DUAL GOLDIE DIMENSION(*)

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SOMMARIO. - Si fa uso del processo di dualizzazione nella categoria degli
R-moduli per provare alcuni teoremi riguardanti i moduli di dimen-
sione duale di Goldie finita, che hanno il loro corrispondente nel caso
dei moduli artiniani. In particolare si dà un teorema di struttura per
i moduli complementati di dimensione duale di Goldie finita. Come
applicazione ai gruppi abeliani, si prova poi che gli \( \mathbb{Z} \)-moduli di dimen-
sione duale di Goldie finita sono esattamente gli \( \mathbb{Z} \)-moduli artiniani.

SUMMARY. - We make use of the dualization process in the category of
R-modules to give a few theorems about modules of finite dual Goldie
dimension that have their familiar counterpart in artinian modules. In
particular, we give a structure theorem for complemented modules of
finite dual Goldie dimension. As an application to abelian groups, we
prove that \( \mathbb{Z} \)-modules of finite dual Goldie dimension are exactly the
artinian \( \mathbb{Z} \)-modules.

1. Preliminary definitions and Lemmas.

\( R \) denotes throughout this article a ring with \( 1 \neq 0 \) and \( R \-
modules are unitary left \( R \)-modules. A submodule \( N \) of an \( R \)-module
\( M \) is said to be superfluous in \( M \), in symbols \( N \preceq M \), if \( N + A \neq M \)
for every proper submodule \( A \) of \( M \).

A minimal complement of a submodule \( A \) of an \( R \)-module \( M \) is a
minimal member of the family of all submodules \( X \) of \( M \) such that
\( X + A = M \).

Minimal complements do not always exist; an \( R \)-module \( M \) is
therefore defined to be complemented if every submodule of \( M \) has
a minimal complement in \( M \). It is easy to see that \( B \) is a minimal
complement of a submodule \( A \) of \( M \) if and only if \( A + B = M \) and

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A \cap B \hookrightarrow B. The following property of minimal complements is easily verified:

Let \(X\) be a submodule of an \(R\)-module \(M\). If \(X\) is a minimal complement in \(M\), then for all submodules \(U\) of \(X\), \(X/U \hookrightarrow M/U\) implies \(X = U\). Conversely, if \(M\) is complemented and \(X/U \hookrightarrow M/U\) implies \(X = U\), then \(X\) is a minimal complement in \(M\).

An \(R\)-module \(M\) is said to be strongly complemented if for any two submodules \(A\) and \(B\) of \(M\) such that \(M = A + B\), \(A\) has a minimal complement contained in \(B\).

A finite family \(A_1, \ldots, A_n\) of submodules of an \(R\)-module \(M\) is said to be coindependent if \(A_i \neq M\) and \(A_i + S_{in} = M\) for every \(i = 1, \ldots, n\), where \(S_{in} = \cap_{j \neq i} A_j\). A family \((A_i)_{i \in I}\) of submodules of an \(R\)-module \(M\) is coindependent if every finite subfamily is coindependent.

Every family of submodules of an \(R\)-module \(M\) contains a maximal coindependent subfamily. If \((N_i)_{i \in I}\) is a maximal coindependent family of maximal submodules of an \(R\)-module \(M\), then the radical \(\text{rad} M\) is equal to \(\cap_i N_i\).

An \(R\)-module \(M\) is said to be hollow if \(M \neq 0\) and every proper submodule of \(M\) is superfluous in \(M\). Equivalently, an \(R\)-module \(M\) is hollow if singletons of proper submodules of \(M\) are the only coindependent families of submodules of \(M\).

The following three lemmas will be used in the next section. The proofs follow by duality and therefore will not be given.

**Lemma 1.1.** Let \(A_1, A_2, \ldots\) be a countable family of submodules of an \(R\)-module \(M\). The following four statements are equivalent:

a) The family of submodules \(A_1, A_2, \ldots\) is coindependent;

b) The family of submodules \(A_1, A_2, \ldots A_n\) is coindependent for every \(n \geq 1\);

c) \(A_n + S_{nn} = M\) for every \(n \geq 1\);

d) \(M = S_{1n} + \ldots + S_{nn}\) for every \(n \geq 1\).

