Doubly Multipliable Matrices(*)

by P. Rubió and J. Gelonch (in Catalunya)(**)

Sommario. - Si studiano i sottospazi nei quali le matrici doppiantemente multiplicabili \((A, B)\) si scompongono secondo i loro divisorii elementari, e l'effetto di \(A\) e \(B\) come applicazioni tra essi. Nasce il concetto di coppia tra matrici doppiantemente multiplicabili e, sulla base di questo, si studia quando due matrici \(P\) e \(Q\) possono scomporvisi nella forma \(AB\) e \(BA\), rispettivamente, con una costruzione esplicita delle matrici \(A\) e \(B\). Infine, si studia l'equivalenza tra coppie di matrici doppiantemente multiplicabili.

Summary. - We study the subspaces into which the doubly multipliable matrices \((A, B)\) decompose according to their elementary divisors, and the effect of \(A\) and \(B\) as applications between them. There appears the concept of coupling between doubly multipliable matrices and on the basis of this we study when two matrices \(P\) and \(Q\) can decompose into the form \(AB\) and \(BA\) respectively, with an explicit construction of the matrices \(A\) and \(B\). Finally, we study the equivalence between doubly multipliable pairs of matrices.

Introduction.

Let \(A\) be a matrix \(m \times n\) and \(B\) of the order \(n \times m\) in \(\mathbb{C}\). We can consider \(A\) as a morphism of \(\mathbb{C}^n\) in \(\mathbb{C}^m\) and \(B\) of \(\mathbb{C}^m\) in \(\mathbb{C}^n\). It is clear that the products \(AB\) and \(BA\), which are endomorphisms of \(\mathbb{C}^m\) and \(\mathbb{C}^n\) respectively, have meaning. The article by Harley Flanders [2] contains a description of the relation between these products for two arbitrary matrices \(A\) and \(B\). In [9] we can find another demonstration of the theorem of Flanders, together with a necessary and sufficient condition for two matrices \(P\) and \(Q\) to be simultaneously decomposed as a product of two matrices \(A\) and \(B\). [1] makes a study of the behaviour of the matrices \(AB\) and \(BA\) on \(\mathbb{C}^m\) and \(\mathbb{C}^n\).

(*) Pervenuto in Redazione il 22 ottobre 1992.
(**) Indirizzo degli Autori: Universitat Politècnica de Catalunya (Spain).
Our aim is to further explore the results of [1], studying the effect of the matrices $A$ and $B$ as morphisms between $\mathbb{C}^n$ and $\mathbb{C}^m$, finding the subspaces in which these spaces decompose, and the effective construction of matrices $A$ and $B$ for pairs of matrices $P$ and $Q$, which meet the necessary conditions.

Some of the results which can already be found in the references quoted have been repeated following the line of work presented here, as we are interested in emphasizing from the outset the role of the matrices $A$ and $B$.

**Theorem 1.** The elementary divisors $(\lambda - \lambda_0)^p$ with $\lambda_0 \neq 0$ are the same in $AB$ as in $BA$. Moreover, $A$ and $B$ are isomorphisms between the subspaces generated by these divisors. (The first part can also be seen in [1], [2] and [9]).

**Proof.** Let $x \neq 0$ be a vector with minimum polynomial $(\lambda - \lambda_0)^p$. We have: $0 = B(AB - \lambda_0)^px = (BA - \lambda_0)^pBx \Rightarrow (\lambda - \lambda_0)^p$ is a polynomial which cancels the vector $Bx$ (with respect to $BA$).

Furthermore, if $(\lambda - \lambda_0)^{p-1}$ cancelled $Bx$, a parallel reasoning would indicate that $(\lambda - \lambda_0)^{p-1}$ cancels $ABx$, and therefore

$$0 = (AB - \lambda_0)^p x = (AB - \lambda_0)^{p-1}(AB - \lambda_0)x$$
$$= (AB - \lambda_0)^{p-1}ABx - (AB - \lambda_0)^{p-1}\lambda_0x$$
$$= -\lambda_0(AB - \lambda_0)^{p-1}x$$

and, as $\lambda_0 \neq 0$, we are left with $(AB - \lambda_0)^{p-1}x = 0$ !!!

Thus, the minimum polynomial of $Bx$ is $(\lambda - \lambda_0)^p$.

We construct the subspaces

$I_1 = \{x|\exists k \in \mathbb{N} \text{ con } (AB - \lambda_0)^k x = 0\} \subset \mathbb{C}^m$

$I_1' = \{x'|\exists k \in \mathbb{N} \text{ con } (BA - \lambda_0)^k x' = 0\} \subset \mathbb{C}^n$

We have just seen that $B(I_1) \subset I_1'$. Repeating the process symmetrically, $A(I_1') \subset I_1$.

Let us now see that $B$ is injective. For this, we take a basis of the subspace $I_1$, $\langle x_1, x_2, \ldots x_r \rangle = I_1$ and the vectors $y_i = Bx_i$ for $i = 1, \ldots, r$; it is a question of checking that the set of vectors $\{y_i\}$ is linearly independent. Indeed:

If $\Sigma \alpha_i y_i = 0$, then $B(\Sigma \alpha_i x_i) = 0$. Let us take $z = \Sigma \alpha_i x_i$ and let us suppose $z \neq 0$. It is obvious that $z \in I_1$, and therefore there
exists $k \in \mathbb{N}$ such that $(AB - \lambda_0)^k z = 0$ and $(AB - \lambda_0)^{k-1} z \neq 0$. As we have seen above, this means that $(BA - \lambda_0)^k Bz = 0$ and $(BA - \lambda_0)^{k-1} Bz \neq 0$, which is impossible, since $Bz = 0$. Therefore, $z = 0 \Rightarrow \alpha_i = 0 \forall i \Rightarrow \{y_i\}$ is a linearly independent set, as we wished to see. Thus, $B$ is injective.

Because $B$ is injective, $\dim I_i \leq \dim I'_i$.

Of course, $A$ is also injective: $\dim I'_1 \leq \dim I_1$. The conclusion is clear: $A$ and $B$ are isomorphisms between $I_1$ and $I'_1$.

Given another eigenvalue $\lambda_1 \neq 0$, we find that $A$ and $B$ are isomorphisms between the respective subspaces $I_2$ and $I'_2$. But as $I_1 \cap I_2 = I'_1 \cap I'_2 = \{0\}$, $A$ and $B$ are isomorphisms between $I_1 \oplus I_2$ and $I'_1 \oplus I'_2$.

Repeating the procedure, it will result that $A$ and $B$ are isomorphisms between two subspaces $I$ and $I'$ with minimum polynomial

$$(\lambda - \lambda_0)^{p_0}(\lambda - \lambda_1)^{p_1} \cdots (\lambda - \lambda_h)^{p_h}$$

the minimum common multiple of all the elementary divisors with a non-nil root.

We will thus have $\mathbb{C}^m = I \oplus F$, $\mathbb{C}^n = I' \oplus F'$ with $I \simeq I'$, $F$ and $F'$ being the subspaces corresponding to the vectors with minimum polynomial $\lambda^k$.

