MEASURES, OUTER MEASURES AND NORMAL LATTICES (*)

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also investigate certain "outer" measures associated with a measure on \( A(\mathbb{L}) \), and study their behaviour in the case where \( \mathbb{L} \) is normal; this also generalizes the work of others (see [3]) to the case of more general measures.

Our notation and terminology is consistent with that of the previous referred to papers. We give a brief review of just those concepts which are relevant to this paper in section 2 for the reader's convenience. Related material can be found in [2,3,6,7].

2. Background and Notation.

We begin by reviewing some notation and terminology which is fairly standard. We supply background material for the reader's convenience.

Let \( X \) be an abstract set, and \( \mathbb{L} \) a lattice of subsets of \( X \) containing \( X \) and \( \emptyset \). The complement of a set \( E \) of \( X \) will be denoted by \( E' \). A delta lattice is one that is closed under countable intersections, and the delta lattice generated by \( \mathbb{L} \) is denoted by \( \delta \mathbb{L} \). \( \mathbb{L} \) is countably paracompact (cp) if for every sequence \( L_n \epsilon \mathbb{L} \) and \( L_n \downarrow \emptyset \) there exists \( A_n \sim \downarrow \emptyset \) st \( A_n \sim \supseteq L_n \) and \( A_n \sim \varepsilon \mathbb{L} \). By a lattice being complement generated (cg) means that if \( L \epsilon \mathbb{L} \) there exists \( L_n \epsilon \mathbb{L} \) st \( L = \cap L_n \cap n = 1, \ldots \). Note that cg implies cp since if \( A_n \downarrow \emptyset \), \( A_n \epsilon \mathbb{L} \) then \( A_n = \cap L_n \cap n = 1, \ldots \). Call \( B_n = \cap L_n \cap n = 1, \ldots, N \), and \( L_K \sim \cap B_K \), \( K = 1, \ldots, N \), \( L_K \sim \supseteq A_K \) \( L_K \sim \epsilon \mathbb{L} \) and \( L_K \sim \downarrow \emptyset \). \( A(\mathbb{L}) \) will denote the algebra generated by \( \mathbb{L} \) and \( \sigma(\mathbb{L}) \) the smallest \( \sigma \)-algebra generated by \( \mathbb{L} \).

Let \( M(\mathbb{L}) \) denote the set of non-trivial bounded non-negative real valued finitely additive measures on \( A(\mathbb{L}) \), and let \( M(\sigma^*, \mathbb{L}) \) denote those elements of \( M(\mathbb{L}) \) that are sigma-smooth on \( (\mathbb{L}) \), i.e. \( L_n \downarrow \emptyset L_n \epsilon \mathbb{L}, \mu \epsilon M(\sigma^*, \mathbb{L}) \) then \( \lim \mu(L_n) = 0 \) as \( n \) goes to infinity. Let \( M(\sigma, \mathbb{L}) \) denote those elements of \( M(\mathbb{L}) \) that are sigma-smooth on \( A(\mathbb{L}) \), i.e. \( L_n \downarrow \emptyset L_n \epsilon A(\mathbb{L}), \mu \epsilon M(\sigma, \mathbb{L}) \) then \( \lim \mu(L_n) = 0 \) as \( n \) goes to infinity. This is equivalent to \( \mu \) being \( \sigma \)-smooth on \( A(\mathbb{L}) \). \( MR(\mathbb{L}) \) will stand for those measures on \( A(\mathbb{L}) \) that are \( \mathbb{L} \)-regular on \( A(\mathbb{L}) \), i.e. if \( \mu \epsilon MR(\mathbb{L}) \) then \( \mu(A) = \operatorname{Sup} \mu(L) \) where \( A \epsilon A(\mathbb{L}) \), \( L \epsilon \mathbb{L} \) and \( A \supseteq L \). This is equivalent to \( \mu \) being \( \mathbb{L} \)-regular on \( \mathbb{L}' \). \( MR(\sigma^*, \mathbb{L}) \) will denote those measures that are sigma-smooth and \( \mathbb{L} \)-regular on \( A(\mathbb{L}) \). \( I(\mathbb{L}) \) will refer to those measures of \( M(\mathbb{L}) \) that
are two valued \(\{0,1\}\) on \(A(L)\). \(I(\sigma^* L)\) will denote those two valued measures that are sigma-smooth on \(L\). \(IR(L)\) will denote those two valued measures that are \(L\)-regular on \(A(L)\). \(IR(\sigma^*, L)\) will denote those two valued measures that are sigma-smooth and \(L\)-regular on \(A(L)\).

A lattice is said to be normal if for \(L_1, L_2 \in L\) and \(L_1 \cap L_2 = \emptyset\) there exists \(L_3, L_4 \in L\) st \(L_3 \supseteq L_1L_4 \supseteq L_2\) and \(L_3 \cap L_4 = \emptyset\). A lattice is said to be countably compact (cc) if \(\mu \in I(L)\) implies that \(\mu \in I(\sigma^*, L)\). A lattice is almost countably compact (acc) if \(\mu \in IR(L')\) implies that \(\mu \in I(\sigma^*, L)\). We now prove a result needed in the sequel.

