f(B))$. So we have proved $\pi_{\beta N}(B^*) = Cl_{\beta N}(f(B)) \setminus N = (f(B))^*$, which is infinite. Suppose $B = \{b_n\}$ is a sequence in $N$ converging to $y \in \alpha N \setminus N$. Then $\pi_{\beta N}(B^*) = \{y\}$, contradiction $\Diamond$

If $\mathcal{F} \subseteq C^*(N)$ is countable, then $\omega_{\mathcal{F}} N$ is metrizable, hence it has a wealth of convergent sequences. The following proposition indicates that if $\mathcal{F}$ is “nearly” countable, then the wealth of nontrivial convergent sequences persists. We recall that the cardinal $s$ is the minimum cardinality of a splitting family in $\mathcal{P}(N)$.

**Proposition 1.6.** Let $\mathcal{F} \subseteq C^*(N)$ with $|\mathcal{F}| < s$ and let $\alpha N = \omega_{\mathcal{F}} N$. 

**Proof.** We can put $\alpha N = N \cup S(f) \subseteq N \cup \beta N$. Let $\tilde{f}$ denote the extension of $f$ to $\beta N$. First suppose that there is $n \in N$ such that $A = g^{-1}(n) = f^{-1}(n)$ is infinite. Then $n \in S(f) = \alpha N \setminus N$ and one has $\tilde{f}(A^*) = \{n\}$. Then $\pi_{\beta N}(A^*) = \{n\}$, hence, by lemma 1.2, $\omega + 1 \not\rightarrow \alpha N$.

Now, let $g$ be finite-to-one. Then $g(N)$ is infinite. We can replace $g(N)$ by $N$ (and $Cl_{\beta N}(g(N))$ by $\beta N$), hence, without loss of generality, we can suppose $g$ surjective. Then, obviously, $\alpha N \setminus N = S(f) = N^*$, so $\omega + 1 \not\rightarrow \alpha N \setminus N$. Now, let $B$ be an infinite subset of $N$. Then $f(B) = g(B)$ is infinite. One has $\tilde{f}(B^*) = \pi_{\beta N}(B^*) \subseteq \alpha N \setminus N = N^*$. Clearly one has $\tilde{f}(B^*) = \tilde{f}(Cl_{\beta N}(B)) \setminus N$. Moreover, since $\tilde{f}$ is closed, one has $\tilde{f}(Cl_{\beta N}(B)) = Cl_{\beta N}(\tilde{f}(B)) = Cl_{\beta N}(f(B))$. So we have proved $\pi_{\beta N}(B^*) = Cl_{\beta N}(f(B)) \setminus N = (f(B))^*$, which is infinite. Suppose $B = \{b_n\}$ is a sequence in $N$ converging to $y \in \alpha N \setminus N$. Then $\pi_{\beta N}(B^*) = \{y\}$, contradiction $\Diamond$
Then, each infinite closed subset of $\alpha \mathbb{N}$ and every open subset of $\alpha \mathbb{N}$ which intersects $\alpha \mathbb{N} \setminus \mathbb{N}$ contains a copy of $\omega + 1$.

Proof. Since $w(\alpha \mathbb{N}) < s$, by [14, Thm. 6.1] every closed subset of $\alpha \mathbb{N}$ is sequentially compact. If $U$ is open in $\alpha \mathbb{N}$ and $p \in U \cap (\alpha \mathbb{N} \setminus \mathbb{N})$, then $U$ contains a closed neighborhood of $p$ which must be infinite. \hfill \Box

Note that, in the above theorem, $\mathbb{N}$ can be replaced by any locally compact space of weight $\mu < s$.