By restriction of $A$ and $B$ to the subspaces $I$ and $I'$ ($A : I' \rightarrow I$, $B : I \rightarrow I'$), we construct the automorphisms $AB : I \rightarrow I$ and $BA : I' \rightarrow I'$ which are similar: $AB = B^{-1}(BA)B$, and which therefore have the same elementary divisors. \qed

**Proposition 1.** If $E_k \subset \mathbb{C}^m$ is a cyclical subspace with minimum polynomial $\lambda^k$, then $B(E_k)$ is cyclical with minimum polynomial $\lambda^k$ or $\lambda^{k-1}$.

(The proof may also be found in [1]. We present it here because it is performed with a different approach in the line of this work).

**Proof.** Let $e, (AB)e, \ldots, (AB)^{k-1} e$ be a basis of $E_k$. Its image by $B$ generates $B(E_k) : \langle Be, (BA)Be, \ldots, (BA)^{k-1} Be \rangle = B(E_k)$. The image of these vectors by $A$ is $(AB)e, (AB)^2 e, \ldots, (AB)^{k-1} e, (AB)^k e = 0$. It is thus deduced that at least the vectors $Be$, $(BA)Be, \ldots, (BA)^{k-2} Be$ are independent.

As for the last vector, $(BA)^{k-1} Be$, two things may happen:
1. \((BA)^{k-1}Be = 0\)

In this case \(B(E_k)\) is cyclical, generated by \(Be\) and with minimum polynomial \(\lambda^{k-1}\).

2. \((BA)^{k-1}Be \neq 0\)

Let us see, by reductio ad absurdum, that it cannot be a linear combination of the above.

If it were \((BA)^{k-1}Be = \alpha_1 Be + \alpha_2 (BA)Be + \ldots + \alpha_{k-1} (BA)^{k-2}Be\), we would be left with:

\[
B[\alpha_1 e + \alpha_2 (AB)e + \ldots + \alpha_{k-1} (AB)^{k-2}e - (AB)^{k-1}e] = 0
\]

Applying \(A\) to this equality,

\[
\alpha_1 (AB)e + \alpha_2 (AB)^2e + \ldots + \alpha_{k-1} (AB)^{k-1}e - 0 = 0 \Rightarrow \\
\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_{k-1} = 0
\]

remembering the linear independence of these vectors.

Thus, \((BA)^{k-1}Be = 0 \) !!!

We thus have that \(Be\), \((BA)Be\), \ldots, \((BA)^{k-1}Be\) form a basis of \(B(E_k)\). Moreover, \((BA)^k Be = B(AB)^k e = 0\), which shows that the minimum polynomial of \(B(E_k)\) is \(\lambda^k\).

\[\Diamond\]

**Lemma 1.** If \(E_k \subset \mathbb{C}^m\) is a cyclical subspace of annihilating polynomial \(\lambda^k\) and \(E'\) is an \(AB\)-invariant subspace such that \(E_k \cap E' = \{0\}\) and \(B(E_k) \cap B(E') \neq \{0\}\), then the annihilating polynomial of \(B(E_k)\) is \(\lambda^k\).

**Proof.** Let \(x \in B(E_k) \cap B(E')\), \(x \neq 0\). As \(x \in B(E_k)\), there exists a vector \(y \in E_k\) such that \(x = By\).

Let \(e\), \((AB)e\), \ldots, \((AB)^{k-1}e\) be a basis of \(E_k\). We have that

\[
y = \sum_{i=0}^{k-1} \alpha_i (AB)^i e
\]

and therefore,

\[
x = \sum_{i=0}^{k-1} \alpha_i (BA)^i Be \Rightarrow Ax = \sum_{i=0}^{k-1} \alpha_i (AB)^{i+1} e = \sum_{i=0}^{k-2} \alpha_i (AB)^{i+1} e
\]
since \((AB)^k e = 0\). Furthermore, \(x \in B(E') \Rightarrow Ax \in AB(E') \subset E'\), ad \(E'\) is \(AB\)-invariant.

We are left with \(Ax \in E_k \cap E' \Rightarrow Ax = 0 \Rightarrow \alpha_i = 0\) for \(i = 0, \ldots, k - 2 \Rightarrow x = \alpha_{k-1}(BA)^{k-1}Be.\) We know that \(x \neq 0\), which gives us \((BA)^{k-1}Be \neq 0\). We are in the second case of the above proposition.

\[\Diamond\]

**Proposition 2.** Let \(E_0\) be the subspace of the vectors annulled by some power of \(AB\). We can obtain a decomposition of \(E_0\) in a direct sum of irreducible subspaces \(E_0 = \oplus E_i\), such that \(B(E_0) = \oplus B(E_i)\).

**Proof.** Let \(E_0 = E_1 \oplus E_2 \oplus \ldots \oplus E_p\), the \(E_i\) being the subspaces given by the elementary divisors \(\lambda^k\) in decreasing order of degrees. Let us take as seen that \(B(E_1 \oplus \ldots \oplus E_i) = B(E_1) \oplus \ldots \oplus B(E_i)\) for \(i \geq 1\). Let \(\lambda^k\) be the minimum polynomial of the subspace \(E_{i+1}\). Let us suppose that \(B(E_{i+1})\) does not form a direct sum with \(B(E_1 \oplus \ldots \oplus E_i)\). There then exists \(x \in B(E_{i+1}) \cap B(E_1 \oplus \ldots \oplus E_i), x \neq 0.\) Taking \(E' = E_1 \oplus \ldots \oplus E_i\), and \(e\) the generating vector of \(E_{i+1}\), we can take, in Lemma 1, \(x = (BA)^{k-1}Be.\)

Furthermore, \(x \in B(E') \Rightarrow x = By\) with \(y = y_1 + \ldots + y_i\), where each \(y_j \in E_j.\)

It is obvious that \(Ax = (AB)^k e = 0 \Rightarrow ABy = 0 \Rightarrow ABy_1 + \ldots + ABy_i = 0\), being \(ABy_j \in E_j\) (as they are invariants) \(\Rightarrow ABy_j = 0.\)

Let \(\lambda e_j\) and \(e_j\) be the minimum polynomial and the generating vector of the subspace \(E_j\). As \(y_j \in E_j\), it is clear that

\[y_j = \sum_{h=0}^{k_j-1} \alpha_h (AB)^h e_j\]

From \(ABy_j = 0\) it is deduced that

\[\sum_{s=0}^{k_j-2} \alpha_s (AB)^{s+1} e_j = 0 \Rightarrow \alpha_0 = \ldots = \alpha_{k_j-2} = 0 \Rightarrow y_j = \alpha_{k_j-1} (AB)^{k_j-1} e_j\]

Due to the ordering taken in the subspaces \(E_r\), we know that \(k_j \geq k.\)

We will put
\[ y_j = \alpha_{k_j-1}(AB)^{k_j-1}(AB)^{k_j-k}e_j \]

\[ \tilde{y}_j = \alpha_{k_j-1}(AB)^{k_j-1}e_j (\in E_j) \]

With this, \( y = y_1 + \ldots + y_i = (AB)^{k-1}(\tilde{y}_1 + \ldots + \tilde{y}_i) \). We define \( \tilde{y} = \tilde{y}_1 + \ldots + \tilde{y}_i \in E_1 \oplus \ldots \oplus E_i \) and \( \tilde{e} = e - \tilde{y} \). We also take \( \tilde{E}_{i+1} = (\tilde{e}, (AB)\tilde{e}, \ldots, (AB)^{k-1}\tilde{e}) \).