**Theorem 2.1.** \(L\) almost countably compact and \(\mu \in MR(L')\) implies \(\mu \in M(\sigma^*, L)\).

**Proof.** Look at \(FB(\mu)|L' = \{L'|L \notin L, \mu (L') = \mu (X)\}\). Then \(FB(\mu)|L'\) is a filter base and thus is contained in an ultrafilter \(UF(\mu)|L'\).

Associated with \(UF(\mu)|L'\) is a measure \(\mu_1 \in IR(L')st\mu_1(L') = 1\) if \(L' \in UF(\mu)|L'\). In particular \(\mu_1(L') = 1\) if \(L' \in FB(\mu)|L'\). Assume that \(\mu\) is not in \(M(\sigma^*, L)\), then there exists \(L_n \in L\) \(L_n \downarrow \emptyset\) and lim \(\mu (L_n) > \varepsilon > 0\) for some \(\varepsilon > 0\). Further since \(L\) is acc \(\mu_1(L_n) = 1\) or \(\mu_1(L_n) = 0\) for \(n > N\). Since \(\mu\) is \(L'\) regular there exists \(L_n \sim L' \in L'\) \(st L_n \supseteq L_n \sim L'\) and \(\mu (L_n) - \mu (L_n \sim L') < \varepsilon/2\) thus \(\mu (L_n \sim L') > \varepsilon/2\) for \(n > N\). But this implies that \(L_n \sim L' \cap L' \neq \emptyset\) for \(L' \in FB(\mu)|L'\) and thus \(L_n \sim L' \notin UF(\mu)|L'\).

Therefore \(1 = \mu_1(L_n \sim L') \leq \mu_1(L_n) = 0\) for \(n > N\). A contradiction \(\mu \in M(\sigma^*, L)\).

Note: For \(\mu, \mu_1 \in M(L)\) we write \(\mu \leq \mu_1(L)\) if \(\mu (L) \leq \mu_1(L)\) for all \(L \in L\) and \(\mu (X) = \mu_1(X)\).

A fact used repeatedly throughout this paper is this: If \(\mu \in M(L)\) then there exists a \(\nu \in MR(L)\) st \(\mu \leq \nu (L)\) [see, 6]. If \(\mu \in I(L)\) then there exists a \(\nu \in IR(L)\) st \(\mu \leq \nu (L)\).

Normality of a lattice is equivalent to the following measure theoretic condition: If \(\mu \in I(L), \mu_1, \nu_1, \nu_2 \in MR(L), \mu \leq \mu_1(L)\) and \(\mu \leq \nu_2(L)\) then \(\mu_1 = \nu_2\). We then have the following theorem which holds for normal lattices.

**Theorem 2.2.** If \(L\) is a normal lattice, \(\mu \in M(L), \nu_1, \nu_2 \in MR(L), \nu_1, \nu_2 \in MR(L), \nu_1, \nu_2 \in MR(L)\),
\( \nu \leq \nu_1(L) \) and \( \nu \leq \nu_2(L) \) then \( \nu_1 = \nu_2(L) \).

Assume that \( \nu_1 \neq \nu_2 \), then \( \exists \mu \in \mathcal{L} \) st \( \nu_1(L) > \nu_2(L) \) or \( \nu_2(L') > \nu_1(L') \). Since \( \nu_1, \nu_2 \) are regular \( \exists \mu_2 \in \mathcal{L} \) for \( \varepsilon > 0 \) st \( L' \supseteq L' \) and \( \nu_1(L') - \nu_1(L') < \varepsilon \), \( \nu_2(L') - \nu_2(L') < \varepsilon \). Since \( L \cap L' = \emptyset \) and \( L \) is normal \( \exists L_1, L_2 \in \mathcal{L} \) st \( L_1 \supseteq L \) and \( L_2 \supseteq L' \) and \( L_1 \cap L_2 = \emptyset \) or \( L_1 \cup L_2 = X \). We have then that \( L' \supseteq L_1 \supseteq L_2 \supseteq \varepsilon \) or \( L' \supseteq L_2 \supseteq L_1 \supseteq \varepsilon \). Since \( \nu_1(L) > \varepsilon > \nu_2(L') \) \( \nu_2(L) \) thus
\[
\mu(X) = \mu(L_1) + \mu(L_2) - \mu(L_1 \cap L_2) \leq \nu_1(L') + \nu_2(L) - \varepsilon - \mu(L_1 \cap L_2).
\]

Letting \( \varepsilon \downarrow 0 \) we have that
\[
\mu(X) \leq \nu_1(L') + \nu_2(L)
\]

But \( \nu_1(L') + \nu_2(L) < \nu_1(L') + \nu_1(L) = \nu_1(X) = \mu(X) \) a contradiction.