**Lemma 1.2.** Let \(A_1, A_2, \ldots\) be a coindependent family of submodules of an \(R\)-module \(M\) and \(B_1, \ldots, B_n\) another family of submodules
of $M$ such that $A_i \subseteq B_i$ and $B_i/A_i \hookrightarrow M/A_i$ for every $i = 1, \ldots, n$. Then $\cap_i B_i/\cap_i A_i \hookrightarrow M/\cap_i A_i$.

**Lemma 1.3.** Let $M$ be an $R$-module in which every family of coindicependent submodules is finite. Then for every proper submodule $A$ of $M$ there exists a submodule $U$ of $M$ containing $A$ such that $M/U$ is hollow.


In Goldie [2], a uniquely determined integer, rank$M \geq 0$ is associated with every $R$-module $M$ in which every independent family of submodules is finite. This concept of dimension has been dualized by several authors [1,3,4,5,6,7] in different not always equivalent ways. The treatment in [4,5] is based on dualizing the notion of independence associating thereby an integer, corank $M \geq 0$ with every $R$-module $M$ in which every coindicependent family of submodules is finite. To these two articles we add under mild restrictions a few theorems that correspond to familiar statements concerning artinian modules. The slightly modified definition of coindicependence given here permits dualizing word by word the theorems and proofs on dimension as stated in the original article [2].

With the definition of coindicependence of families of submodules of an $R$-module given in the previous section, the dual of the Goldie Dimension Theorem takes the following form:

**Theorem 2.1.** Let $M$ be an $R$-module in which every coindicependent family of submodules is finite. Then there exists an integer $n \geq 0$ with the following properties:

a) $M$ has a coindicependent family of submodules $U_1, \ldots, U_n$ such that $M/U_1, \ldots, M/U_n$ are hollow and $U_1 \cap \ldots \cap U_n \hookrightarrow M$.

b) Every coindicependent family of submodules of $M$ has at most $n$ elements.

c) A submodule $U$ of $M$ is superfluous in $M$ if and only if there exists a coindicependent family of submodules $U_1, \ldots, U_n$ of
such that \( M/U_1, \ldots, M/U_n \) are hollow and \( U \subseteq U_1 \cap \cdots \cap U_n \).

The uniquely determined integer \( n \) is called the corank of the \( R \)-module \( M \), and we write \( \text{corank} \ M = n \). If \( M \) has infinite co-independent families of submodules, we write \( \text{corank} \ M = \infty \).

The proof of Theorem 2.1 follows by duality from that of Theorem 1.07 in [2] and therefore will not be given. We point out, however, that it requires a repeated application of Lemmas 1.1, 1.2 and 1.3.

The following remarks are easy consequences of Theorem 2.1 and the definition of coindependence.

i) Isomorphic \( R \)-modules have equal coranks.

ii) \( \text{corank} \ M = 0 \) if and only if \( M = 0 \).

iii) \( \text{corank} \ M = 1 \) if and only if \( M \) is hollow.

iv) If \( N \) is a submodule of the \( R \)-module \( M \), then \( \text{corank} \ M/N \leq \text{corank} \ M \). Equality holds if \( N \hookrightarrow M \), and if \( \text{corank} \ M \) is finite, the converse holds.

v) If \( M \) is a direct sum of a family of submodules \( (N_i)_{i \in I} \) then \( \text{corank} \ M = \sum_i \text{corank} \ N_i \).

vi) Let \( M \) be an \( R \)-module of finite corank. If \( N_1, \ldots, N_r \) is a maximal coindependent family of maximal submodules of \( M \), then \( 0 \leq r \leq \text{corank} \ M \) and \( \text{rad} \ M = N_1 \cap \cdots \cap N_r \). By coindependence, \( M/\text{rad}M \) is isomorphic to the semisimple \( R \)-module \( (M/N_1) \oplus \cdots \oplus (M/N_r) \), and so the integer \( r \) is uniquely determined and depends only on \( M \). \( r \) is called the torsion corank of \( M \) and will be denoted by \( r(M) \). The equality \( r(M) = \text{corank} M \) holds if and only if \( \text{rad} M \hookrightarrow M \), and since \( M/\text{rad}M \) is semi-simple artinian, \( \text{rad} M \hookrightarrow M \) if and only if \( M \) is finitely generated. It follows that the corank of the \( R \)-module \( R \) is equal to the corank of the right \( R \)-module \( R \). For if either corank is finite, then it is equal to the length of the semi-simple ring \( R/\text{rad} R \). Hence it makes sense to define the corank of the ring \( R \) as the corank of the \( R \)-module \( R \). In case \( R \) is commutative and
corank $R < \infty$, it follows that corank $R$ is equal to the number of maximal ideals of $R$.