It is clear that \( E_1 \oplus \ldots \oplus E_i \oplus E_{i+1} = E_1 \oplus \ldots \oplus E_i \oplus E_{i+1} \), since the vectors \( \tilde{e}, (AB)\tilde{e}, \ldots, (AB)^{k-1}\tilde{e} \) are still independent (if \( \Sigma \alpha_s(AB)^s \tilde{e} = 0, \Rightarrow \Sigma \alpha_s (AB)^s e - \Sigma \alpha_s(AB)^s \tilde{y} = 0 \), the first addend being a vector of \( E_{i+1} \) and the second one of \( E_1 \oplus \ldots \oplus E_i \); this means that each of them must be nil, and therefore \( \alpha_s = 0 \ \forall s \).

Thus \( B(\tilde{E}_{i+1}) = (B\tilde{e}, (BA)B\tilde{e}, \ldots, (BA)^{k-1}B\tilde{e}) \).

But \( (BA)^{k-1}B\tilde{e} = (BA)^{k-1}Be - (BA)^{k-1}B\tilde{y} = x - B(AB)^{k-1}\tilde{y} = x - By = x - x = 0 \).

With this, the minimum polynomial of \( B(\tilde{E}_{i+1}) \) is \( \lambda^{k-1} \). According to Lemma 1, this tells us that \( B(\tilde{E}_{i+1}) \cap B(E_1 \oplus \ldots \oplus E_i) = \{0\} \), and therefore,

\[ B(E_1 \oplus \ldots \oplus E_i \oplus \tilde{E}_{i+1}) = B(E_1) \oplus \ldots \oplus B(E_i) \oplus B(\tilde{E}_{i+1}) \]

Repeating the process, we will obtain the stated decomposition.

\[ \Box \]

Consequences of these results.

With all these results, we can take

\[ \mathbb{C}^m \simeq E \oplus E_0 = E \oplus E_1 \oplus \ldots \oplus E_p \text{ and } \mathbb{C}^n \simeq E' \oplus E'_0 \]

such that if the minimum polynomial of \( \mathbb{C}^m \) with respect to \( AB \) is \( \lambda^kp(\lambda) \) and that of \( \mathbb{C}^n \) is \( \lambda^{k'}q(\lambda) \) with respect to \( BA \), it will be \( E = \text{Ker } p(\lambda), E' = \text{Ker } q(\lambda), p(\lambda) = q(\lambda) \) and \( A, B \) being isomorphisms between \( E \) and \( E' \), according to Theorem 1.

Proposition 1 assures us that \( k' \) has the same value as \( k - 1, k \) or \( k+1 \). Indeed: If in the minimum polynomial there appears the factor \( \lambda^{k'} \), it is because there is some vector which has \( \lambda^{k'} \) as a minimum polynomial. Let \( I \) be the subspace generated by it. This means that (Prop. 1) the minimum polynomial of subspace \( A(I) \subset \mathbb{C}^m \) is \( \lambda^{k'} \) or \( \lambda^{k'-1} \), and therefore \( k' \leq k \) or \( k' - 1 \leq k \). Symmetrically, it can be
deduced that \( k \leq k' \) or \( k - 1 \leq k' \). In any case, we can state that \( k - 1 \leq k' \leq k + 1 \).

The first part of the proof of Theorem 1 tells us, besides, that \( B(E_0) \subset E'_0 \).

**Proposition 3.** Each subspace \( B(E_i) \) is a maximal cyclic or is contained in a maximal cyclic of one degree more.

**Proof.** Let \( E_i = \langle e, (AB) e, \ldots, (AB)^{k-1} e \rangle \). The vector \( Be \) is the generator of \( B(E_i) \). Let us remember that it is possible that \( (BA)^{k-1} Be = 0 \). If \( B(E_i) \) is maximal, there remains nothing to say. Let us suppose, then, that it is not: \( \exists F \) cyclic such that \( B(E_i) \subset F \). As we know, it is possible to find a generator vector in \( F \), \( e' \), such that

\[
F = \langle e', (BA)e', \ldots, (BA)^r e', Be, \ldots, (BA)^{k-1} Be \rangle
\]

with of course \( Be = (BA)^r e' \).

With this,

\[
A(F) = \langle Ae', (AB) Ae', \ldots, (AB)^r Ae', (AB)e, \ldots, (AB)^{k-1} e \rangle
\]

Let \( \bar{e} = (BA)^r e' \). We know that \( (BA)\bar{e} = Be \). Moreover,

of \( A\bar{e} \in E_0 = E_1 \oplus \ldots \oplus E_i \oplus \ldots \oplus E_p \) is \( A\bar{e} = z_1 + \ldots + z_i + \ldots + z_p \) and then

\[
(BA)\bar{e} = Be = Bz_1 + \ldots + Bz_i + \ldots + Bz_p \in B(E_1) \oplus \ldots \oplus B(E_p)
\]

and necessary \( Bz_j = 0 \) \( \forall j = 1, \ldots, p, j \neq i, Be = Bz_i \).

Furthermore,

\[
z_i \in E_i \Rightarrow z_i = \sum_{s=0}^{k-1} \gamma_s (AB)^s e \Rightarrow Bz_i = \sum_{s=0}^{k-1} \gamma_s (BA)^s Be \text{ and then}
\]

\[
\sum_{s=0}^{k-1} \gamma_s (BA)^s Be = Be \Rightarrow \gamma_0 = 1, \quad \gamma_1 = \ldots = \gamma_{k-2} = 0
\]

We cannot state anything on the coefficient \( \gamma_{k-1} \) since it is possible that \( (BA)^{k-1} Be = 0 \). With this

\[
z_i = e + \gamma (AB)^{k-1} e
\]
Let \( \tilde{E}_i \) be the subspace engendered by \( A\tilde{e} \). We will see that \( E_i \) can be replaced with \( \tilde{E}_i \) for all purposes. Firstly, do we still find \( E_0 = E_1 \oplus \ldots \oplus \tilde{E}_i \oplus \ldots \oplus E_p \)?

Let us take \( y_j \in E_j, j = 1, \ldots, p, j \neq i \) and \( \tilde{y}_i \in \tilde{E}_i \) such that \( y_1 + \ldots + \tilde{y}_i + \ldots + y_p = 0 \). We must see that all are nil.

