NODAL REGIONS FOR SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEMS (*)

by LUISA DI PIAZZA (in Palermo)
and CATERINA MANISCALCO (in Cosenza) (**)

SOMMARIO. - *In questo lavoro, mediante la teoria di Morse, viene data una stima del numero delle regioni nodali delle soluzioni del problema* $-\Delta u = \lambda c(x) u + |u|^{p-2} u$ *in* Ω, $u \in H_0^1(\Omega)$, *dove* $\Omega \subset \mathbf{R}^N$, $N \geq 3$, *è un aperto connesso, limitato e regolare,* $p \in (2, 2N/(N-2)]$, $c(x) \in L^q(\Omega)$, $q > p/(p-2)$ *e* $\lambda \in \mathbf{R}$.

SUMMARY. - *In this paper we are concerned with the problem* $-\Delta u = \lambda c(x) u + |u|^{p-2} u$ *in* Ω, $u \in H_0^1(\Omega)$, *where* $\Omega \subset \mathbf{R}^N$, $N \geq 3$, *is a smooth bounded domain,* $p \in (2, 2N/(N-2)]$, $c(x) \in L^q(\Omega)$, $q > p/(p-2)$ *and* $\lambda \in \mathbf{R}$. *Using the Morse theory, we estimate the number of the nodal regions of the solutions of the above problem.*

1. Introduction.

This note deals with the problem of estimating the number of the nodal regions of the solutions of the nonlinear elliptic problem

$$(1.1) \qquad \begin{cases} -\Delta u = \lambda c(x) u + |u|^{p-2} u & \text{in } \Omega \\ u \in H_0^1(\Omega) \end{cases}$$

where $\Omega \subset \mathbf{R}^N$, $N \geq 3$, is a bounded smooth domain, λ is a real parameter, $2 < p \leq 2^* = 2N/(N-2)$ and $c(x) \in L^q(\Omega)$, $q > p/(p-2)$.

(*) Pervenuto in Redazione il 23 novembre 1991.
Lavoro svolto nell'ambito dei programmi nazionali di ricerca del M.U.R.S.T. .

(**) Indirizzi degli Autori: L. Di Piazza: Dipartimento di Matematica ed Applicazioni – Università degli Studi di Palermo – Via Archirafi, 34 – 90123 Palermo (Italy);
C. Maniscalco: Dipartimento di Matematica – Università degli Studi della Calabria – 87036 Arcavacata di Rende (CS) (Italy).

Throughout this paper $\|\ \|, |\ |_r$ denote respectively the norms in $H_0^1(\Omega)$ and $L^r(\Omega)$, $(1 \le r \le +\infty)$, 2^* is the critical Sobolev exponent for the embedding $H_0^1(\Omega) \xrightarrow{j} L^r(\Omega)$, namely the exponent such that j is continuous but not compact, and S defined by

$$S = \inf\{\|u\|^2 : \quad u \in H_0^1(\Omega), \quad |u|_{2^*} = 1\}$$

is the best constant for the Sobolev embedding $H_0^1(\Omega) \xrightarrow{j} L^{2^*}(\Omega)$.

Solving problem (1.1) is equivalent to finding critical points of the energy functional

$$(1.2) \qquad f_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega c(x)u^2 dx - \\ -\frac{1}{p}\int_\Omega |u|^p dx, \ u \in H_0^1(\Omega).$$

Since $c(x) \in L^q(\Omega)$, $q > p/(p-2)$, standard computations show that $f_\lambda \in C^2(H_0^1(\Omega), \mathbf{R})$ and that

$$(1.3) \ d^2 f_\lambda(u) \cdot [w_1, w_2] = \int_\Omega \nabla w_1 \nabla w_2 \, dx - \lambda \int_\Omega c(x) w_1 w_2 \, dx - \\ -(p-1)\int_\Omega |u|^{p-2} w_1 w_2 \, dx, \quad w_1, w_2 \in H_0^1(\Omega).$$

If u is a critical point of f_λ, we denote by $Z(u)$ the number of the nodal regions of u, i.e.

$$Z(u) \equiv \# \ \{\text{connected components of } \Omega \backslash u^{-1}(\{0\})\},$$

and by $i(u)$ the Morse index of u, i.e. the number of the negative eigenvalues (repeated according to their multiplicity) of the operator $f''_\lambda(u)$ defined by $(f''_\lambda(u)w_1, w_2) = d^2 f_\lambda(u) \cdot [w_1, w_2]$.