vii) An $R$-module $M$ of finite corank is weakly complemented in the sense that if $M$ is a sum $A_1 + A_2$ of two proper submodules $A_1$ and $A_2$, then there exists a submodule $B$ of $A_2$ such that $M = A_1 + B$ and $A_1 \cap B \hookrightarrow M$.

Artinian $R$-modules are obvious examples of modules of finite corank. The converse is not in general true. Take, for example, $R$ to be a non-artinian local ring. We show at the end of this article that $\mathbb{Z}$-modules of finite corank are necessarily artinian. However, the following result is true for any ring $R$.

**Theorem 2.2.** Let $M$ be an $R$-module. The following statements are equivalent:

a) $M$ is artinian;

b) corank $M$ is finite and every superfluous submodule of $M$ is artinian;

c) Every submodule of $M$ has finite corank and every non-zero subquotient module of $M$ has a simple submodule;

d) corank $M$ is finite and $M/U$ is finitely cogenerated for every superfluous submodule $U$ of $M$.

**Proof.** Certainly an artinian module satisfies (b), (c) and (d). We show that each of (b), (c) and (d) implies (a).

b) $\Rightarrow$ a) Suppose that corank $M = n$. By Theorem 2.1, there exists a coindependent family $U_1, \ldots, U_n$ of submodules of $M$ such that $M/U_1, \ldots, M/U_n$ are hollow and $U_1 \cap \ldots \cap U_n \hookrightarrow M$. By coindependency, there exists an exact sequence

$$0 \rightarrow U_1 \cap \ldots \cap U_n \rightarrow M \rightarrow (M/U_1) \oplus \ldots \oplus (M/U_n) \rightarrow 0$$

of $R$-module homomorphisms. Hence $(M/U_1) \oplus \ldots \oplus (M/U_n)$ is isomorphic to $M/(U_1 \cap \ldots \cap U_n)$. Since $U_1 \cap \ldots \cap U_n \hookrightarrow M$, it follows that every superfluous submodule of $(M/U_1) \oplus \ldots \oplus (M/U_n)$
is artinian. The same is true for the submodules $M/U_1, \ldots, M/U_n$ of $(M/U_1) \oplus \cdots \oplus (M/U_n)$, and since every $M/U$ is hollow, it follows that every $M/U_i$ is artinian. Since $U_1 \cap \cdots \cap U_n \hookrightarrow M$, it follows that $U_1 \cap \cdots \cap U_n$ is an artinian $R$-module. The exactness of the above sequence implies that $M$ is an artinian $R$-module.

c) $\Rightarrow$ a) Let $M/N$ be a non-zero quotient module of $M$ and prove that $M/N$ is finitely cogenerated. It is easily seen that every submodule and every quotient module of $M$ satisfies property (c). In particular, soc $M/N$ is a non-zero finitely generated submodule of $M/N$. If $K/N$ is a non-zero submodule of $M/N$, then soc $K/N \neq 0$, and so $K/N \cap$ soc $(M/N) \neq 0$. Thus soc $M/N$ is a finitely generated essential submodule of $M/N$, that is $M/N$ is finitely cogenerated.

d) $\Rightarrow$ a) Let $M/N$ be a non-zero quotient module of $M$. Since $M$ is weakly complemented, there exists a submodule $K$ of $M$ such that $M = K + N$ and $K \cap N \subseteq M$. By assumption $M/(K \cap N)$ is finitely cogenerated. Hence so is the submodule $K/(K \cap N)$. Since $M/N \cong K/(K \cap N)$, $M/N$ is also finitely cogenerated.