As \( \tilde{E}_i = \langle A\tilde{e}, (AB)A\tilde{e}, \ldots, (AB)^{k-1}A\tilde{e} \rangle \), we have:

\[
\tilde{y}_i = \alpha A\tilde{e} + y_i \text{ with } y_i = \sum_{s=1}^{k-1} \delta_s(AB)^s e
\]

for \( (AB)^s A\tilde{e} = A(BA)^{s-1}BA\tilde{e} = (AB)^s e \ s = 1, \ldots, k - 1 \) then \( \tilde{y}_i = \alpha(z_1 + \ldots + z_i + \ldots + z_p) + \sum_{s=1}^{k-1} \delta_s(AB)^s e = \alpha z_1 + \ldots + \alpha e + \alpha \gamma(AB)^{k-1} e + \sum_{s=1}^{k-1} \delta_s(AB)^s e + \ldots + \alpha z_p \) and is 
\[
0 = y_1 + \ldots + \tilde{y}_i + \ldots + y_p = (y_1 + \alpha z_1) + \ldots + (\alpha e + \alpha \gamma(AB)^{k-1} e + \sum_{s=1}^{k-1} \delta_s(AB)^s e) + \ldots + y_p + \alpha z_p \in E_1 \oplus \ldots \oplus E_i \oplus \ldots \oplus E_p \text{ of where } \\
y_j + \alpha z_j = 0, j = 1, \ldots, p, j \neq i, \\
\alpha e + \alpha \gamma(AB)^{k-1} e + \sum_{s=1}^{k-1} \delta_s(AB)^s e = 0 .
\]

From the second equality it is deduced that \( \alpha = 0 \) (it is the only coefficient of the vector \( e \)) and therefore \( y_i = 0 \) and \( \tilde{y}_i = 0 \) and also \( y_j = 0 \ \forall j \neq i \).

The sum is therefore still direct.

The vector \( A\tilde{e} \) has the same minimum polynomial as \( e \), since all the vectors \( z_j, j \neq i \) are cancelled with \( AB \) (let us remember that \( Bz_j = 0 \)). Thus, the subspace \( \tilde{E}_i \) corresponds to the same elementary divisor as \( E_i \). Moreover, \( B(\tilde{E}_i) = B(E_i) \).

Let us redefine \( E_i = \tilde{E}_i \) and \( e = A\tilde{e} \). With this,

\[
A(F) = \langle A\tilde{e}', \ldots, (AB)^{r-1}A\tilde{e}', e, (AB)e, \ldots, (AB)^{k-1}e \rangle \text{ and } E_i \subset A(F)
\]

As it is easy to check, each subspace \( E_i \) is maximal within \( E_0 \). Therefore, the vectors \( A\tilde{e}', \ldots, (AB)^{r-1}A\tilde{e}' \) cannot exist. In other words, the subspace \( F \) is necessarily of the form

\[
F = \langle \tilde{e}, Be, \ldots, (BA)^{k-1}Be \rangle
\]

one degree superior to that of \( B(E_i) \). Moreover, \( A(F) = E_i \). \( \diamond \)
THEOREM 2. There are decompositions in the direct sum of irreducibles of the subspaces \( E_0 = E_1 \oplus E_2 \oplus \ldots \oplus E_p \) and \( E'_0 = E'_1 \oplus E'_2 \oplus \ldots \oplus E'_q \) such that \( B(E_j) \subset E'_j \), \( A(E'_j) \subset E_j \) for \( j = 1, \ldots r \) \((r \leq \min(p, q))\), \( A \) or \( B \) being a monomorphism, the other with a kernel of dimension 1 and the dimensions of \( E_j \) and \( E'_j \) equal or different by one; furthermore, \( \forall i > r \) we have that \( \dim E_i = 1 \), \( B(E_i) = \{0\} \) and \( \dim E'_i = 1 \), \( A(E'_i) = \{0\} \).

Proof. Let \( E_1, \ldots, E_r \) be the irreducibles of the decomposition of \( E_0 \) with \( B(E_j) \neq \{0\} \). According to proposition 3, the subspace \( B(E_j) \) is maximal or is contained in a maximal cyclic of a dimension one more than \( B(E_j) \). Let \( E'_j \) be the maximal subspaces such that \( B(E_j) \subset E'_j \). For a given subspace \( E_j \), let \( e \) be its generating vector:

\[
E_j = \langle e, (AB)e, \ldots, (AB)^{k-1}e \rangle
\]

Then,

\[
B(E_j) = \langle Be(BA)Be, \ldots, (BA)^{k-2}Be, (BA)^{k-1}Be \rangle
\]

where the last vector may be nil.

If \( E'_j = B(E_j) \), then \( A(E'_j) = \langle (AB)e, \ldots, (AB)^{k-1}e \rangle \subset E_j \). In this case, if \( (BA)^{k-1}Be \neq 0 \), \( B \) is a monomorphism (in fact it is an isomorphism) and we will have \( \text{Ker} (A) = \langle (BA)^{k-1}Be \rangle \), and thus \( \dim \text{Ker} (A) = 1 \); furthermore, \( \dim E_j = \dim E'_j \). If \( (BA)^{k-1}Be = 0 \), then \( A \) is a monomorphism and \( \text{Ker} (B) = \langle (AB)^{k-1}e \rangle \), since \( \dim E_j = \dim E'_j + 1 \).

If \( E'_j \neq B(E_j) \), is \( E'_j = \langle \bar{e}, Be, \ldots, (BA)^{k-2}Be, (BA)^{k-1}Be \rangle \), with \( A\bar{e} = e \). Then \( A(E'_j) = E_j \).

If \( (BA)^{k-1}Be \neq 0 \), \( B \) is a monomorphism and \( \text{Ker} (A) = \langle (BA)^{k-1}Be \rangle \), since \( \dim E'_j = \dim E_j + 1 \).

If \( (BA)^{k-1}Be = 0 \), \( A \) is a monomorphism (in fact it is an isomorphism) and we have \( \text{Ker} (B) = \langle (AB)^{k-1}e \rangle \), since \( \dim E'_j = \dim E_j \).

We still have to see that the subspaces \( E'_j \) form a direct sum. Let us take \( y'_j \in E'_j \) such that \( \Sigma y'_j = 0 \). From this, \( \Sigma Ay'_j = 0 \). As \( Ay'_j \in E_j \forall j \rightarrow Ay'_j = 0 \forall j \), since the subspaces \( E_j \), form a direct sum. It may happen that \( y'_j \in B(E_j) \) or that \( y'_j \notin B(E_j) \) (if
\((B(E_j) \neq E_j')\). Let us analyze this second case: \(B(E_j) \neq E_j'\).

\[ y_j' \in E_j' \implies y_j' = \alpha_0 e + \sum_{s=0}^{k-1} \alpha_{s+1}(BA)^s Be \implies \]

\[ A y_j' = \alpha_0 A e + \sum_{s=0}^{k-2} \alpha_{s+1}(AB)^{s+1} e = \sum_{s=0}^{k-1} \alpha_s (AB)^s e \]

since \(Ae = e\). We know that \(Ay_j' = 0\) and this means that \(\alpha_s = 0\)
\(\forall s \leq k - 1\), so \(y_j' = \alpha_k (BA)^{k-1} Be \in B(E_j)\).

In any case, then \(y_j' \in B(E_j)\). But the subspaces \(B(E_j)\) form a
direct sum, and then the \(E_j'\) do too.