Recently Benci and Fortunato [BF] have investigated the same question for (1.1) under the condition $c(x) = 1$ and, using Morse theory have proved that, if $\lambda_n \le \lambda < \lambda_{n+1}$, ($\lambda_i$, $i \in \mathbf{N}^+$, being the i-th eigenvalue of $-\Delta$ with zero Dirichlet boundary data), then there exists at least a solution of (1.1) with $Z(u) \le i(u) = n + 1$.

A relation between the Morse index and the nodal regions of solutions for elliptic problems have been proved also by C.V. Coffmann [C] when $N = 1$ and by Bahry and P.L. Lions [BL] for equations with superlinear nonlinearities.

In order to study (1.1), we consider the linear problem related to (1.1)

$$-\Delta u = \nu c(x) u\,, \quad u \in H_0^1(\Omega)\,. \tag{1.4}$$

We suppose that the measure of the set $T_0 = \{x \in \Omega : c(x) = 0\}$ is zero. Then it is well known that the eigenspace of (1.4) corresponding to zero is $\{0\}$.

Set $T_1 = \{x \in \Omega : c(x) > 0\}$ and $T_2 = \{x \in \Omega : c(x) < 0\}$. Manes and Micheletti in [MM] proved that, if the measure of T_1 (resp. T_2) is positive, then the positive (resp. negative) eigenvalues of (1.4) are a divergent sequence

$$0 < \nu_1 < \nu_2 \le \nu_3 \le \dots$$

$$(\text{resp.} \quad \dots \mu_3 \le \mu_2 < \mu_1 < 0)\,.$$

Let us remark that, when the measure of the set T_1 (resp. T_2) is zero, there are no positive (resp. negative) eigenvalues of (1.4). Under this assumption, it is easy to see that, if $p < 2^*$ and $c(x)$ is a smooth function (e.g. an Hölder continuous function), then, for every $\lambda \ge 0$ (resp. $\lambda \le 0$), the problem (1.1) possesses a positive solution. On the contrary, if Ω is starshaped, $p = 2^*$, $c(x) \le 0$ a.e. (resp. $c(x) \ge 0$ a.e.) and $\lambda \ge 0$ (resp. $\lambda \le 0$), using the Pohozaev identity, it is not difficult to show that the problem (1.1) has no solution.

In what follows if $p < 2^*$ we assume that the condition below holds

(H_1) *if $\lambda > 0$ (resp. $\lambda < 0$), the measure of the set T_1 (resp. T_2) is positive.*

While if $p = 2^*$, in order to overcome the lack of compactness of f_λ, we need the stronger condition

(H_2) *if $\lambda > 0$ (resp. $\lambda < 0$), there is a ball $B_\rho(x_0) = \{x \in \mathbf{R}^N : |x - x_0| \le \rho\} \subseteq \Omega$, such that $\inf_{x \in B_\rho(x_0)} c(x) > 0$ (resp. $\sup_{x \in B_\rho(x_0)} c(x) < 0$).*

The main results of the paper are the following theorems.

THEOREM 1.1. *Let* $p = 2^*$ *and suppose that* (H_2) *holds. If* $N \geq 5$ *and* $\nu_n \leq \lambda < \nu_{n+1}$ *or* $\mu_{n+1} < \lambda \leq \mu_n$, $n \in \mathbf{N}^+$, *then the problem* (1.1) *has at least a non trivial solution* u *with* $Z(u) \leq i(u) = n+1$. *If* $N = 4$ *the same conclusion holds when* $\nu_n < \lambda < \nu_{n+1}$ *or* $\mu_{n+1} < \lambda < \mu_n$, $n \in \mathbf{N}^+$.

THEOREM 1.2. *Let* $2 < p < 2^*$, $N \geq 3$. *If* (H_1) *holds and* $\nu_n \leq \lambda < \nu_{n+1}$ *or* $\mu_{n+1} < \lambda \leq \mu_n$, $n \in \mathbf{N}^+$, *then the problem* (1.1) *has at least a non trivial solution* u *with* $Z(u) \leq i(u) = n+1$.