**Theorem 2.3.** Let $N$ be a submodule of an $R$-module $M$.

a) If $N$ and $M/N$ have finite coranks, then $M$ has finite corank.

b) corank $M \leq$ corank $M/N +$ corank $N$.

c) If $N$ is a minimal complement in $M$, then

$$\text{corank} M = \text{corank} M/N + \text{corank} N \quad \text{(*)}$$

If corank $M$ is finite then the converse is also true.

d) If corank $M$ is finite then (*) is true for every submodule $N$ of $M$ if and only if $M$ is semi-simple.

**Proof.** a) Suppose, to the contrary, that corank $M$ is not finite, and let $C_1, C_2, \ldots$ be a countably infinite coindependent sequence of submodules of $M$. By Lemma 1.1, the sequence

$$U_1 = C_1, U_2 = C_2 \cap C_3, \ldots, U_n = C_{t+1} \cap \cdots \cap C_{t+n}, \ldots, t = \frac{n(n-1)}{2}$$
of submodules of $M$ is coindependent. Since corank $M/N$ is finite, 
the sequence $(N + U_1), (N + U_2), \ldots$ of submodules of $M$ is not 
coindependent and so $N + U_n = M$ for almost all $n$. Choose $n$ 
so that $n > \text{corank } N$ and $N + U_n = M$. Then $M/U_n$ is an 
epimorphic image of $N$, and so corank $M/U_n \leq \text{corank } N$. On the other hand, 
$C_{t+1}/U_n, \ldots, C_{t+n}/U_n$ is a coindependent family of submodules of 
$M/U_n$ having $n$ elements, and so corank $M/U_n \geq n \geq \text{corank } N$. 
This contradiction shows that corank $M$ is finite.

b) If corank $M = \infty$, it follows from (a) that corank $N = \infty$ 
or corank $M/N = \infty$. So suppose corank $M$ is finite. Then $M$ is 
weakly complemented, and hence there exists a submodule $K$ of $M$ 
such that $M = N + K$ and $N \cap K \hookrightarrow M$. Hence 
\[
\text{corank } M = \text{corank } M/(N \cap K) = \text{corank } N/(N \cap K) + \text{corank } K/(N \cap K).
\]
Since $K/(N \cap K) \cong M/N$ and corank $N/(N \cap K) \leq \text{corank } N$, it 
follows that corank $M \leq \text{corank } M/N + \text{corank } M$.

c) If $N$ is a minimal complement in $M$, then $K$ in (b) may be 
chosen such that $N \cap K \hookrightarrow N$, and so corank $N = \text{corank } N/(N \cap \ K)$. Conversely, if corank $M$ is finite and corank $M = \text{corank } N + \text{corank } M/N$, then corank $N$ is finite, and therefore $N \cap K \hookrightarrow N$ so that $N$ is a minimal complement in $M$.

d) This follows from (c) since an $R$-module $M$ is semi-simple if 
and only if every submodule of $M$ is a minimal complement in $M$.

**Lemma 2.4.** Let $A_1$ and $A_2$ be submodules of an $R$-module $M$ 
such that $A_1 \subseteq A_2$. If $A_1$ is a minimal complement in $M$ and $A_2/A_1$ 
is a minimal complement in $M/A_1$, then $A_2$ is a minimal complement 
in $M$.

**Proof.** Let $B_1$ be a submodule of $M$ such that $A_1 + B_1 = M$ 
and $A_1 \cap B_1 \hookrightarrow A_1$ and $B_2/A_1$ a submodule of $M/A_1$ such that 
$A_2 + B_2 = M$ and $(A_2 \cap B_2)/A_1 \hookrightarrow A_2/A_1$. Then $M = A_2 + (B_1 \cap B_2)$ 
and it remains to prove that $A_2 \cap B_1 \cap B_2 \hookrightarrow A_2$. The natural 
isomorphisms 
\[
(A_2 \cap B_2)/A_1 \cong (A_2 \cap B_1 \cap B_2)/(A_1 \cap B_1)
\]
and 
\[
A_2/A_1 \cong (A_2 \cap B_1)/(A_1 \cap B_1)
\]
together with \((A_2 \cap B_2)/A_1 \hookrightarrow A_2/A_1\) imply that
\[
(A_2 \cap B_1 \cap B_2)/(A_1 \cap B_1) \hookrightarrow (A_2 \cap B_1)/(A_1 \cap B_1).
\]
Hence \((A_2 \cap B_1 \cap B_2)/(A_1 \cap B_1) \hookrightarrow A_2/(A_1 \cap B_1)\).
Since \((A_1 \cap B_1) \hookrightarrow A_1 \subseteq A_2\), it follows that \(A_2 \cap B_1 \cap B_2 \hookrightarrow A_2\).