Let us now take the \(E_i\) such that \(B(E_i) = \{0\}\). It is evident
that \(\dim E_i = 1\), according to proposition 1. Let \(E_0''\) be an invariant
subspace for \(BA\), supplementary to \(E_1' \oplus \ldots \oplus E_{r'}'\), within \(E_0'\). For
all \(y_0' \in E_0''\) we have:

\[ Ay_0' \in E_0 \implies Ay_0' = y_1 + \ldots + y_{r'} + y_{r'+1} + \ldots + y_p \text{ with } y_j \in E_j \]

As \(B(y_{r'+1}) = \ldots = B(y_p) = 0\), since \(B(E_j) = \{0\}\) for \(j > r'\), we have

\[ BA y_0' = B y_1 + \ldots + B(y_{r'}) \in (E_1' \oplus \ldots \oplus E_{r'}') \cap E_0'' \implies \]

\[ BA y_0' = 0, \quad By_j = 0 \quad j = 1, \ldots, r'. \]

From the first equality we deduce that \(E_0''\) has \(\lambda\) as its annihilating
polynomial. From the others, we conclude the existence of \(y_j' \in E_j'\)
such that \(Ay_j' = y_j\) (in the first part of the proof it has already been
seen with the roles of \(A\) and \(B\) inverted). We can put

\[ Ay_0' = A(\ldots + y_{r'}) + y_{r'+1} + \ldots + y_p \]

It is clear that the subspace \(E_0''\) decomposes into grade one cyclics:

\[ E_0'' = \langle v_1 \rangle \oplus \ldots \oplus \langle v_q \rangle \]

For each vector \(v_i\) we obtain the vectors \(y_j'_{v_i}\), as has been done for
the vector \(y_0'\), and we define \(w_i = v_i - (y_1'_{v_i} + \ldots + y_{r'}'_{v_i})\). The subspace
\(F = \langle w_1 \rangle \oplus \ldots \oplus \langle w_q \rangle\) is still supplementary to \(E_j' \oplus \ldots \oplus E_{r'}'\), and
also \(Aw_i \in E_{r'+1} \oplus \ldots \oplus E_p\), since

\[ Aw_i = Av_i - A(\ldots + y_{r'}_{v_i}) = y_{(r'+1)}_{v_i} + \ldots + y_{p_i} \]
It is clear that \( A(F) = \langle Aw_1, \ldots, Aw_q \rangle = \langle Aw_1, \ldots, Aw_r'' \rangle \), eliminating those that are nil and those that are a linear combination of the other ones (if there are any). We define \( E_{r'+1} = \langle Aw_1, \ldots, E_{r'+r''} = E_r = \langle Aw_r'' \rangle \). All of them are grade 1 cyclic subspaces, since \( (AB)Aw_i = A(BA)w_i = A(0) = 0 \). We will put \( e_{r'+i} = Aw_i \).

With this we have that \( E_{r'+1} = \langle e_{r'+1} \rangle, E_{r'+i}' = \langle w_i \rangle \), all of dimension 1, with \( B(E_i) = \{0\} \) and \( A(E_i') = E_i \). The subspaces \( E_1, \ldots E_r \) and \( E_1', \ldots, E_r' \) fulfill the stated proposal.

We complete the decomposition into irreducibles of \( E_0 \):

\[
E_0 = E_1 \oplus \ldots \oplus E_r \oplus E_{r+1} \oplus \ldots \oplus E_p
\]

with \( \dim E_{r+1} = \ldots = \dim E_p = 1 \) and \( B(E_{r+1} = \ldots = B(E_p) = \{0\} \).

In \( E_0'' \) we have \( E_0'' = E_{r'+1} \oplus \ldots \oplus \langle w_{r+1} \rangle \oplus \ldots \oplus E_r \oplus \langle w_q \rangle \), since

\[
Aw_{r+i} = \sum_{j=1}^{r''} \alpha_{ji}Aw_j
\]

We will take

\[
e_{r'+1}'' = w_{r+i} - \sum_{j=1}^{r''} \alpha_{ji}w_j
\]

The subspaces \( \langle e_{r'+i}'' \rangle \) may replace the \( \langle w_{r+i} \rangle \) in the direct sum, and of course \( A(e_{r'+i}'') = 0 \).

We thus obtain the decomposition of the statement, taking

\[
E_{r+1}' = \langle e_{r'+i}'' \rangle.
\]

\[\diamond\]

### Couplings.

Let \( P \) and \( Q \) be square matrices of the respective orders \( m \) and \( n \). If there exist matrices \( A \) (of the order \( m \times n \)) and \( B \) (of the order \( n \times m \)) such that \( P = AB \) and \( Q = BA \), Theorem 2 ensures a decomposition

\[
C^m = E \oplus E_1 \oplus \ldots \oplus E_p
\]

\[
C^n = E' \oplus E_1' \oplus \ldots \oplus E_q'
\]
where $E \simeq E'$ with minimum polynomial

$$(\lambda - \lambda_0)^{p_0}(\lambda - \lambda_1)^{p_1} \ldots (\lambda - \lambda_k)^{p_k} \ (\lambda_i \neq 0 \ \forall i)$$

and the subspaces $E_i, E'_j$ are irreducible cyclics with minimum polynomial $\lambda^k$, giving strong relations between the two decompositions, characterized through $A$ and $B$.

We will see that these relations are sufficient to ensure that, given $P$ and $Q$, $\exists A$ and $B$ such that $P = AB$ and $Q = BA$.

We will say that two square matrices $P$ and $Q$ of the respective orders $m$ and $n$ are *couplable* if there exist decompositions of $\mathbb{C}^m$ and $\mathbb{C}^n$,

$$\mathbb{C}^m = E \oplus E_1 \oplus \ldots \oplus E_p$$

$$\mathbb{C}^n = E' \oplus E'_1 \oplus \ldots \oplus E'_q$$

such that

a) in $E$ and $E'$ the matrices $P$ and $Q$ have the same elementary divisors

$$(\lambda - \lambda_i)^{p_i} \ \lambda_i \neq 0$$

b) the minimum polynomials of the subspaces $E_i$ and $E'_j$ are $\lambda^k$

c) $\exists r \leq \min(p, q)$ such that:

- $|\dim E_i - \dim E'_i| \leq 1 \ \forall i \leq r$.
- $\dim E_j = \dim E'_k = 1 \ \forall j > r, k > r$.

Given two couplable matrices $P$ and $Q$, we will call any arrangement of the subspaces $E_i$ and $E'_j$ which fulfills the above conditions, indicating one of the subspaces $E_j$ or $E'_j$ as principal if $\dim E_j = \dim E'_j$ (this will later involve choosing the meaning of a certain monomorphism), a $(P, Q)$-coupling.

We will call each of the pairs of subspaces which are related within a $(P, Q)$-coupling a *coupled pair*. We will also consider as coupled pairs those formed by $E_{r+1} \oplus \ldots \oplus E_p$ with $\{0\}$ and $\{0\}$ with $E'_{r+1} \oplus \ldots \oplus E'_q$, if they exist, as well as the pair $(E, E')$.