Example 1.7. It is of course possible to lose copies of $\omega + 1$ in the supremum of a family of compactifications. In fact $\beta \mathbb{N}$ is the supremum of all 2-point compactifications. However, it is even possible to lose copies of $\omega + 1$ in the supremum of two compactifications. Suppose $A$, $B$ are disjoint infinite subsets of $\mathbb{N}$. Let $\alpha \mathbb{N}$ be the compactification formed from $\beta \mathbb{N}$ by identifying $A^*$ to a point, and let $\gamma \mathbb{N}$ be formed by identifying $B^*$ to a point. Each of these compactifications contains a copy of $\omega + 1$. However $\alpha \mathbb{N} \vee \gamma \mathbb{N} = \beta \mathbb{N}$.

Now we consider two compactifications of an arbitrary Tychonoff space $X$. Suppose that $\alpha X \leq \gamma X$ and that $\omega + 1 \hookrightarrow \alpha X$. It should be the case that, if $\gamma X$ is not “too far” above $\alpha X$, then $\omega + 1 \hookrightarrow \gamma X$. In fact:

Theorem 1.8. Let $\alpha X \leq \gamma X$ and suppose, for each $p \in \alpha X$, one of the following is true:

a. $\pi^{-1}_{\gamma \alpha} (p)$ is first-countable;

b. $|\pi^{-1}_{\gamma \alpha} (p)| < 2^1$;

c. $w(\pi^{-1}_{\gamma \alpha} (p)) < s$.

Then $\omega + 1 \hookrightarrow \alpha X$ implies $\omega + 1 \hookrightarrow \gamma X$.

Proof. First suppose each fiber of $\pi_{\gamma \alpha}$ is finite. Let $\{ p_n \ | \ n < \omega \}$ a nontrivial sequence in $\alpha X$ converging to a point $p$. Choose $q_n \in \pi^{-1}_{\gamma \alpha} (p_n)$ and let $Q$ be the set of limit points of $\{ q_n \}$. One has $Q \subseteq \pi^{-1}_{\gamma \alpha} (p)$, hence $Q$ is finite. Then some subsequence of $\{ q_n \}$ converges. Now suppose $A = \pi^{-1}_{\gamma \alpha} (q)$ is an infinite fiber. Then, by a., b or c., $A$ contains a nontrivial convergent sequence. \hfill \Box

Note that, if $\alpha X$, $\gamma X$ are as in the above theorem, then $\omega + 1 \hookrightarrow \alpha X \setminus X$ implies $\omega + 1 \hookrightarrow \gamma X \setminus X$.  

Let \( \alpha X, \gamma X \) be compactifications of \( X \) and let \( \mathcal{F} \subseteq C^*(X) \). One has

\[
C_\gamma(X) = \overline{C_\alpha(X) \cup \mathcal{F}}
\]

if and only if \( \gamma X \) is the smallest compactification greater than or equal to \( \alpha X \) to which every element of \( \mathcal{F} \) extends. In this case the family \( \mathcal{F} \) of the extensions separates points in every fiber \( \pi_{\gamma\alpha}^{-1}(p) \), where \( p \in \alpha X \setminus X \). In fact, if two points in \( \pi_{\gamma\alpha}^{-1}(p) \) were not separated by \( \mathcal{F} \), then we could identify them to obtain a compactification still greater than or equal to \( \alpha X \), to which each function in \( \mathcal{F} \) extends and at the same time smaller than \( \gamma X \). This contradicts the minimality of \( \gamma X \).

**Corollary 1.9.** Let \( C_\gamma(X) = \overline{C_\alpha(X) \cup \mathcal{F}} \), where \( |\mathcal{F}| < s \). If \( \omega + 1 \leftrightarrow \alpha X \), then \( \omega + 1 \leftrightarrow \gamma X \).

**Proof.** Let \( A = \pi_{\gamma\alpha}^{-1}(p) \) be an arbitrary fiber. Then \( \mathcal{F} \) separates points of \( A \). Since \( A \) is compact, this implies that \( \omega(A) < s \) and we can apply the above theorem. \( \diamond \)

We recall that, if \( \gamma X = \alpha X \vee \delta X \), then \( \pi_{\gamma\alpha} \) and \( \pi_{\gamma\delta} \) separate points of \( \gamma X \setminus X \), hence \( \pi_{\gamma\delta} \) is injective on the fibers of \( \pi_{\gamma\alpha} \).