**Theorem 2.5.** Let \(M\) be an \(R\)-module.

a) If corank \(M\) is finite, then \(M\) satisfies DCC on minimal complements.

b) If \(M\) is strongly complemented and satisfies DCC on minimal complements then \(M\) satisfies ACC on minimal complements.

c) If \(M\) is strongly complemented and satisfied ACC on minimal complements then corank \(M\) is finite.

**Proof.** a) Let \(T_1 \supseteq T_2 \supseteq \ldots\) be a strictly descending infinite sequence of minimal complements in \(M\). Then \(T_n\) is not superfluous in \(M\) and \(T_n/T_{n+1}\) is not superfluous in \(M/T_{n+1}\) for all \(n = 1, 2, \ldots\). It follows that there exists a sequence \(A_1, A_2, \ldots\) of proper submodules of \(M\) such that \(A_n \supseteq T_{n+1}\) and \(A_n + T_n = M\) for all \(n = 1, 2, \ldots\). Then for all \(n \geq 1\), we have \((A_1 \cap \ldots \cap A_{n-1}) + A_n \supseteq T_n + A_n = M\). It follows that \(M\) has an infinite coincidently family of submodules which is not so.

b) Let \(T_1 \subseteq T_2 \subseteq \ldots\) be an ascending sequence of minimal complements in \(M\). Let \(K_1\) be a minimal complement of \(T_1\) in \(M\). Then \(M = T_1 + K_1 = T_2 + K_1\), and so there exists a submodule \(K_2\) of \(K_1\) such that \(K_2\) is a minimal complement in \(M\). By induction, we obtain a descending sequence, \(K_1 \supseteq K_2 \supseteq \ldots\) of minimal complements in \(M\). By assumption there exists an integer \(n\) such that \(K_n = K_{n+1} = \ldots\). We prove that \(T_n = T_{n+1} = \ldots\). For this it suffices to prove that \((T_{n+1}/T_n) \hookrightarrow M/T_n\). So let \(A/T_n \subseteq M/T_n\) be such that \(T_{n+1}/T_n + A/T_n = M/T_n\). Then \(T_{n+1} + A = M = T_n + K_n\). From \(T_{n+1} \supseteq T_n\) we conclude that \(T_{n+1} = T_n + (K_n \cap T_{n+1})\) and so
\[
M = T_{n+1} + A = T_n + (K_n \cap T_{n+1}) + A = (K_n \cap T_{n+1}) + A.
\]
But
\[ K_n \cap T_{n+1} = K_{n+1} \cap T_{n+1} \rightarrow K_n \subseteq M. \]

It follows that \( A = M \).

c) Suppose, to the contrary, that \( \text{corank} \ M \) is not finite and let \( C_1, C_2, \ldots \) be a countably infinite coindpendent sequence of submodules of \( M \). Starting with \( A_0 = 0 \), we prove by induction the existence of a strictly ascending infinite sequence \( A_0 \subseteq A_1 \subseteq \ldots \) of minimal complements in \( M \) such that \( M/A_n \) has infinite corank for every \( n = 0, 1, \ldots \). This is clear when \( n = 0 \). The transition from the \( k \)-th step to the \((k + 1)\)-st step takes place as follows:

Let \( B_1/A_k, B_2/A_k, \ldots \) be a countably infinite coindpendent sequence of submodules of \( M/A_k \). Since \( M \) is strongly complemented, so is the quotient module \( M/A_k \) of \( M \) by the minimal complement \( A_k \). Hence \( M/A_k = B_1/A_k + B_2/A_k \) implies the existence of a minimal complement \( A_{k+1}/A_k \) of \( B_1/A_k \) in \( M/A_k \) such that \( A_{k+1}/A_k \subseteq B_2/A_k \). By coindependence of the sequence \( B_1/A_k, B_2/A_k, \ldots \), it follows that \( A_k \) is a proper submodule of \( A_{k+1} \), and \( A_{k+1} \) is a proper submodule of \( B_2 \). By Lemma 2.4, \( A_{k+1} \) is a minimal complement in \( M \) and it remains to verify that corank \( M/A_{k+1} \) is not finite. A straight-forward calculation shows that \( A_{k+1} + (B_1 \cap B_n) \neq M \) and

\[
(A_{k+1} + (B_1 \cap B_2)) \cap \ldots \cap (A_{k+1} + (B_1 \cap B_{n-1})) = (A_{k+1} + (B_1 \cap B_n)) = M
\]

for every \( n \geq 1 \), so that the infinite sequence \( B_2 = A_{k+1} + (B_1 \cap B_2), A_{k+1} + (B_1 \cap B_3), \ldots, A_{k+1} + (B_1 \cap B_n), \ldots \) of submodules \( M \) is coindependent. Hence corank \( M/A_{k+1} \) is not finite.

\textbf{Theorem 2.6} Let \( M \) be a complemented \( R \)-module. Then

a) \( \text{corank} \ M = n < \infty \) if and only if \( M \) is an irredundant sum, \( M = H_1 + \ldots + H_n \) and \( n \) hollow submodules \( H_1, \ldots, H_n \). If \( M = H'_1 + \ldots + H'_n \) is another irredundant sum of \( n \) hollow submodules \( H'_1 \ldots H'_n \) then for every integer \( m, 0 \leq m \leq n \), there exist \( n - m \) distinct indices \( i_1, \ldots, i_{n-m} \) such that

\[
M = H_1 + \ldots + H_m + H'_{i_1} + \ldots + H'_{i_{n-m}}.
\]
b) If $M = H_1 + \ldots + H_n$ is an irredundant sum of $n$ hollow submodules, then

i) $\text{rad } M = \text{rad} H_1 + \ldots + \text{rad} H_n,$

ii) $M/\text{rad} M$ is isomorphic to the semi-simple $R$-module $H_1/\text{rad} H_1 \oplus \ldots \oplus H_n/\text{rad} H_n$. The length $r(M)$ of $M/\text{rad} M$ is equal to the number of cyclic submodules among $H_1, \ldots, H_n$. If these are $H_1, \ldots, H_r$ and $N_i = H_1 + \ldots + H_{i-1} + T_i + H_{i+1} + \ldots + H_n$, where $T_i$ is the maximal submodule of $H_i$, $1 \leq i \leq r$, then every $N_i$ is a maximal submodule of $M$, the intersection $N_1 \cap \ldots \cap N_r$ is irredundant and is equal to $\text{rad } M$.

Proof. a) Assume $M = H_1 + \ldots + H_n$ is an irredundant sum of $n$ hollow submodules. The mapping

$$f : H_1 \oplus \ldots \oplus H_n \to M$$

given by

$$f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$$

is an epimorphism with kernel $K = K_1 \oplus \ldots \oplus K_n$ where $K_i = H_i \cap (H_1 + \ldots + H_{i-1} + H_{i+1} + \ldots + H_n)$, $i = 1, \ldots, n$. Since the sum $H_1 + \ldots + H_n$ is irredundant, $K_i \neq H_i$ and so $K_i \not\subseteq H_i$ for every $i = 1, \ldots, n$. It follows that $K \not\subseteq H_1 \oplus \ldots \oplus H_n$ and so corank $M = \text{corank}(H_1 \oplus \ldots \oplus H_n) = n$.