2. We start proving a lemma which gives an upper bound to the number of the nodal regions of a solution u of (1.1), through the Morse index of u.

Analogous results were obtained in [BF] and in [BL].

Our proof contains also an estimate of the Morse index of the restriction of u to each nodal region.

LEMMA 2.1. *Let* u *be a solution of* (1.1). *Then we have*

$$Z(u) \leq i(u) < +\infty . \tag{2.1}$$

Proof. (2.1) trivially holds if $u \equiv 0$. Let u be a non trivial solution of (1.1). Since $c(x) \in L^{N/2}$, $N \geq 3$, using a result of Brezis and Kato [BK], we obtain $u \in L^t(\Omega)$, for every t, $1 \leq t < +\infty$, so, by classical results $u \in C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$.

Consider the linearized problem

$$-\Delta v = \eta[\lambda c(x) + (p-1)|u|^{p-2}]v , \quad v \in H_0^1(\Omega) . \tag{2.2}$$

It is easy to see that $\eta \neq 0$ is an eigenvalue of (2.2) if and only if $(\eta - 1)/\eta$ is an eigenvalue of $f_\lambda''(u)$.

So, since $c(x) \in L^{N/2+\sigma}(\Omega)$, $\sigma > 0$, the spectrum of $f_\lambda''(u)$ is discrete (see [CH], [MM]) and $i(u) < +\infty$.

Now, let Ω_j, $j = 1,\dots,s$ be the connected components of $\Omega \backslash u^{-1}(\{0\})$ and set

$$u_j(x) = \begin{cases} u(x) & x \in \Omega_j \\ 0 & x \notin \Omega_j \end{cases} \qquad j = 1,2,\dots,s\,.$$

Obviously $u_j \in H_0^1(\Omega) \cap H_0^1(\Omega_j) \cap C^{0,\alpha}(\bar{\Omega}_j)$, $0 < \alpha < 1$, is a solution of the problem

$$(2.3) \qquad -\Delta v = \lambda c(x) v + |v|^{p-2} v\,, \quad v \in H_0^1(\Omega_j)\,.$$

Let i_j be the Morse index of u_j, i.e. the number of the negative eigenvalues of the restriction of $f''_\lambda(u_j)$ to $H_0^1(\Omega_j)$.

If we denote with $\gamma_k[h]$ the k-th eigenvalue of the linear problem $-\Delta v = \gamma h v$, $v \in H_0^1(\Omega_j)$, $h \in L^{N/2+\sigma}(\Omega_j)$, $\sigma > 0$, from

$$\int_{\Omega_j} \nabla u_j \nabla \varphi dx = \int_{\Omega_j} (\lambda c(x) + |u_j|^{p-2}) u_j \varphi dx\,, \quad \varphi \in H_0^1(\Omega_j)\,,$$

we get that $\gamma_k[\lambda c(x) + |u_j|^{p-2}] = 1$, for some k.

Since u_j does not change sign in Ω_j, it is $k = 1$ (see [MM]).

Moreover, by the comparison property of the eigenvalues we deduce

$$1 = \gamma_1[\lambda c(x) + |u_j|^{p-2}] > \gamma_1[\lambda c(x) + (p-1)|u_j|^{p-2}]\,.$$

Then $i_j \geq 1$ (see [A]) and (2.1) holds. ◊

To prove theorem 1.1 and theorem 1.2 we shall use the following result contained in [B] and in [BF].

THEOREM 2.2. *Let I be a C^2 functional on a real Hilbert space E and let $E = W \oplus V$, where W is an n-dimensional space and $V = W^\perp$.*

Suppose that for each critical point u of I, $I''(u)$ has a discrete spectrum and that I satisfies the Palais-Smale condition (P.S.) *in* $]-\infty, \beta[$, $(\beta > 0)$, *i.e.:*

any sequence $\{u_m\} \subset E$, such that $I(u_m) \to c$, $c < \beta$, and $I'(u_m) \to 0$, has a converging subsequence.