We next introduce some set functions that are inner or outer measures or that have some of the properties of such set functions.

Let \( \mu \in M(\sigma, \mathcal{L}) \), then define for \( E, X \supseteq E \mu^*(E) = \inf \Sigma \mu(A_i) \) where \( i = 1, 2, \ldots, \cup A_i \supseteq E \cup A_i \in \mathcal{E} \mathcal{A}(\mathcal{L}) \). Then \( \mu^* \) is an outer measure whose restriction to \( A(\mathcal{L}) \) agrees with \( \mu \).

Let \( \mu \in M(\sigma^*, \mathcal{L}) \), then define for \( E, X \supseteq E \mu''(E) = \inf \Sigma \mu(L_i) \) where \( i = 1, 2, \ldots, \cup L_i \supseteq E \cup L_i \in \mathcal{L} \). Then \( \mu'' \) is an outer measure.

Let \( \mu \in M(\mathcal{L}) \), then define for \( E, X \supseteq E \mu'(E) = \inf \mu(L') \supseteq E \mathcal{L} \). Then \( \mu' \) is a finitely subadditive "outer measure".

Let \( \mu \in M(\mathcal{L}) \) and \( X \supseteq E \), then define \( \mu_i(E) = \sup \mu(L) \supseteq L \mathcal{L} \). Then \( \mu_i \) has the properties of a finitely superadditive inner measure.

For \( \mu \in M(\mathcal{L})_1 \mathcal{L} \mathcal{L} \) if \( \mu(L') = \sup \mu'(L') \supseteq L \supseteq \varepsilon \mathcal{L} \) then \( \mu \) is said to be weakly regular and the set of such measure is denoted by \( MW(\mathcal{L}) \).

3. Measure Theoretic Consequences of Normality.

In this section we investigate measure theoretic properties of normal lattices.

**Theorem 3.1.** \( \mathcal{L} \) normal \( \mu \in M(\mathcal{L}) \) \( \nu \in M(\mathcal{L}) \) \( \mu \leq \nu \mathcal{L} \mathcal{L} \) then \( \nu(L') = \sup \mu(L') \supseteq L \supseteq \varepsilon \mathcal{L} \).
Proof. We have the following $\nu(L') > \nu(L \sim) \geq \mu(L \sim)$ for any $L' \supseteq L \sim$ and $L, L \sim \in \mathbb{L}$. Since $\nu \in MR(\mathbb{L})$ for $\varepsilon > 0$ chooses $L \sim \in \mathbb{L}$, $L' \supseteq L \sim$ st $\nu(L') - \nu(L \sim) < \varepsilon$. $L \cap L \sim = \emptyset$ and since $\mathbb{L}$ is normal there exists $L_1, L_2 \in \mathbb{L}$ st $L_1' \supseteq L \sim L_2' \supseteq L \sim$ and $L_1' \cap L_2' = \emptyset$. But this implies that $L' \supseteq L_1' \supseteq L_2 \supseteq L \sim$ and $\nu(L') \geq \nu(L_1') \geq \mu(L_1) \geq \nu(L_2) \geq \nu(L \sim)$. Therefore since $\nu(L') - \nu(L \sim) < \varepsilon$, $L' \supseteq L_1$ and since $\varepsilon$ is arbitrary, $\nu(L') = \sup \mu(L \sim)L' \supseteq L \sim$, $L \sim \in \mathbb{L}$.

Theorem 3.2. If $\mathbb{L}$ is normal, $\mu_1 \leq \mu_2(\mathbb{L})$, $\mu_1 \in M(\sigma^*, \mathbb{L})$ and $\mu_2 \in MR(\mathbb{L})$ then $\mu_2 \in M(\sigma^*, \mathbb{L}')$.

Proof. Let $A_n \in \mathbb{L}$ st $A_n \uparrow \emptyset$ but $\lim \mu_2(A_n) \neq 0$. Then since $\mu_2$ is regular for $\varepsilon > 0$ there exists $B_n \in \mathbb{L}$ st $\mu_2(A_n) - \mu_2(B_n) < \varepsilon/2$ $A_n \supseteq B_n$ and without the loss of generality we can assume that $B_n \downarrow \emptyset$. Then $A_n \cap B_n = \emptyset$ and since $\mathbb{L}$ is normal, there exists $C_n$, $D_n \in \mathbb{L}$ st $C_n \supseteq B_n$, $D_n \supseteq A_n$ and $C_n \cap D_n = \emptyset$. Then $C_n \supseteq D_n \supseteq A_n$ or $C_n \supseteq A_n \supseteq D_n \supseteq C_n$. Since $A_n \downarrow \emptyset$ can assume without loss of generality that $D_n \downarrow \emptyset$, $C_n \downarrow \emptyset$, $\mu_1(D_n) \geq \mu_1(C_n) \geq \mu_2(C_n)$ and since $\mu_1 \in M(\sigma^*, \mathbb{L})$ as $n$ goes to infinity $\lim \mu_1(D_n) = 0$. Therefore $\lim \mu_1(C_n) = \lim \mu_2(C_n) = 0$, and since $C_n \supseteq B_n \lim \mu_2(B_n) = 0$. Thus for $n > N$ $\mu_2(B_n) < \varepsilon/2$, which implies that $\mu_2(A_n) < \varepsilon$, and letting $\varepsilon \downarrow 0 \lim \mu_2(A_n' = 0$ and $\mu_2 \in M(\sigma^*, \mathbb{L}')$.