Conversely, assume that corank $M = n$. If $n = 1$, then $M$ is hollow. So assume $n > 1$, By lemma 1.3 there exists a submodule $U_1$ of $M$, such that $M/U_1$ is hollow. Let $H_1$ be a minimal complement of $U_1$ in $M$. By induction, pick at the $k$-th step in case $M/(H_1 + \ldots + H_{k-1})$ is not hollow, a submodule $U_k$ of $M$ containing $H_1 + \ldots + H_{k-1}$ such that $M/U_k$ is hollow and let $H_k$ be a minimal complement of $U_k$ in $M$. By Lemma 1.3, the family of submodules $U_1, \ldots, U_k$ is coinddependent, since

$$U_k + (U_1 \cap \ldots \cap U_{k-1}) \supseteq (H_1 + \ldots + H_{k-1}) + (U_1 \cap \ldots \cap U_{k-1}),$$

and by induction, this last sum is equal to $M$. Hence, by Theorem 2.1, this inductive process must stop for some $k \leq n$. Since corank
$M = n$, it follows from the result of the previous paragraph that the process stops when $k = n$. Thus $M/(H_1 + \ldots + H_{n-1})$ is hollow. If $H_n$ is a minimal complement of $H_1 + \ldots + H_{n-1}$ in $M$, then $H_n$ is hollow and the sum $M = H_1 + \ldots + H_n$ is irredundant.

Suppose now that $M = H'_1 + \ldots + H'_n$ is another irredundant sum of $n$ hollow submodules. Let $G_m = H_1 + \ldots + H_m$, $0 \leq m \leq n$. Then on the one hand $M/G_m = (G_m + H_{m+1})/G_m + \ldots + (G_m + H_n)/G_m$ is an irredundant sum of $n-m$ hollow submodules of $M/G_m$ implying thereby that corank $M/G_m = n-m$. On the other hand, $M/G_m = (G_m + H'_1)/G_m + \ldots + (G_m + H'_n)/G_m$ is a sum of $n$ zero or hollow submodules of $M/G_m$. It follows that $m$ of these submodules may be deleted to yield an irredundant sum of $n-m$ hollow submodules. Thus there exist distinct indices $i_1, \ldots, i_{n-m}$ such that $M = H_1 + \ldots + H_m + H'_{i_1} + \ldots + H'_{i_{n-m}}$.

b) Since the kernel $K$ of the epimorphism $f$ defined in (a) is superfluous in $H_1 \oplus \ldots \oplus H_n$, $f$ maps $\text{rad}(H_1 \oplus \ldots \oplus H_n) = \text{rad}H_1 \oplus \ldots \oplus \text{rad}H_n$ onto $\text{rad}M$. Hence $\text{rad} M = \text{rad}H_1 + \ldots + \text{rad}H_n$. This proves (i). To prove (ii), note that the kernel of the restriction of $f$ to $H_1 \oplus \ldots \oplus \text{rad}H_n$ is also equal to $K$ and so $f$ induces an isomorphism from $H_1/\text{rad}H_1 \oplus \ldots \oplus H_n/\text{rad}H_n$ onto $M/\text{rad}M$. $H_i/\text{rad}H_i = 0$ in case $H_i$ is not finitely generated and $H_i/\text{rad}H_i$ is a simple $R$-module in case $H_i$ is cyclic. This shows that the torsion corank $r(M)$ of $M$ is equal to the number of cyclic submodules among $H_1, \ldots H_n$. To prove the remaining part of (ii) we may assume that $r(M) > 0$. For, if $r(M) = 0$, then $\text{rad} M = \text{rad}H_1 + \ldots + \text{rad}H_n = H_1 + \ldots + H_n = M$. For every $i$, $1 \leq i \leq r$, the quotient module $M/N_i$ is an epimorphic image of the simple $R$-module $H_i/T_i$, and since $T_i$ is superfluous in $H_i$ and the sum $M = H_1 + \ldots + H_n$ irredundant, it follows that $N_i \neq M$. Hence $M/N_i$ is also a simple $R$-module and so $N_i$ is a maximal submodule of $M$. For the same reason, the intersection $N_1 \cap \ldots \cap N_r$ is irredundant or equivalently the family of maximal submodules $N_1, \ldots, N_r$ is coindependent. The uniqueness of $r(M)$ implies $\text{rad} M = N_1 \cap \ldots \cap N_r$.