**Corollary 1.10.** Let \( \gamma X = \alpha X \vee \delta X \) and suppose \( |\delta X \setminus X| < 2^\omega \). Then \( \omega + 1 \leftrightarrow \alpha X \) implies \( \omega + 1 \leftrightarrow \gamma X \).

**Proof.** Since \( \pi_{\gamma\delta} \) is injective on the fibers of \( \pi_{\gamma\alpha} \), one has \( |\pi_{\gamma\alpha}^{-1}(p)| < 2^\omega \) for each \( p \in \alpha X \). \( \diamond \)

**Example 1.11.** Note that \( \alpha X \leq \gamma X \) and \( \omega + 1 \leftrightarrow \gamma X \) do not imply \( \omega + 1 \leftrightarrow \alpha X \). To see this, let \( K \) be a copy of \( \beta\mathbb{N} \) which is contained in \( \mathbb{N}^* \) and let \( K_1 \cong \mathbb{N}^* \) be the set of nonisolated points of \( K \). Then \( \gamma \mathbb{N} = \beta\mathbb{N}/K_1 \) contains a convergent sequence. Clearly \( \beta\mathbb{N}/K < \gamma \mathbb{N} \) and \( \omega + 1 \not\leftrightarrow \beta\mathbb{N}/K \).

There is a class of compactifications of \( \mathbb{N} \) having the property that convergent sequences cannot disappear as you descend in the lattice. A compactification \( \alpha X \) of a locally compact space \( X \) is said to be **singular** provided \( K = \alpha X \setminus X \) is a retract of \( \alpha X \) \([4]\). If \( f : X \to K \) is the restriction of a retraction, then the topology on \( \alpha X \) may be realized by taking as a base the collection of all open subsets of \( X \) together with sets of the form \( U \cup (f^{-1}(U) \setminus F) \) where \( U \) is open in \( K \) and \( F \) is an arbitrary compact subset of \( X \).
Theorem 1.12. Suppose $\gamma N$ be a singular compactification of $N$ such that $\omega + 1 \hookrightarrow \gamma N$. If $\alpha N \leq \gamma N$, then $\omega + 1 \hookrightarrow \alpha N$.

Proof. The proof is easy if the sequence in $\gamma N$ can be chosen in $N$. Suppose not and let $\{p_n\}$ be a nontrivial sequence in $\gamma N \setminus N$ which converges to $p \in \gamma N \setminus N$. Let $f : X \to \gamma X \setminus X$ be, as above, the restriction of a retraction of $\gamma X$ to the remainder. Choose $q_n \in f^{-1}(p_n)$, for each $n$ and let $V = U \cup (f^{-1}(U) \setminus F)$ be a basic neighborhood of $p$. Since all but finitely many of $p_n$ are in $U$, all but finitely many of $q_n$ are in $V$. Thus $q_n \to p$. ◇

Remark 1.13. It is easy to see that, if $\alpha X \leq \gamma X$ and $\pi^{-1}_{\gamma \alpha}(p)$ is finite for each $p \in \alpha X$, then $\omega + 1 \hookrightarrow \gamma X$ implies that $\omega + 1 \hookrightarrow \alpha X$.

2. We now turn to what is probably a more interesting question: when do compactifications of $N$ contain copies of $\beta N$?

First we note the following:

Remark 2.1. If $\beta N \hookrightarrow \alpha N$, then $\beta N \hookrightarrow \alpha N \setminus N$. In fact, if $h$ is the embedding, then $h(N^*) \cap (\alpha N \setminus N)$ is an infinite closed subset of $h(N^*)$, so it contains a copy of $\beta N$.