Let us see that if $P$ and $Q$ are couplable, there exist $A$ and $B$ (of the respective orders $m \times n$ and $n \times m$) such that $P = AB$ and $Q = BA$. Furthermore, $A$ and $B$ will be isomorphisms between $E$ and $E'$ and the conditions laid out in Theorem 2 will be fulfilled.
When \( \text{dim } E_j = \text{dim } E_j' \) we will make the monomorphism be \( A \) if \( E_j' \) has been indicated as principal or \( B \) if the principal subspace is \( E_j. \) Of course, the possible plurality of \((P, Q)\)-couplings will provide different pairs of matrices \((A, B)\) such that \( P = AB \) and \( Q = BA. \) The existence of the matrices \( A \) and \( B \) will be checked by means of their effective construction.

With \( P \) and \( Q \) reduced to the Jordan form, arranging the blocks according to Theorems 1 and 2, and considering the correspondence between divisors given by the \((P, Q)\)-coupling chosen, we will have:

1) Coupled pairs between the subspaces of diagonal blocks in \( P \) and \( Q \) with annuler \((\lambda - \lambda_0)^p\) for \( \lambda_0 \neq 0:\)

\[
\begin{pmatrix}
\lambda_0 & 0 & \ldots & 0 & 0 \\
1 & \lambda_0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \lambda_0 & 0 \\
0 & 0 & \ldots & 1 & \lambda_0 \\
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
\lambda_0 & 0 & \ldots & 0 & 0 \\
1 & \lambda_0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \lambda_0 & 0 \\
0 & 0 & \ldots & 1 & \lambda_0 \\
\end{pmatrix}
\]

Let \( e_1, \ldots, e_p \) be the basis of the first subspace and \( v_1, \ldots, v_p \) of the second.

We will take \( B \) defined by

\[
B(e_1) = v_1, \ldots, B(e_p) = v_p
\]

As \( AB = P, (AB)e_i = \lambda_0 e_i + e_{i+1} \) for \( i = 1, \ldots, p - 1 \) and \( (AB)e_p = \lambda_0 e_p, \) as indicated by the matrix \( P, \) it is clear that it must be

\[
\begin{align*}
A(v_1) &= AB e_1 = \lambda_0 e_1 + e_2 \\
A(v_2) &= \lambda_0 e_2 + e_3 \\
& \vdots \\
A(v_{p-1}) &= \lambda_0 e_{p-1} + e_p \\
A(v_p) &= \lambda_0 e_p
\end{align*}
\]

Let us observe that \((BA)v_i = B(\lambda_0 e_i + e_{i+1}) = \lambda_0 v_i + v_{i+1}\) for \( i = 1, \ldots, p - 1 \) and \((BA)v_p = B(\lambda_0 e_p) = \lambda_0 v_p,\) fully in agreement with the corresponding form of the matrix \( Q.\)
2) Coupled pairs of the type

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

with the first matrix of the order \( p \) and the second of the order \( p - 1 \).

In these coupled pairs \( A \) is always a monomorphism.

The morphism \( B \) is defined according to the relations

\[
B(e_1) = v_1, B(e_2) = v_2, \ldots, B(e_{p-1}) = v_{p-1}, B(e_p) = 0
\]

With this, it must be

\[
A(v_1) = e_2, A(v_2) = e_3, \ldots, A(v_{p-1}) = e_p
\]

3) Coupled pairs of the type

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
\]

with the two matrices of the order \( p \).

If we choose \( A \) as a monomorphism:

\[
B(e_i) = v_{i+1} \text{ for } i = 1, \ldots, p - 1; \quad B(e_p) = 0
\]

\[
A(v_i) = e_i \text{ for } i = 1, \ldots, p
\]

If the monomorphism is \( B \):

\[
B(e_i) = v_i \text{ for } i = 1, \ldots, p
\]

\[
A(v_i) = e_{i+1} \text{ for } i = 1, \ldots, p - 1; \quad A(v_p) = 0
\]
4) Coupled pairs of the type

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix} \leftrightarrow 
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 \\
\end{pmatrix}
\]

with the first matrix of the order \( p - 1 \) and the second of the order \( p \).

In these, the monomorphism is \( B \). The definition is:

\[
B(e_i) = v_{i+1} \text{ for } i = 1, \ldots, p - 1 \\
A(v_i) = e_i \text{ for } i = 1, \ldots, p - 1; \ A(v_p) = 0
\]

5) Coupled pairs of the type

\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\end{pmatrix} \leftrightarrow \emptyset \quad B(e_i) = 0, \quad A(0) = 0, \quad A \text{ monomorphism }.
\]

\[
\emptyset \leftrightarrow 
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\end{pmatrix} \quad B(0) = 0, \quad A(v_i) = 0; \quad B \text{ monomorphism }.
\]

Taking \( C^m \) and \( C^n \) the basis used in these coupled pairs, the matrices \( B \) and \( A \) will be composed of diagonal blocks of the following types:

Type 1) The matrix of \( B \) is the identity. That of \( A \) is the type

\[
\begin{pmatrix}
\lambda_0 & 0 & \cdots & 0 & 0 \\
1 & \lambda_0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \lambda_0 & 0 \\
0 & 0 & \cdots & 1 & \lambda_0 \\
\end{pmatrix}
\]

In other words, the diagonal block corresponding to the matrix \( B \) is \( I_p \), and that corresponding to \( A, \lambda_0 I_p + H_p, I_p \) being the identity
of order \( p \) and \( H_p \) the matrix, also of the order \( p \),

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0 \\
\end{pmatrix}
\]

Type 2) The block \( B \) corresponding to subspaces that form this type of coupled pair is of the order \((p - 1) \times p\):

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0 \\
\end{pmatrix} = \begin{pmatrix} I_{p-1} & 0 \end{pmatrix}
\]

That corresponding to the matrix \( A \) is of the order \( p \times (p - 1) \):

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix} 0 \\
\vdots \\
I_{p-1} \end{pmatrix}
\]

Type 3) When \( A \) is a monomorphism, the block of \( B \) is \( H_p \), and that of \( A \) is \( I_p \); if \( B \) is a monomorphism, that of \( B \) is \( I_p \) and that of \( A \) is \( H_p \).

Type 4) The block of \( B \) is of the order \( p \times (p - 1) \) and that of \( A \) is \((p - 1) \times p\)

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & 0 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ddots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ddots & 1 & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0 \\
\end{pmatrix}
\]
respectively.

We thus have the following theorem:

**Theorem 3.** Given two square matrices $P$ and $Q$ with respective orders $m$ and $n$, there exist matrices $A$ and $B$ of orders $m \times n$ and $n \times m$, respectively, such that $P = AB$ and $Q = BA$ if and only if $P$ and $Q$ are couplable.

**Succession of nil power.**

For any square matrix $P$, let $k_1 \geq k_2 \geq \ldots \geq k_p > 0$ be the ordered succession of the exponents of its elementary divisors in the form $\lambda^k$. We will call this succession made infinite by adding zeroes, the *succession of nil power*:

$$k_1 \geq k_2 \geq \ldots \geq k_p > k_{p+1} = k_{p+2} = \ldots = 0$$

It is clear that the matrix $P$ will be fully described if we know its elementary divisors $(\lambda - \lambda_0)^{p_0}$ with $\lambda_0 \neq 0$ and its *succession of nil power*.