Corollary 3.1. If $\mathbb{L}$ is normal and countably paracompact, $\mu_1 \in M(\sigma^*, \mathbb{L})$, $\mu_2 \in MR(\mathbb{L})$ and $\mu_1 \leq \mu_2(\mathbb{L})$ then $\mu_2 \in MR(\sigma^*, \mathbb{L})$.

Proof. From theorem 3.2 $\mu_2 \in M(\sigma^*, \mathbb{L}')$ but since $\mathbb{L}$ is cp $M(\sigma^*, \mathbb{L}) \supseteq M(\sigma^*, \mathbb{L}')$. Therefore $\mu_2 \in MR(\sigma^*, \mathbb{L})$.

Corollary 3.2. If $\mathbb{L}$ is normal and countably paracompact, $\mu_1 \leq \mu_2(\mathbb{L})$, $\mu_1 \in M(\sigma^*, \mathbb{L})$, $\mu_2 \in M(\mathbb{L})$ then $\mu_2 \in M(\sigma^*, \mathbb{L})$.

Proof. Let $\mu_3 \in MR(\mathbb{L})$ be such that $\mu_1 \leq \mu_2 \leq \mu_3(\mathbb{L})$ then $\mu_1 \leq \mu_3(\mathbb{L})$ and by corollary 3.1 $\mu_3 \in MR(\sigma^*, \mathbb{L})$, therefore $\mu_2 \in M(\sigma^*, \mathbb{L})$.

Theorem 3.3. Suppose for each $\mu \in M(\sigma^*, \mathbb{L})$ there exists a unique $\nu \in MR(\mathbb{L})$ st $\mu \leq \nu(\mathbb{L})$. Then this property together with almost countably compact of $(\mathbb{L})$ implies the normality of $\mathbb{L}$.
Proof. Let $\mu E M(L)$ then $\mu E M(L')$ and there exists $\nu E MR(L')$ st $\mu \leq \nu(L')$. Then $\nu \leq \mu(L)$ and thus there exists a $\nu_1 E MR(L)$ st $\nu \leq \mu \leq \nu_1(L)$. Now since $L$ is almost countably compact, by theorem 2.1 $\nu E M(\sigma^*, L)$, and $\nu \leq \nu_1(L)$. By the hypothesis of the theorem $\nu_1$ is the unique element of $MR(L)$ such that this holds. Therefore it is thus also the unique $\nu_1 E MR(L)$ st $\mu \leq \nu_1(L)$. Then for any $\mu E M(L)$, there exists a unique $\nu_1 E MR(L)$ st $\mu \leq \nu_1(L)$. But this is equivalent measure theoretically to $L$ being normal by theorem 2.2.

Note you can reduce the first assumption in theorem 3.3 to simply: For $\mu E I(\sigma^*, L')$ there exists a unique $\nu E I(L)$ st $\mu \leq \nu(L)$. In addition if $L$ is countably paracompact or complement generated (which implies countably paracompact see section 2), and if also $L$ is normal and almost countably compact, then $L$ is countably compact. First $L$ being almost countably compact implies that $M(\sigma^*, L) \supseteq MR(L')$. Thus for any $\mu E M(L)$, $\mu E M(L')$ there exists a $\nu E MR(L')$ st $\mu \leq \nu(L')$ or $\nu \leq \mu(L) \nu E M(\sigma^*, L)$. Since $L$ is normal and countably paracompact by corollary 3.1 there exists a unique $\nu_1 E MR(\sigma^*, L)$ st $\nu \leq \nu_1(L)$ and thus a unique $\nu_1 E MR(\sigma^*, L)$ st $\mu \leq \nu_1(L)$. Therefore $\mu E M(\sigma^*, L)$ for all $\mu E M(L)$. In particular for $\mu E I(L)$ this implies that $\mu E I(\sigma^*, L)$ and thus $L$ is cc.

4. Consequences of normality and set functions $\mu'$, $\mu''$, $\mu_i$ and $\mu^\wedge$.

In this section we discuss consequences of a lattice being normal in terms of the set functions $\mu'$, $\mu''$, $\mu_i$ and $\mu^\wedge$ (to be defined later) where $\mu E M(L)$ or $\mu E M(\sigma^*, L)$ as the case may be.