It is well known that if the continuous image of a topological space contains a copy of $\beta N$ then the space must as well. So we have:

Proposition 2.2. If $\beta N \hookrightarrow \alpha X \leq \gamma X$ then $\beta N \hookrightarrow \gamma X$.

In section 1 we gave examples of compactifications of the form $\beta N / F$ which are contrasequential. For this kind of compactifications one has:

Proposition 2.3. Let $F$ be a closed subset of $N^*$ such that $\omega + 1 \not\hookrightarrow \alpha N = \beta N / F$. Then every infinite closed subset of $\alpha N$ contains a copy of $\beta N$.

Proof. Put $\pi_{\beta \alpha}(F) = \{p\}$ and let $H$ be an infinite closed subset of $\alpha N$. The proof is trivial if $p \notin H$. So suppose $p \in H$ and let $H_1 = \pi_{\beta \alpha}^{-1}(H)$. If there is an open subset $U$ of $\beta N$ such that $F \subset U$ and $H_1 \setminus U$ is infinite, we are done. In fact, in this case, $H_1 \setminus U$ must contain a copy $B$ of $\beta N$ and
so \( \pi_{\beta\alpha}(B) \cong B \) is a copy of \( \beta N \) contained in \( H \). Then let us suppose that \( H_1 \setminus U \) is finite for each open \( U \) containing \( F \). Since \( H_1 \setminus F \) is infinite, it contains a countable discrete subset \( \{ p_n \mid n < \omega \} \). Let \( q_n = \pi_{\beta\alpha}(p_n) \), for each \( n \). Under our assumption, for every open subset \( U \) of \( \beta N \) containing \( F \), all but finitely many \( p_n \) are in \( U \). This clearly implies \( q_n \to p \) in \( \alpha N \), contradiction.

\[ \diamond \]

**Example 2.4.** As in Example 1.12, let \( K \) be a copy of \( \beta N \) which is contained in \( N^* \) and let \( K_1 \cong N^* \) be the set of nonisolated points of \( K \). Then \( \gamma N = \beta N / K_1 \) contains a convergent sequence and \( \gamma N > \alpha N = \beta N / K \), which has the property that every infinite closed subset contains a copy of \( \beta N \). Therefore, that property can be lost “going up” in the lattice.

**Proposition 2.5.** Let \( g \) be a map from \( N \) into \( N \) and let \( f : N \to \beta N \) be the composition of \( g \) and the inclusion map. Let \( \alpha N \) be the minimum of the compactifications of \( N \) to which \( f \) extends. Then \( \alpha N \) has the property that every infinite closed subset contains a copy of \( \beta N \) if and only if \( g \) is is finite-to-one.

**Proof.** Let \( f \) be finite-to-one, so that, by Thm. 1.5, \( \omega + 1 \not\to \alpha N \). The hypothesis implies \( \alpha N \setminus N \cong N^* \) (see section 1). Let \( F \) be an infinite closed subset of \( \alpha N \). If \( F \cap (\alpha N \setminus N) \) is infinite, then \( F \) contains a copy of \( \beta N \). But, if \( F \cap (\alpha N \setminus N) \) were finite, then \( F \) would contain a convergent sequence, contradiction. The converse follows directly from Theorem 1.5. \( \diamond \)

The next theorem asserts that if \( \beta N \) can be realized as the supremum of two compactification one of which is “simple”, then the other must be “complex”.

**Theorem 2.6.** Suppose \( \beta N = \alpha N \cup \gamma N \) and \( \omega + 1 \not\to \alpha N \). Then \( \beta N \not\to \gamma N \).

**Proof.** If \( \omega + 1 \not\to \alpha N \), then, by Thm. 1.8, some fiber \( A = \pi^{-1}_\beta(p) \) is infinite. Since \( A \) is closed, it must contain a copy of \( \beta N \). But \( \pi_{\beta\alpha} \) is injective on \( A \), hence its restriction to \( A \) is an embedding. Thus \( \beta N \not\to \gamma N \). \( \diamond \)
Now, as the results of a theorem of Shapirovskii's, we see that it is impossible to create copies of $\beta N$ from small collections of simple compactifications.