Moreover assume that there exist constants $R_1, R_2, R_3 > 0$, with $R_1 > R_3$, and $z \in V$, $||z|| = 1$, such that

$$\sup I(Q) < \beta \tag{2.4}$$

$$I(u) \geq \zeta > 0 \,, \quad u \in W^{\perp},\ ||u|| = R_3 \tag{2.5}$$

$$I(u) \leq 0 \,, \quad u \in \partial Q \tag{2.6}$$

where Q is the set

$$Q = \{y + tz : y \in W,\ ||y|| \leq R_2,\ t \in [0, R_1]\} \,.$$

Then I possesses a critical point u with Morse index

$$i(u) \leq n + 1 \,.$$

Moreover $\zeta \leq I(u) \leq \sup I(Q)$. ◇

3. Proof of theorem 1.1.

First of all we notice that, if $p = 2^*$, f_λ satisfies the (P.S.) condition in the energy range $]-\infty, \frac{1}{N} S^{N/2}[$ (see [BN], [CFS]).

We denote by v_j and m_j normalized eigenfunctions corresponding respectively to the eigenvalues ν_j and μ_j of the problem (1.4) and by $M(\nu_j)$ and $M(\mu_j)$ respectively the corresponding eigenspaces.

By classical results of regularity the functions v_j and m_j, $j = 1, 2, \ldots$, belong to $C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$. Besides they are a complete ortonormal system in $H_0^1(\Omega)$ (see [MM]).

Let us introduce the sets

$$H_+^1 = \overline{\bigoplus_{j \geq n+1} M(\nu_j) \oplus \bigoplus_{j \in \mathbb{N}^+} M(\mu_j)} \,, \quad H_-^1 = \bigoplus_{j \leq n} M(\nu_j) \tag{3.1}$$

$$H_+^2 = \overline{\bigoplus_{j \geq n+1} M(\mu_j) \oplus \bigoplus_{j \in \mathbb{N}^+} M(\nu_j)} \,, \quad H_-^2 = \bigoplus_{j \leq n} M(\mu_j) \tag{3.2}$$

where the closure is taken in $H_0^1(\Omega)$.

Now, if x_0 and $B_\rho \equiv B_\rho(x_0)$ are respectively the point and the ball in hypothesis (H_2), we set for each $\mu > 0$

$$\psi_\mu(x) = \phi(x)\,u_\mu(x) \tag{3.3}$$

where $\phi \in C_0^\infty(B_\rho)$, $\phi(x) = 1$ in $B_{\rho/2} \equiv B_{\rho/2}(x_0)$, and

$$u_\mu(x) = \frac{[N(N-2)\mu]^{(N-2)/4}}{[\mu + |x - x_0|^2]^{(N-2)/2}}.$$

Moreover, let us denote by P_+^i, P_-^i, $i = 1,2$, the projector operators on the space H_+^i and H_-^i respectively.

So, arguing as in [BF], we set

$$\tilde{\psi}_\mu^i = \frac{P_+^i \psi_\mu}{||P_+^i \psi_\mu||} = \frac{\psi_\mu - P_-^i \psi_\mu}{||\psi_\mu - P_-^i \psi_\mu||}, \quad i = 1,2 \tag{3.4}$$

and

$$Q_\mu^i = \{u^- + t\tilde{\psi}_\mu^i : u^- \in H_-^i, \quad ||u^-|| \leq R_2, \quad t \in [0, R_1]\}.$$

We claim that, when $N \geq 5$ and $\nu_n \leq \lambda < \nu_{n+1}$ (resp. $\mu_{n+1} < \lambda \leq \mu_n$), $n \in \mathbf{N}^+$, the assumptions of theorem 2.2. are verified if we put $\beta = \frac{1}{N} S^{N/2}$, $V = H_+^1$ (resp. H_+^2), $W = H_-^1$ (resp. H_-^2), $Q = Q_\mu^1$ (resp. Q_μ^2), with suitable μ, R_1, R_2, R_3. The same statement holds, if $N = 4$ and $\nu_n < \lambda < \nu_{n+1}$ (resp. $\mu_{n+1} < \lambda < \mu_n$).

In order to prove our claim we recall the following estimates, for $\mu \to 0$ (see [BN], [CFP], [F])

$$||\psi_\mu||^2 = S^{N/2} + O(\mu^{(N-2)/2}) \text{ (1)} \tag{3.5}$$

$$|\psi_\mu|_{2^*}^{2^*} = S^{N/2} + O(\mu^{N/2}) \tag{3.6}$$

$$|\psi_\mu|_1 = O(\mu^{(N-2)/4}). \tag{3.7}$$

---

(1) Here and in the sequel we denote by $O(\mu^\alpha)$, $\alpha>0$, a function g such that $|g(\mu)| <$ const. μ^α near $\mu = 0$.