**Theorem 4.1.** If $L$ is normal, $\mu E M(L)$, $\nu E MR(L)$, $\mu \leq \nu(L)$ then $\mu \leq \nu = \nu' = \mu'(L)$.

**Proof.** From theorem 3.1 we know that $\nu(L') = \sup \mu(L \sim)$, $L, L' \sim E L$ and $L' \supseteq L \sim$. But this implies that $\nu(L) = \inf \mu(L \sim') = \mu'(L)L, L \sim E L, L \sim' \supseteq L$. Also $\nu = \nu'$ since $\nu E MR(L)$. Therefore $\mu \leq \nu = \nu' = \mu'(L)$. 
THEOREM 4.2. If \( \mathcal{L} \) is normal and \( \mu \in \text{MEW}(\mathcal{L}) \) then \( \mu \in \text{MEM}(\mathcal{L}) \).

Proof. Let \( \mu \in \text{MEW}(\mathcal{L}) \), then there exists a \( \nu \in \text{MEM}(\mathcal{L}) \) st \( \mu \leq \nu(\mathcal{L}) \). From theorem 4.1 \( \nu = \mu(\mathcal{L}) \). Since \( \mu \in \text{MEW}(\mathcal{L}) \) then \( \mu'(L') = \text{sup} \mu'(L \sim) = \text{sup} \nu(L \sim) = \nu(L') \), \( L, L \sim \in \mathcal{L}, L' \supseteq L \sim \) since \( \nu \) is regular. Therefore \( \mu = \nu \) and \( \mu \in \text{MEM}(\mathcal{L}) \).

THEOREM 4.3. Let \( \mathcal{L} \) be normal and countably paracompact. Let \( \mu \in \text{M}(\sigma^*, \mathcal{L}) \) then \( \mu' = \mu''(\mathcal{L}) \).

Proof. Since \( \mathcal{L} \) is normal and countably paracompact from corollary 3.1 there exists a \( \nu \in \text{MEM}(\sigma^*, \mathcal{L}) \) st \( \mu \leq \nu(\mathcal{L}) \). Then \( \mu \leq \nu \leq \nu'' = \nu' \leq \mu' \leq \mu(\mathcal{L}) \) and by theorem 4.1 \( \nu' = \mu'(\mathcal{L}) \) and \( \nu = \nu''(\mathcal{L}) \) since \( \nu \in \text{MEM}(\sigma^*, \mathcal{L}) \) thus \( \mu'' = \mu' \).

THEOREM 4.4. Let \( \mathcal{L} \) be a normal lattice of subsets of a set \( X \).
Let \( X \supseteq E_1, E_2 \) st there exists \( L_1, L_2 \in \mathcal{L}, L_1 \supseteq E_1, L_2 \supseteq E_2 \) and \( L_1 \cap L_2 = \emptyset \) then for \( \mu \in \text{M}(\mathcal{L}) \) \( \mu'(E_1 \cup E_2) = \mu'(E_1) + \mu'(E_2) + \mu'(E_2) \) and for \( \mu \in \text{M}(\sigma^*, \mathcal{L}) \) \( \mu''(E_1 \cup E_2) = \mu''(E_1) + \mu''(E_2) \) holds.

Proof. First \( \mu' \) is finitely additive outer measure, thus \( \mu'(E_1 \cup E_2) \leq \mu'(E_1) + \mu'(E_2) \) holds. Since \( \mathcal{L} \) is normal and \( L_1 \cap L_2 = \emptyset \) there exists \( L_1 \sim, L_2 \sim \in \mathcal{L} \) st \( L_1 \sim \supseteq L_1, L_2 \sim \supseteq L_2, L_1 \sim \cap L_2 \sim = \emptyset \), \( L_1 \sim \supseteq L_1 \supseteq E_1, L_2 \sim \supseteq L_2 \supseteq E_2 \). Then \( \mu'(L_1 \sim \cup L_2 \sim) = \mu'(L_1 \sim) + \mu'(L_2 \sim) \geq \mu(E_1) + \mu(E_2) \). Now if \( L' \supseteq E_1 \cup E_2 \) then \( L' \cap L_1 \sim \supseteq E_1, L' \cap L_2 \sim \supseteq E_2 \) and \( L' \supseteq (L' \cap L_1 \sim) \cup (L' \cap L_2 \sim) \) and \( (L' \cap L_1 \sim) \cap (L' \cap L_2 \sim) = \emptyset \). Therefore we can restrict coverings of \( E_1 \cup E_2 \) to ones where \( L_1 \sim \supseteq E_1, L_2 \sim \supseteq E_2 \), and \( L_1 \sim \cap L_2 \sim = \emptyset \). Therefore \( \mu'(E_1 \cap E_2) = \mu'(E_1) + \mu'(E_2) \). Similarly if \( \cup A_n \supseteq E_1 \cup E_2, \mu \in \text{M}(\sigma^*, \mathcal{L}), A_n \in \mathcal{L}, n = 1, 2, \ldots, \infty \) then \( \cup (A_n' \cap L_1 \sim) \supseteq E_1, \cup (A_n' \cap L_2 \sim) \supseteq E_2, ((\cup (A_n' \cap L_1 \sim)) \cup (\cup (A_n' \cap L_2 \sim))) = \emptyset \) and \( \cup A_n' \supseteq ((\cup (A_n' \cap L_1 \sim)) \cup (\cup (A_n' \cap L_2 \sim))) \) \( n = 1, 2, \ldots, \infty \). Thus \( \Sigma \mu(A_n') \geq \Sigma \mu(A_n' \cap L_1 \sim) + \Sigma \mu(A_n' \cap L_2 \sim) \). Therefore \( \mu''(E_1 \cap E_2) = \mu''(E_1) + \mu''(E_2) \) holds.