**Theorem 2.7.** [12, Cor. 3]. If $\beta N \hookrightarrow \prod_{\lambda < \kappa} Y_\lambda$ and $\kappa < cf(c)$ then there is $\lambda_0 < \kappa$ such that $\beta N \hookrightarrow Y_{\lambda_0}$.

The pertinence of this to our considerations follows:

**Corollary 2.8.** Let $X$ be any Tychonoff space and let $\alpha X = \sup\lbrace \alpha_\lambda X \mid \lambda < \kappa \rbrace$ with $\kappa < cf(c)$. If $\beta X \not\hookrightarrow \alpha_\lambda X$ for each $\lambda$, then $\beta X \not\hookrightarrow \alpha X$.

**Proof.** We know that $\alpha X$ is (homeomorphic to) a subset of $\prod_{\lambda < \kappa} \alpha_\lambda X$.

**Theorem 2.9.** If $\beta N = \alpha N \lor \gamma N$ and $\beta N \not\hookrightarrow \gamma N$, then every infinite closed subset of $\alpha N$ contains a copy of $\beta N$.

**Proof.** One has $\beta N \subseteq \alpha N \times \gamma N$ and $\pi_{\beta \alpha}$ is the restriction of the first projection. Put $G = \pi_{\beta \alpha}^{-1}(F)$. Then $G \subseteq F \times \gamma N$. Since $\beta N \hookrightarrow G$, by theorem 2.7 $\beta N \hookrightarrow F$.

**Corollary 2.10.** Let $\beta N = \alpha N \lor (\sup\lbrace \gamma_\lambda N \mid \lambda < \kappa \rbrace)$ with $\kappa < cf(c)$. If $\beta N \not\hookrightarrow \gamma_\lambda N$ for each $\lambda$, then every infinite closed subset of $\alpha N$ contains a copy of $\beta N$.

**Theorem 2.11.** Suppose $\mathcal{F}$ is a subset of $C^*(N)$ such that $|\mathcal{F}| < c$. Suppose also that $\alpha N$ satisfies

$$C^*(N) = \langle C_\alpha(N) \cup \mathcal{F} \rangle.$$  

Then each infinite closed subset of $\alpha N$ contains a copy of $\beta N$.

**Proof.** Clearly, one has $\beta N = \alpha N \lor \omega_{\mathcal{F}} N$. We know that $w(\omega_{\mathcal{F}} N) < c$ (see Section 1) and this implies $\beta N \not\hookrightarrow \omega_{\mathcal{F}} N$. Then we can apply Theorem 2.9.

**Remark 2.12.** In Theorem 2.6 we could replace the hypothesis $\beta N = \alpha N \lor \gamma N$ by $\delta N = \alpha N \lor \gamma N$, where $\delta N$ is a compactification such that every infinite closed subset contains a copy of $\beta N$. We can do the same
with Theorem 2.9 and Cor. 2.10. Also, in Theorem 2.11, we can replace $C^*(\mathbb{N})$ by $C_\emptyset(\mathbb{N})$, where $\emptyset \mathbb{N}$ is a compactification which satisfies the same condition.

In order to give a nontrivial example of a compactification $\alpha \mathbb{N}$ and of a family $\mathcal{F}$ of functions which, compatibly, satisfy the hypotheses of Theorem 2.11, we need the following Lemma:

**Lemma 2.13.** Let $X$ be a normal space. Suppose $F_1, F_2$ are closed subsets of $X$ and let $h : F_1 \to F_2$ be a homeomorphism such that, $\forall x \in F_1 \cap F_2$ one has $h(x) = x$. Let $Y$ be the quotient space obtained by identifying $x$ and $h(x)$ $\forall x \in F_1$. Then $Y$ is Hausdorff.