Moreover easy calculations show that

$$(3.8)\qquad |\psi_\mu|_r^r = \begin{cases} O(\mu^{[2N-(N-2)r]/4}) & \text{for } 2^*/2 < r < 2^* \\ O(\mu^{r/2}) & \text{for } 1 < r < 2 \text{ and } N = 4\,. \end{cases}$$

The next lemmas hold

LEMMA 3.1. *If ψ_μ and $\tilde{\psi}_\mu^i$ $(i = 1,2)$ are defined as in* (3.3) *and* (3.4), *then, as $\mu \to 0$, we have*

$$(3.9)\qquad \|P_+^i\psi_\mu\|^2 = \begin{cases} S^{N/2} + O(\mu^{1+\alpha}), \quad \alpha > 0^{(2)}\,, & \text{if } N \geq 5 \\ S^2 + O(\mu) & \text{if } N = 4 \end{cases}$$

$i = 1,2$,

$$(3.10)\qquad |\tilde{\psi}_\mu^i|_{2^*-1}^{2^*-1} = \begin{cases} 0(\mu^b) & \text{if } N \geq 5 \\ 0(\mu^{1/2}) & \text{if } N = 4 \end{cases} \qquad i = 1,2\,,$$

where $b = \min\left\{\dfrac{N-2}{4},\ \dfrac{(1+\alpha)(N+2)}{2(N-2)}\right\}$,

$$(3.11)\qquad \int_\Omega c(x)|\tilde{\psi}_\mu^i(x)|^2 dx \geq \begin{cases} k_1\mu + O(\mu^{1+\alpha}) & \text{if } N \geq 5 \\ k_2\mu|\log\mu| + O(\mu) & \text{if } N = 4 \end{cases}$$

$i = 1,2$, *when* $\inf\limits_{x\in B_\rho} c(x) > 0$,

$$(3.12)\qquad \int_\Omega c(x)|\tilde{\psi}_\mu^i(x)|^2 dx \leq \begin{cases} -k_3\mu + O(\mu^{1+\alpha}) & \text{if } N \geq 5 \\ -k_4\mu|\log\mu| + O(\mu) & \text{if } N = 4 \end{cases}$$

$i = 1,2$, *when* $\sup\limits_{x\in B_\rho} c(x) < 0$,

$$(3.13)\qquad |P_+^i\psi_\mu|_{2^*}^{2^*} = \begin{cases} S^{N/2} + O(\mu^{1+\alpha}) & \text{if } N \geq 5 \\ S^2 + O(\mu) & \text{if } N = 4 \end{cases} \qquad i = 1,2\,,$$

(2) In what follows with the same symbol α we will indicate different positive exponents.

$$(3.14)\qquad |\tilde{\psi}^i_\mu|_1 = \begin{cases} O(\mu^{(1+\alpha)/2}) & \text{if } N \geq 5 \\ O(\mu^{1/2}) & \text{if } N = 4 \end{cases} \qquad i = 1,2\ ,$$

where k_s ($s = 1, \ldots, 4$) are suitable positive constants [(3)].

LEMMA 3.2. *If $u = u^- + t\tilde{\psi}^i_\mu$, with $u^- \in H^i_-$ ($i = 1,2$) and $t \in \mathbf{R}$, for μ small enough, we have*

$$(3.15)\qquad |u|^{2^*}_{2^*} \geq |t\tilde{\psi}^i_\mu|^{2^*}_{2^*} + \frac{1}{2}|u^-|^{2^*}_{2^*} - t^{2^*}A(\mu)$$

where $A(\mu) = \begin{cases} O(\mu^{1+\alpha}) & \text{if } N \geq 5 \\ O(\mu^{2/3}) & \text{if } N = 4\ . \end{cases}$

Proof of lemma 3.1. Let us set $P^1_-\psi_\mu = \sum_{k=1}^n a_k v_k$ and $P^2_-\psi_\mu = \sum_{k=1}^n b_k m_k$.