DEFINITION 4.1. Denote by \( S \mu'' \) the class of all subsets of a set \( X \) that are \( \mu'' \) measurable, i.e. \( E \in S \mu'' \) if \( \mu''(F) = \mu''(F \cap E) + \mu''(F \cap E') \) for all \( F \supseteq X \).
THEOREM 4.5. Let $X$ be a set and $\mathbb{L}$ a lattice of subsets of $X$ that is normal, and $\mu \in I(\sigma^*, \mathbb{L})$. Also let $A \in \mathbb{L}$, $A = \cap B_k \cap B_k \in \mathbb{L}$, $k = 1, 2, \ldots, \infty$, then $A \in E\mu''$.

Proof. We have that $A' = \cup B_k$. Define $C_n = \cup B_m = 1, 2, \ldots, n$, $A' \supseteq C_n$. Then $A \cap C_n = \emptyset$ and since $\mathbb{L}$ is normal there exists $D_n$, $E_n \in \mathbb{L}$ st $D_n \supseteq A$, $E_n \supseteq C_n$ and $D_n \cap E_n = \emptyset$. Further we can arrange so that $D_n \downarrow$ and $E_n \uparrow$. Then $\cup D_n \supseteq A$, $\cup E_n \supseteq A'$ or $A' \supseteq \cap D_n$ and $A \supseteq \cap E_n n = 1, 2, \ldots, \infty D_n \uparrow$ and $E_n \downarrow$. If $\mu(E_n) \neq 1$ for all $n$ then $\mu(E_n) = 0$ for $n \geq M$. Therefore $\mu(D_n') = 0$ for $n \geq M$ or $\mu(D_n') = 1$ for $n \geq M$ and $A' \supseteq D_n$. Then $A' \supseteq \cap D_k$ for $k = M, M + 1, \ldots, \infty$ thus $A \in E\mu''$ from a result that for $\mu \in I(\sigma^*, \mathbb{L})S\mu'' = \{E|E \supseteq \cap L_n, n = 1, 2, \ldots, \infty, \mu(L_n) = 1 \text{ all } n \text{ or } E' \supseteq \cap L_n, n = 1, 2, \ldots, \infty, \mu(L_n) = 1 \text{ all } n\}$.

We next introduce a new set function $\mu^\wedge$ and point out relations to previously introduced set functions such as $\mu_i$, $\mu^*$ and $\mu''$. We next investigate $\mu^\wedge$ in detail in case $\mathbb{L}$ is normal. We then prove that $\mu^\wedge$ is a finitely additive set function if the lattice $\mathbb{L}$ is normal. We also show that the restriction to $A(\mathbb{L})$ is a regular measure.

DEFINITION 4.2. Let $\mu \in M(L)$ then define for $X \supseteq E \mu^\wedge(E) = \inf \mu_i(L')$, $L' \supseteq E$. Le\mathbb{L}$.

THEOREM 4.6. For $\mu \in M(L)$ $\mu = \mu_i \leq \mu^\wedge \leq \mu'(\mathbb{L})$ and $\mu^\wedge = \mu_i \leq \mu = \mu'(\mathbb{L}')$.

Proof. The proofs follow immediately from the definitions and will be omitted.

We now prove a property of $\mu_i$.

THEOREM 4.7. $\mu_i$ is finitely additive on $\mathbb{L}'$.

Proof. Let $A', B' \in \mathbb{L}$, $A' \cap B' = \emptyset$ then from the definition of $\mu_i$ and for $\varepsilon > 0$ there exists $L_1, L_2 \in \mathbb{L}$ st $A' \supseteq L_1$, $B_1 \supseteq L_2$ and $\mu(L_1) + \mu(L_2) + \varepsilon \geq \mu(A') + \mu_i(B')$ or $\mu_i(A' \cup B') + \varepsilon \geq \mu_i(A') + \mu_i(B')$ and letting $\varepsilon \downarrow 0 \mu(A' \cup B') \geq \mu_i(A') + \mu_i(B')$. Conversely look at $\mu_i(A' \cup B')$. Then there exists a $L \in \mathbb{L}$ for $\varepsilon > 0$ st $\mu(L) + \varepsilon \geq \mu_i(A' \cup B')$ and $A' \cup B' \supseteq L$. Then since $A' \cap B' = \emptyset$, $A' \supseteq L \cap B$, $B' \supseteq L \cap A$ and
\((L \cap B) \cup (L \cap A) = L\). Thus \(\mu(L) + \epsilon = \mu(A \cap L) + \mu(B \cap L) + \epsilon \geq \mu_i(A' \cup B')\). Letting \(\epsilon \downarrow 0\), \(\mu_i(A' \cup B') = \mu_i(A') + \mu_i(B')\) and \(\mu_i\) is finitely additive on \(L'\).