We have the following proposition:

**Proposition 4.** Let $P$ and $Q$ be two square matrices with respective successions of nil power

$$k_1 \geq k_2 \geq \ldots \geq k_p > k_{p+1} = k_{p+2} = \ldots = 0$$

$$k'_1 \geq k'_2 \geq \ldots \geq k'_q > k'_{q+1} = k'_{q+2} = \ldots = 0$$

We have that $P$ and $Q$ are couplable if and only if

a) They have the same elementary divisors $(\lambda - \lambda_0)^{p_0}$, $\lambda_0 \neq 0$.
b) $|k_h - k'_h| \leq 1 \ \forall h$.

**Proof.** ($\Leftarrow$) Let $E_i \subset \mathbb{C}^m$ be the cyclic subspaces corresponding to the elementary divisors $\lambda^k$, and $E'_i \subset \mathbb{C}^n$ those corresponding to $\lambda^k$.

The pairs $(E_i, E'_i)$, together with $(E_{q+1} \oplus \ldots \oplus E_p, \{0\})$ (if $p > q$) or $(\{0\}, E'_{p+1} \oplus \ldots \oplus E'_q)$ (if $p < q$), are coupled pairs and define a $(P, Q)$-coupling.
(⇒) Section a) forms part of the definition of \((P, Q)\)-coupling. Let us see section b).

Let
\[
\mathbb{C}^m = E \oplus E_1 \oplus \ldots \oplus E_p
\]
\[
\mathbb{C}^n = E' \oplus E'_1 \oplus \ldots \oplus E'_q
\]
be the decomposition given by a given \((P, Q)\)-coupling.

We rearrange the subspaces \(E_i\) and \(E'_j\) by decreasing order of degrees of their minimal polynomial (their dimension):
\[
E_{i_1}, E_{i_2}, \ldots , E_{i_p}
\]
\[
E'_{j_1}, E'_{j_2}, \ldots , E'_{j_q}
\]

We define \(k_h = \text{dim } E_{j_h}\), \(k'_h = \text{dim } E'_{j_h}\), constructing with them the successions of nil power of \(P\) and \(Q\):
\[
k_1 \geq k_2 \geq \ldots \geq k_p > k_{p+1} = k_{p+2} = \ldots = 0
\]
\[
k'_1 \geq k'_2 \geq \ldots \geq k'_q > k'_{q+1} = k'_{q+2} = \ldots = 0
\]

Let us suppose that for some value of \(h\) it is \(k_h - k'_h > 1\). According to the definition of \((P, Q)\)-coupling, the subspaces of \(\mathbb{C}^n\) which form coupled pairs with \(E_{i_1}, E_{i_2}, \ldots , E_{i_{h-1}}\) must be in front of \(E_{j_h}\), and are therefore \(E'_{j_1}, E'_{j_2}, \ldots , E'_{j_{h-1}}\) (not necessarily in this order). The subspace \(E_{i_h}\) must also couple with one of these, which is clearly impossible. Therefore \(k_h - k'_h \leq 1\) \(\forall h\).

A parallel reasoning leads us to \(k'_h - k_h \leq 1\) \(\forall h\). \(\diamondsuit\)

**Representative of a \((P, Q)\)-coupling.**

We will say that the pair of matrices \((A, B)\) is a representative of the \((P, Q)\)-coupling

\[
\Omega = \{(E, E'), (E_1, E'_1) \ldots (E_r, E'_r), (E_{r+1} \oplus \ldots \oplus E_p, \{0\}) \}
\]
\[
\{(\{0\}, E'_{r+1} \oplus \ldots \oplus E'_q)\}
\]
if:

i) \(A\) and \(B\) are isomorphisms between \(E\) and \(E'\).
ii) $A(E_i') \subset E_i$, $B(E_i) \subset E_i'$, for $i \leq r$, being one of the two a monomorphism and the other with the nucleus of dimensions 1. If \( \dim E_i = \dim E_i' \), the monomorphism will be suitable as indicated by the \((P, Q)\)-coupling.

iii) $A(E_i') = B(E_i) = \{0\}, \forall i > r$.

iv) $AB = P$, $BA = Q$.

Equivalent couplings.

Let

$$
\Omega_1 = \{(E, E'), (E_1, E_1'), \ldots, (E_r, E_r'), (E_{r+1} \oplus \ldots \oplus E_p, \{0\}), \}
\{(0), E_{r+1}' \oplus \ldots \oplus E_q'\}
$$

$$
\Omega_2 = \{(F, F'), (F_1, F_1'), \ldots, (F_r, F_r'), (F_{r+1} \oplus \ldots \oplus F_p, \{0\}), \}
\{(0), F_{r+1}' \oplus \ldots \oplus F_q'\}
$$

be a \((P_1, Q_1)\)-coupling and a \((P_2, Q_2)\)-coupling respectively, $P_1$ and $P_2$ being square matrices of the order $m$ and $Q_1$ and $Q_2$ square matrices of the order $n$.

We say that $\Omega_1$ is equivalent to $\Omega_2(\Omega_1 \sim \Omega_2)$ if:

i) The matrices $P_1$ and $P_2$ have the same elementary divisors \((\lambda - \lambda_0)^p\) with $\lambda_0 \neq 0$ (and consequently to $Q_1$ and $Q_2$ also).

ii) There is a rearrangement of the coupled pairs such that

$$
\dim E_{i_k} = \dim F_{j_k}
\dim E_{i_k}' = \dim F_{j_k}'
$$

for $1 \leq k \leq r$

and such that the pairs $(E_{i_k}, E_{i_k}')$, $(F_{j_k}, F_{j_k}')$ have the same principal subspaces at the same side, if $\dim E_{i_k} = \dim E_{i_k}'$.

It is clear that in these conditions $P_1$ is similar to $P_2(P_1 \sim P_2)$ and $Q_1$ is similar to $Q_2(Q_1 \sim Q_2)$.
Equivalence of doubly multipliable pairs of matrices.

We say that two pairs of doubly multipliable matrices \((A_1, B_1)\) and \((A_2, B_2)\) are equivalent \(((A_1, B_1) \sim (A_2, B_2))\) if there exist basis changes which transform one pair into another.

**Lemma 2.** Given two pairs of multipliable square matrices \((A_1, B_1)\) and \((A_2, B_2)\) with \(B_1\) and \(B_2\) invertible and such that \(A_1 B_1 \sim B_2 A_2\), then the two pairs are equivalent.

**Proof.** We have that

\[
A_1 B_1 = T^{-1} B_2 A_2 T \Rightarrow A_1 = T^{-1} B_2 A_2 T B_1^{-1} = (B_2^{-1} T)^{-1} A_2 (T B_1^{-1})
\]

Defining \(S = B_2^{-1} T\) and \(R = T B_1^{-1}\), we are left with

\[
R^{-1} B_2 S = (B_1 T^{-1}) B_2 (B_2^{-1} T) = B_1 T^{-1} T = B_1
\]

and therefore

\[
A_1 = S^{-1} A_2 R
\]

\[
B_1 = R^{-1} B_2 S
\]

which shows that the pairs of matrices considered are equivalent.