**Proof.** Let $q$ be the quotient map and $F = F_1 \cup F_2$. Let $y, z$ be distinct point of $Y$. The only nontrivial case if when $y, z \in q(F)$. Let $y_1, z_1 \in F_1$ be such that $q(y_1) = y$, $q(z_1) = z$. Let $U, V$ be open subsets of $F_1$ such that $y_1 \in U, z_1 \in V$ and $\overline{U} \cap \overline{V} = \emptyset$. The hypotheses imply that $U \cup h(U)$, $V \cup h(V)$ are open in $F$, and that $\overline{U} \cup \overline{h(U)}$ and $\overline{V} \cup \overline{h(V)}$ are disjoint closed subsets of $X$. Let $W, T$ be open subsets of $X$ such that $W \cap F = U \cup h(U), T \cap F = V \cup h(V)$. We can choose $W, T$ so that $W \cap T = \emptyset$. In fact, let $W'$ and $T'$ be disjoint open subsets of $X$ which contain $\overline{U} \cup \overline{h(U)}$ and $\overline{V} \cup \overline{h(V)}$ respectively. If necessary, we can replace $W, T$ by $W' \cap W'$ and $T' \cap T'$, respectively. Clearly, $q^{-1}(q(W)) = W$ and $q^{-1}(q(T)) = T$, so that $q(W)$ and $q(T)$ are disjoint open subsets of $Y$ which contain $y$ and $z$ respectively.

**Example 2.14.** Let $Y$ be a $P$-space of the form $Z \cup \{x\}$, where $Z = \{x_\lambda : \lambda < \omega_1\}$, every $x_\lambda \in Z$ is isolated in $Y$ and the neighborhoods of $x$ are sets of the form $\{x\} \cup F$, where $F \subseteq Z$ has countable complement. Now $Y$ can be embedded in $\mathbb{N}^* [15, \text{Thm. 4.4.4}$. Clearly $Y \cup \mathbb{N}$ is regular and Lindelöf and hence normal. Since $Y$ is closed in $Y \cup \mathbb{N}$, $Y$ is $C^*$-embedded in $\mathbb{N}^*$, hence $\text{Cl}_{\mathbb{N}^*}(Y) \cong \beta Y$. Clearly, $\text{Cl}_{\mathbb{N}^*}(Y)$ consists of $\{x\}$ together with all points of $\mathbb{N}^*$ which are in the closure of some countable subset of $Z$.

Let $S, T \subset Y$, with $Z = S \cup T$, $S \cap T = \emptyset$ and $|S| = |T| = \omega_1$. Put $S' = S \cup \{x\}, T' = T \cup \{x\}$. Clearly both $S'$ and $T'$ are copies of $Y$, then $\beta S', \beta T' \subset \mathbb{N}^*$ and there is a homeomorphism $h : \beta S' \to \beta T'$ with $h(x) = x$. Since $\beta S'$ and $\beta T'$ have only $x$ in common, by the above Lemma we can create a compactification $\alpha \mathbb{N}$ of $\mathbb{N}$ by identifying each point $p$.
in $\beta S'$ with the point $h(p)$ in $\beta T'$. Notice that no countable collection of continuous functions can separate every $p$ in $\beta S'$ from the associated points $h(p)$ in $\beta T'$.

We are now in a position to define a collection of functions $\mathcal{F}$ of cardinality $\omega_1$, such that $C^*(\mathbb{N}) = (C^*_\alpha(\mathbb{N}) \cup \mathcal{F})$. We can put $S = \{p_\mu : \mu < \omega_1\}$ and $T = \{q_\mu : \mu < \omega_1\}$, with $h(p_\mu) = q_\mu$. For each $\kappa < \omega_1$ let $F_\kappa = Cl_{\mathbb{N}}(\{p_\mu : \mu < \kappa\})$ and $G_\kappa = Cl_{\mathbb{N}}(\{q_\mu : \mu < \kappa\})$. Now, for each $\kappa < \omega_1$, let $f_\kappa \in C^*(\mathbb{N})$ be such that $f_\kappa(F_\kappa) = \{1\}$ and $f_\kappa(\beta T') = \{0\}$. Likewise let $g_\kappa$ be such that $g_\kappa(G_\kappa) = \{1\}$ and $g_\kappa(\beta S') = \{0\}$. Then $\mathcal{F} = \{f_\kappa : \kappa < \omega_1\} \cup \{g_\kappa : \kappa < \omega_1\}$ is the desired collection of functions.