**THEOREM 4.8.** If \(L\) is normal, then \(\mu^\wedge\) is a finitely subadditive outer measure for \(\mu \in M(L)\).

**Proof.** a) Let \(F \supseteq E\) then \(\mu^\wedge(E) \leq \mu^\wedge(F)\) if \(F \supseteq E\). Since if \(L'_1 \supseteq E, L'_2 \supseteq F, L'_2\) not containing \(L'_1, L'_1 - L'_2 \neq \emptyset\), then \(L'_1 \cap L'_2 \supseteq E, L'_1 \cap L'_2 \in L'\) \(L'_2 \supseteq L'_1 \cap L'_2\) and \(L'_1 \supseteq L'_1 \cap L'_2\). Thus taking infs over all \(L'_\in L\), \(\mu^\wedge(E) \leq \mu^\wedge(F)\) if \(F \supseteq E\).

b) \(\mu^\wedge(\emptyset) = 0 \mu^\wedge(X) = \mu(X)\) obviously.

c) Look at \(\mu^\wedge(E \cup F), X \supseteq E,F\). One need only look at \(L', A', B' \in L\) coverings of \(E \cup F\), where \(A' \supseteq E, B' \supseteq F\), since if \(L' \supseteq E \cup F, L' \in L'\) then for any coverings of \(E\) and \(FA' \supseteq E, B' \supseteq F, C' = (L' \cap A') \cup (L' \cap B') \supseteq E \cup F, C \in L', L' \supseteq C', L' \cap B' \supseteq F\) and \(L' \cap A' \supseteq E\). Next for any \(L \in L\) st \(C' \supseteq L\) if we can prove that there exists \(L_1, L_2 \in L\) st \(A' \supseteq L_1 \cap A' \supseteq L_1, B' \supseteq L_1 \cap B' \supseteq L_2\) and \(L = L_1 \cup L_2\) then for \(\epsilon > 0\) there exists a \(L \in L\) st \(A' \cup B' \supseteq E \cup F \supseteq L\). Thus we have that \(\mu(E \cup F) \leq \mu(L) + \epsilon \leq \mu(L_1) + \mu(L_2) + \epsilon \leq \mu^\wedge(E) + \mu^\wedge(F) + \epsilon\) and letting \(\epsilon \downarrow 0\), \(\mu^\wedge(E \cup F) \leq \mu^\wedge(E) + \mu^\wedge(F)\) and \(\mu^\wedge\) is a finite outer measure.

We now prove the lemma.

**LEMMA 4.1.** If \(A' \cup B' \supseteq L\) for \(L, A, B \in L\) and if the lattice \(L\) is normal then there exists \(L_1, L_2 \in L\) st \(L = L_1 \cup L_2\).

**Proof.** Since \(L' \supseteq A \cap B, (L \cap A) \cap (L \cap B) = \emptyset\) or \(L \cap B = \emptyset\). If \(L \cap B = \emptyset\) or \(L \cap A = A\), then take \(L_1 = L, L_2 = \emptyset, A' \supseteq L_2\) and \(L \neq L_1 \cup L_2\). Otherwise \(L \cap A = \emptyset\) and \(L \cap B \neq \emptyset\) and \((L \cap A) \cap (L \cap B) = \emptyset\). Then since \(L\) is normal there exists \(L_1, L_2 \in L\) st \(L'_1 \supseteq L \cap A, L'_2 \supseteq L \cap B\) or \(L' \cup A' \supseteq L_1, L' \cup B' \supseteq L_2, L'_1 \cap L'_2 = \emptyset\) or \(L_1 \cup L_2 = X\), then \(A' \supseteq A' \cap L \supseteq L \cap L_1, B' \supseteq B' \cap L \supseteq L \cap L_2\) and \((L \cap L_2) \cup (L \cap L_1) = L\).

We first make a definition of \(S\mu^\wedge\) measurable sets and then look at a criterion for a set to be \(\mu^\wedge\) measurable. In the remaining theorems we will assume that \(L\) is normal.

**DEFINITION 4.2.** Denote by \(S\mu^\wedge\) the class of all subsets of a set \(X\).
that are $\mu^\wedge$ measurable, i.e. $E \epsilon S \mu''$ if $\mu^\wedge(F) = \mu^\wedge(F \cap E) + \mu^\wedge(F \cap E')$ for all sets $st \ X \supseteq F$.

**Theorem 4.9.** $E \epsilon S \mu^\wedge if \mu^\wedge(A' \cap E') + \mu^\wedge(A' \cap E) \leq \mu^\wedge(A')$ for $A' \epsilon L'$.