If the invertible matrices are \(A_1\) and \(A_2\), maintaining the other hypotheses the result is still true.

**Proposition 5.** If \((A_1, B_1)\) and \((A_2, B_2)\) are representatives of equivalent couplings, then they are equivalent pairs.

**Proof.** Let \((A_1, B_1)\) be a representative of a \((P_1, Q_1)\)-coupling \(\Omega_1\), and \((A_2, B_2)\) a representative of a \((P_2, Q_2)\)-coupling \(\Omega_2\) where \(\Omega_1 \sim \Omega_2\).

On the coupled pairs of type 1 we have that

\[
A_1 B_1 = P_1 \sim Q_2 = B_2 A_2
\]

\(A_i\) and \(B_i\) being invertible square matrices, because the matrices \(P_i\) and \(Q_i\) are Lemma 2 already ensures the equivalence of the pairs of matrices on coupled pairs of this type.
For coupled pairs of type 2, the matrices $A_1$ and $A_2$, of the order $p \times (p - 1)$ are monomorphisms. Their rank is thus $p - 1$. Also, the matrices $B_1$ and $B_2$, of the order $(p - 1) \times p$, have the nucleus of dimension 1; their rank is also $p - 1$. We can state that $A_1$ is "left-invertible" and $B_1$ is "right-invertible", i.e.

$$\exists M | MA_1 = I_{p-1} \quad \exists N | B_1N = I_{p-1}$$

The similarity between the matrices $P_1 = A_1B_1$ and $P_2 = A_2B_2$ ensures the existence of an invertible matrix $V$ such that $VA_1B_1 = A_2B_2V$.

With all this,

$$VA_1B_1 = A_2B_2V \Rightarrow VA_1B_1N = A_2B_2VN \Rightarrow A_1 = V^{-1}A_2(B_2VN)$$
$$VA_1 = A_2B_2V \Rightarrow A_1B_1 = V^{-1}A_2B_2V \Rightarrow$$
$$\Rightarrow MA_1B_1 = MV^{-1}A_2B_2V \Rightarrow B_1 = (MV^{-1}A_2)B_2V$$

Let us see the matrices $(B_2VN)$ and $(MV^{-1}A_2)$ are inverse to each other:

$$(MV^{-1}A_2)(B_2VN) = M(V^{-1}A_2B_2V)N = MA_1B_1N = I_{p-1}$$

Indeed, defining $R = B_2VN$ (by which $R^{-1} = MV^{-1}A_2$) are inverse to each other:

$$A_1 = V^{-1}A_2R, \quad B_1 = R^{-1}B_2V$$

which ensures that $(A_1, B_1) \sim (A_2, B_2)$ in this type of coupled pairs.

In type 3 coupled pairs the monomorphisms are in reality isomorphisms, which makes it possible to apply Lemma 2 again.

Type 4 coupled pairs are dealt with in parallel to those of type 2, using the similarity $B_1A_1 \sim B_2A_2$ working with the matrices $B_2$ and $A_2$.

Those of type 5 show no problems.

PROPOSITION 6. If $(A_1, B_1), (A_2, B_2)$ are pairs of equivalent doubly multipliable matrices, then they are representatives of equivalent couplings.

Proof. As $(A_1, B_1), (A_2, B_2)$ are equivalent pairs, there exist the invertible matrices $V$ and $W$ such that

$$A_1 = V^{-1}A_2W, \quad B_1 = W^{-1}B_2V$$
Let be

\[ P_1 = A_1 B_1, Q_1 = B_1 A_1, P_2 = A_2 B_2, Q_2 = B_2 A_2 \]

It is clear that

\[ P_1 = V^{-1} P_2 V, \quad Q_1 = W^{-1} Q_2 W \]

Therefore, the four matrices have the same elementary divisors \((\lambda - \lambda_0)^p\) with \(\lambda_0 \neq 0\), and also the same ones as the form \(\lambda^k\) and \(Q_1, Q_2\) too.

Let

\[ \Omega_1 = \{(E, E'), (E_1, E'_1), \ldots, (E_r, E'_r), (E_{r+1} \oplus \ldots \oplus E_p, \{0\}),\]

\[ (\{0\}, E'_{r+1} \oplus \ldots \oplus E'_{q})\}\]

be the \((P_1, Q_1)\)-coupling of which \((A_1, B_1)\) is the representative.

We build

\[ \Omega_2 = \{(F, F'), (F_1, F'_1), \ldots, (F_r, F'_r), (F_{r+1} \oplus \ldots \oplus F_p, \{0\}),\]

\[ (\{0\}, F'_{r+1} \oplus \ldots \oplus F'_{q})\}\]

since

\[ F = V(E), F' = W(E'); \quad F_i = V(E_i), F'_i = W(E'_i) \ \forall i \]

Each \(F_i\) has the same minimum polynomial as the corresponding \(E_i\), and the same occurs for \(F'_i\) and \(E'_i\). It is also obvious that \(\Omega_1 \sim \Omega_2\) and that the pair \((A_2, B_2)\) is a representative of the coupling \(\Omega_2\).

With the last two propositions, we have the following theorem:

**Theorem 4** (of Equivalence). Two pairs \((A_1, B_1)\) and \((A_2, B_2)\) of doubly multipliable matrices are equivalent if and only if there are representatives of equivalent couplings.
Doubly Multipliable Matrices

Rank of the matrices $A$ and $B$.

Given a $(P, Q)$-coupling, Theorem 4 ensures that all its representatives are equivalent. Therefore, if $(A_1, B_1)$ and $(A_2, B_2)$ are representatives of the same $(P, Q)$-coupling, we can ensure that $\text{rg}(A_1) = \text{rg}(A_2)$ and $\text{rg}(B_1) = \text{rg}(B_2)$. The construction of the matrices $A$ and $B$, performed for the countable pairs $(P, Q)$, allows us to calculate their rank. Indeed:

Let $n_i$ for $i = 2, 3, 4$ be the number of coupled pairs of type $i$ existing in the $(P, Q)$-coupling with which the matrices $A$ and $B$ have been constructed.

Each coupled pair of type two will produce a nil row in $A$ and a nil column in $B$. Those of type 4 will produce a transposed effect: a nil column in $A$ and a nil row in $B$. The coupled pairs of type 3 will become a nil row and nil column in $A$ if $B$ is a monomorphism, or in $B$ if $A$ is one; let $n'_3$ and $n''_3$ be the respective number of them of each form (of course $n'_3 + n''_3 = n_3$).

The coupled pairs of type 5 produce nil $p - r$ rows and $q - r$ columns in $A$; in matrix $B$ there will be nil $q - r$ columns and $p - r$ rows.

The rank of matrices $A$ and $B$, calculated by rows, is

$$\text{rg}(A) = m - n_2 - n'_3 - p + r$$

$$\text{rg}(B) = n - n_4 - n''_3 - q + r$$

If we calculate it by columns, we have

$$\text{rg}(A) = n - n_4 - n'_3 - q + r$$

$$\text{rg}(B) = m - n_2 - n''_3 - p + r$$

From these equalities it is deduced that for any $P, Q$-coupling,

$$m - n_2 - p = n - n_4 - q.$$
REFERENCES