Note that, if $\alpha\mathbb{N}$ and $\mathcal{F}$ satisfy the hypotheses of Theorem 2.11, then $\pi_{\beta\alpha}^{-1}(p)$ is finite for each $p \in \alpha\mathbb{N}$. In fact, one has $u(\pi_{\beta\alpha}^{-1}(p)) \leq |\mathcal{F}| < c$, whereas infinite closed subsets of $\beta\mathbb{N}$ must be of weight $c$. Then Theorem 2.11 could be deduced from Theorem 2.16 below. To prove it, we need the following lemma:

**Lemma 2.15.** If $\alpha\mathbb{N}$ is a compactification of $\mathbb{N}$ such that $\pi_{\beta\alpha}^{-1}(p)$ is finite $\forall p \in \alpha\mathbb{N}$, then there exists $m \in \mathbb{N}$ such that $|\pi_{\beta\alpha}^{-1}(p)| < m, \forall p$.

**Proof.** Suppose that the cardinality of the fibers is unbounded, so that, $\forall n \in \mathbb{N}$ there exists $y_n \in \alpha\mathbb{N} \setminus \mathbb{N}$ such that $|\pi_{\beta\alpha}^{-1}(y_n)| > n$. We can choose a discrete $A = \{a_n\} \subset \{y_n\}$ which still has the property $|\pi_{\beta\alpha}^{-1}(a_n)| > n$. For each $a_n$, choose $n$ distinct points $b_1^{(n)}, \ldots, b_n^{(n)} \in \pi_{\beta\alpha}^{-1}(a_n)$. The sets $B_i = \{b_i^{(n)} | n \geq i\}$ are discrete and pairwise disjoint, then their closures in $\beta\mathbb{N}$ are pairwise disjoint and homeomorphic to $\beta\mathbb{N}$. Let $x_1 \in Cl_{\beta\mathbb{N}}(B_1) \setminus B_1$ and let $k \in \mathbb{N}$. Put $B' = \{B_1 \setminus \{b_1^{(1)}, \ldots, b_{k-1}^{(1)}\}\}. Then Cl_{\beta\mathbb{N}}(B') \setminus B' = Cl_{\beta\mathbb{N}}(B_1) \setminus B_1$ and there is the bijection $h : b_i^{(n)} \mapsto b_i^{(k)}$ from $B'$ to $B_k$ with the property $\pi_{\beta\alpha}(b_i^{(n)}) = \pi_{\beta\alpha}(h(b_i^{(n)})), \forall n \geq k$. Then there is a point $x_k \in Cl_{\beta\mathbb{N}}(B_k) \setminus B_k$ such that $\pi_{\beta\alpha}(x_k) = \pi_{\beta\alpha}(x_1)$. Since this is true for every $k \in \mathbb{N}$, $\pi_{\beta\alpha}$ has an infinite fiber. \hfill $\Diamond$

**Theorem 2.16.** If $\alpha\mathbb{N}$ is a compactification of $\mathbb{N}$ such that $\pi_{\beta\alpha}^{-1}(p)$ is finite $\forall p \in \alpha\mathbb{N}$, then every infinite closed subset of $\alpha\mathbb{N}$ contains a copy of $\beta\mathbb{N}$.