**Proof.** If $E \epsilon S \mu^\wedge$ then for $X \supseteq F$ $\mu^\wedge(F) \geq \mu^\wedge(F \cap E) + \mu^\wedge(F \cap E')$ holds. Taking $F = A' \epsilon L'$ $\mu^\wedge(A') \geq \mu^\wedge(E' \cap A') + \mu^\wedge(E \cap A')$ holds for all $A' \epsilon L'$.

Conversely let $L' \supseteq F$ then $\mu_i(L') = \mu^\wedge(L') \geq \mu^\wedge(L' \cap E) + \mu^\wedge(L' \cap E') \geq \mu^\wedge(F \cap E) + \mu^\wedge(F \cap E')$ for any $L' \supseteq F L' \epsilon L'$. Therefore $\mu^\wedge(F) \geq \mu^\wedge(F \cap E) + \mu^\wedge(E' \cap F)$.

**Theorem 4.10.** $S \mu^\wedge \supseteq A(L)$.

**Proof.** We will prove that $S \mu^\wedge \supseteq L'$ and since $S \mu^\wedge$ is an algebra $S \mu^\wedge \supseteq A(L)$. Must now show $\mu^\wedge(A') \geq \mu^\wedge(A' \cap E) + \mu^\wedge(A' \cap E')$ for $A', E' \epsilon L'$ by theorem 4.9. Let $A' \cap E' \supseteq D$ and $A' \cap D' \supseteq F$ for $D$, $F \epsilon L$, then $A' \cap E', A' \cap D' \epsilon L'$. Now $D \cap F = \emptyset$ clearly and $A' \supseteq D \cup F$ therefore $\mu^\wedge(A') = \mu_i(A') \geq \mu(F \cup D) = \mu(D) + \mu^\wedge(D' \cap E) \geq \mu(D) + \mu_i(A' \cap D') = \mu(D) + \mu^\wedge(A' \cap D') \geq \mu^\wedge(D' \cap E) + \mu(D)$ taking sups over all possible $F st A' \cap D' \supseteq F F \epsilon L$ and the last inequality is due to the fact that $D' \supseteq A \cup E$, therefore $D' \supseteq E$ and the monotonicity of $\mu^\wedge$. Taking sups over all possible $D st A' \cap E' \supseteq D$ we have that $\mu^\wedge(A') \geq \mu^\wedge(A' \cap E) + \mu^\wedge(A \cap E')$ and we are done.

We now prove that $\mu^\wedge$ restricted to $A(L)$ is a regular measure.

**Theorem 4.11.** $\nu = \mu^\wedge | A(L) \epsilon M R(L)$ and $\mu^\wedge(X) = \nu(X)$.

**Proof.** To prove that $\nu \epsilon M R(L)$ it suffices to prove that $\nu$ is $L$ regular on $L'$ for $L' \epsilon L'$. Then $\nu(L') = \mu^\wedge(L') = \mu_i(L') = \sup \mu(L_1) L' \supseteq L_1 L_1 \epsilon L'$. But this implies that $\nu$ is $L$ on result of $L'$, therefore $\nu = \mu | A(L) \epsilon M R(L)$. Also $\nu(X) = \mu^\wedge(X) = \inf \mu_i(L') = \mu_i(X) = \sup \mu(L \sim) = \mu(X)$ where the inf is over all $L' \epsilon L'$ and $L' \supseteq X$ and the sup is over all $L \sim \epsilon L st X \supseteq L \sim$. Thus $\mu^\wedge(X) = \nu(X)$.

**Theorem 4.12.** $\mu^\wedge = \mu'(L)$ for $\mu \epsilon M(L)$.
Proof. From theorem 4.11 \( \nu = \mu|MR(\mathbb{L}) \) and \( \nu = \mu^\wedge \geq \mu \). We know from previous work that if \( \mu \leq \nu \) \( \nu \in MR(\mathbb{L}) \) then since \( \mathbb{L} \) is normal from theorem 4.1 then \( \nu = \mu'(\mathbb{L}) \), therefore \( \mu' = \mu^\wedge \) on \( \mathbb{L} \).

Remark. Let \( \mu \in M(\mathbb{L}) \). Define \( \mu_j(E) = \sup \mu'(L)E \supseteq L \), \( L \in \mathbb{L} \). Then immediately from their definitions \( \mu_j \leq \mu_j \leq \mu^\wedge \leq \mu' \) everywhere. In addition, also from their definitions, \( \mu = \mu_j(\mathbb{L}') \) iff \( \mu \in MW(\mathbb{L}) \) and \( \mu = \mu'(\mathbb{L}) \) iff \( \mu \in MR(\mathbb{L}) \). Then by theorem 4.2, if \( \mathbb{L} \) is normal and \( \mu \in MW(\mathbb{L}) \), then \( \mu = \mu_j = \mu' \in MR(\mathbb{L}) \).

References