The irredundant sum representation of a complemented $R$-module of finite corank is generally not direct. It is direct if the $R$-module $M$ is projective. In fact, if $P$ is any complemented projective $R$-module, then $\text{rad} P \subseteq P$, $P/\text{rad} P$ is semi-simple and the projection
map $P \to P/\text{rad } P$ lifts the direct sum decomposition of $P/\text{rad } P$
into simple submodules to a direct sum decomposition of $P$ into hol-
low submodules. Since the endomorphism ring of a hollow projective $R$-module is local, the direct sum decomposition of a complemented projective $R$-module into hollow submodules is unique in the sense of the Krull-Schmidt-Remak Theorem. Since a hollow projective $R$-
module is necessarily cyclic, it follows that a complemented projective $R$-module has finite corank if and only if it is finitely generated.

**Corollary.** If the ring $R$ is a complemented $R$-module, then

a) $\text{corank } R = n < \infty$

b) $R = Re_1 \oplus \ldots \oplus Re_n = e_1 R \oplus \ldots \oplus e_n R$, where $e_1, \ldots, e_n$
are orthogonal idempotents, $1 = e_1 + \ldots + e_n$ and every $Re_i(e_i R)$ is a hollow left (right) ideal of $R$.

c) If $R$ is also commutative, then every $Re_i$ is a local ring with
unit element $e_i$ and $R$ is ring isomorphic to the ring product
$Re_1 \times \ldots \times Re_n$.

d) The $R$-module $R$ is complemented if and only if the right
$R$-module $R$ is complemented.

The proof is clear. We only point out that the equality of the
coranks of the $R$-module $R$ and the right $R$-module $R$ implies that
the right ideals $e_1 R, \ldots, e_n R$ are hollow. (d) permits defining the
notion of a complemented ring without ambiguity. However, the
corollary implies that the notion of a complemented ring coincides
with the notion of semi-perfectness of the ring. The same corollary
gives another proof of the left-right symmetry of semi-perfectness.

A weak point of Theorem 2.6 is the assumption that the $R$-
module $M$ is complemented. Our next result shows that this as-
sumption is redundant in the case of abelian groups.

**Theorem 2.8.** Let $G$ be a non-zero abelian group

a) If $G$ is torsion free, then $\text{corank } G = \infty$. 
b) If corank $G = n < \infty$, then $G$ is a direct sum of $n$ hollow subgroups, and the direct summands are uniquely determined.

c) If corank $G = n < \infty$ then $G$ is complemented.

Proof. a) If $G$ is not reduced, then it contains a direct summand isomorphic to the group $\mathbb{Q}$ of rational numbers which is of infinite corank. So suppose $G$ is reduced, and let $P$ be the set of prime numbers $p$ for which $pG \neq G$. Since for any relatively prime numbers $r$ and $s$, $rG + sG = G$ and $rsG = rG \cap sG$, it follows that the family of subgroup $(pG)_{p \in P}$ is coindependent. Hence if $P$ is infinite, then corank $G = \infty$. If $P$ is finite, then $pG = G$ for an infinite number of primes. Let these be $p_1, p_2, \ldots$, and let $a$ be a non-zero element of $G$. Since $G$ is torsion free, division in $G$ by each of $p_1, p_2, \ldots$ is unique, and hence the elements $Za + p_1^{-1}a, Za + p_1^{-2}a, \ldots$ generate a direct summand of $G/Za$ isomorphic to the hollow group $\mathbb{Z}(p_1^{\infty})$ for every $i = 1, 2, \ldots$. It follows that corank $G/Za = \infty$ and hence corank $G = \infty$.

b) Let $T$ be the torsion part of $G$. Since $G/T$ is torsion free, it follows from (a) that $T = G$. Being a non-zero torsion group, $G$ has a direct summand isomorphic to the hollow group $\mathbb{Z}(p^k)$, $1 \leq k \leq \infty$. By induction $G$ is a direct sum of $n$ hollow subgroups. Being the primary components of $G$, these direct summands are uniquely determined.

c) Being a finite sum of hollow, hence complemented subgroups, $G$ is complemented.

Theorem 4.4 may be restated as follows:

A $\mathbb{Z}$-module has finite corank if and only if it is artinian.

REFERENCES