**Proof.** Let $F$ be an infinite closed subset of $\alpha\mathbb{N}$. First suppose that the set $\{p \in F | |\pi_{\beta\alpha}^{-1}(p)| > 1\}$ is finite. Then, clearly, $\pi_{\beta\alpha}(F)$ contains a copy
$B$ of $\beta \mathbb{N}$ on which $\pi_{\beta \alpha}$ is injective. Hence $\pi_{\beta \alpha}|_B$ an embedding.

Now suppose that $\pi_{\beta \alpha}^{-1}(F)$ contains infinitely many nontrivial fibers. Put $k = \max\{h \in \mathbb{N} |$ there exist infinitely many $y$ in $F$ such that $|\pi_{\beta \alpha}^{-1}(y)| = h\}$. The existence of $k$ is ensured by Lemma 2.15. Let $A = \{a_n | n < \omega\}$ be a discrete subset of $F$ such that $|\pi_{\beta \alpha}^{-1}(a_n)| = k$ for each $n$. Put, for every $n$, $\pi_{\beta \alpha}^{-1}(a_n) = \{y_1^{(n)}, \ldots, y_k^{(n)}\}$ and $B_i = Cl_{\beta \mathbb{N}}(\{y_i^{(n)} | n < \omega\})$. Then $B_1, \ldots, B_k$ are pairwise disjoint copies of $\beta \mathbb{N}$ contained in $\pi_{\beta \alpha}^{-1}(F)$. Clearly, for each $z_1 \in B_1$ there is $z_i \in B_i$, such that $\pi_{\beta \alpha}(z_i) = \pi_{\beta \alpha}(z_1), i = 2, \ldots, k$. Then, by the definition of $k$, there are only finitely many $p \in \pi_{\beta \alpha}(B_1)$ such that $|\pi_{\beta \alpha}^{-1}(p) \cap B_1| > 1$. This implies that $B_1$ contains a closed subset $B \cong \beta \mathbb{N}$ such that $\pi_{\beta \alpha}|_B$ is injective. \hfill \qed

Let $\alpha \mathbb{N}$ be the compactification constructed in Example 2.14. We have already remarked that, if $\mathcal{G} \subset C^*(\mathbb{N})$ satisfies $C^*(\mathbb{N}) = \overline{(C_0(\mathbb{N}) \cup \mathcal{G})}$, then $|\mathcal{G}| \geq \omega_1$. Then, under CH, no family of functions satisfies, with respect to $\alpha \mathbb{N}$, the hypotheses of Theorem 2.11. However, each fiber of $\pi_{\beta \alpha}$ is finite.

We can generalize the above theorem as follows:

**Theorem 2.17.** Suppose that $\alpha \mathbb{N} < \gamma \mathbb{N}$ and $\pi_{\gamma \alpha}$ is finite-to-one. Then $\gamma \mathbb{N}$ has the property that every infinite closed set contains a copy of $\beta \mathbb{N}$ if and only if $\alpha \mathbb{N}$ does.

**Proof.** First suppose that $\alpha \mathbb{N}$ satisfies the requested property. Let $G$ be an infinite closed subset of $\gamma \mathbb{N}$ and put $F = \pi_{\gamma \alpha}(G)$. Then $F$ is closed and infinite, hence it contains a copy of $\beta \mathbb{N}$. But this implies that $G$ contains a copy of $\beta \mathbb{N}$.

Conversely, first observe that, by Theorem 2.16, every image of $\mathbb{N}^*$ with respect to a continuous finite-to-one map contains some copies of $\beta \mathbb{N}$. Now, let $F$ be an infinite closed subset of $\alpha \mathbb{N}$ and let $G = \pi_{\gamma \alpha}^{-1}(F)$. Let $B$ be a copy of $\mathbb{N}^*$ contained in $G$. Since $\pi_{\gamma \alpha}|B$ is finite-to-one, one has $\beta \mathbb{N} \hookrightarrow \pi_{\gamma \alpha}(B) \subset F$. \hfill \qed

**References**

[1] Caterino A., Faulkner G.D. and Vipera M.C., *Two applications of*