Verification of (3.9)

$$(3.16)\quad \|P^1_-\psi_\mu\| = \left(\sum_{k=1}^n a_k^2\right)^{1/2} =$$

$$= \left[\sum_{k=1}^n \nu_k^2 \left(\int_\Omega c(x)\psi_\mu(x)v_k(x)\,dx\right)^2\right]^{1/2} \leq$$

$$\leq \left(\sum_{k=1}^n \nu_k^2 |v_k|^2_\infty\right)^{1/2} \int_\Omega |c(x)\psi_\mu|dx \leq$$

$$\leq \text{const}\,.\, |c|_s |\psi_\mu|_{s/(s-1)}\ ,$$

where $1 < s \leq q$.

Respectively

$$(3.17)\qquad \|P^2_-\psi_\mu\| \leq \left(\sum_{k=1}^n \mu_k^2 |m_k|^2_\infty\right)^{1/2} \int_\Omega |c(x)\psi_\mu|dx \leq$$

$$\leq \text{const}\,.\, |c|_s |\psi_\mu|_{s/(s-1)}\ ,$$

---

(3) In what follows with k_s ($s \in \mathbf{N}$) we will denote positive constants.

where $1 < s \leq q$.

So by (3.8), with suitable r, (3.16) and (3.17) we obtain

$$(3.18) \qquad \|P_-^i \psi_\mu\| = \begin{cases} O(\mu^{(1+\alpha)/2}) & \text{if } N \geq 5 \\ O(\mu^{1/2}) & \text{if } N = 4 \end{cases} \qquad i = 1,2 .$$

Thus by (3.5) and (3.18) it follows

$$\|P_+^i \psi_\mu\|^2 = \|\psi_\mu\|^2 - \|P_-^i \psi_\mu\|^2 = \\ = \begin{cases} S^{N/2} + O(\mu^{1+\alpha}) & \text{if } N \geq 5 \\ S^2 + O(\mu) & \text{if } N = 4 \end{cases} \qquad i = 1,2 .$$

Verification of (3.10)

Arguing as above we obtain

$$(3.19) \qquad |P_-^i \psi_\mu|_\infty = \begin{cases} O(\mu^{(1+\alpha)/2}) & \text{if } N \geq 5 \\ O(\mu^{1/2}) & \text{if } N = 4 \end{cases} \qquad i = 1,2 .$$

For $N \geq 5$, combining (3.8), (3.9) and (3.19) we get

$$|\tilde{\psi}_\mu^i|_{2^*-1}^{2^*-1} = \frac{1}{\|P_+^i \psi_\mu\|^{2^*-1}} |\psi_\mu - P_-^i \psi_\mu|_{2^*-1}^{2^*-1} \leq \\ \leq [S^{N/2} + O(\mu^{1+\alpha})]^{(1-2^*)/2} (|\psi_\mu|_{2^*-1} + |P_-^i \psi_\mu|_{2^*-1})^{2^*-1} \leq \\ \leq [S^{N/2} + O(\mu^{1+\alpha})]^{(1-2^*)/2} \text{const}.[O(\mu^{(N-2)/4}) + O(\mu^{\frac{(1+\alpha)(N+2)}{2(N-2)}})] .$$

Then, for $\mu \to 0$, we have

$$|\tilde{\psi}_\mu^i|_{2^*-1}^{2^*-1} = O(\mu^b) \qquad i = 1,2 ,$$

where $b = \min \left\{ \frac{N-2}{4}, \frac{(1+\alpha)(N+2)}{2(N-2)} \right\} > \frac{1}{2}$.

Analogously, in case $N = 4$, (3.10) follows by (3.8), (3.9) and (3.19).

Verification of (3.11)

Set $\tilde{k} = \inf_{x \in B_\rho} c(x)$. If $N \geq 5$, we have

$$\int_\Omega c(x)\psi_\mu^2 dx = \int_{B_\rho} (c(x)\phi^2(x) - \tilde{k}) \frac{[N(N-2)\mu]^{(N-2)/2}}{[\mu + |x - x_0|^2]^{N-2}} dx +$$
$$+ \int_{\mathbf{R}^N} \tilde{k} \frac{[N(N-2)\mu]^{(N-2)/2}}{[\mu + |x - x_0|^2]^{N-2}} dx -$$
$$- \int_{\mathbf{R}^N \setminus B_\rho} \tilde{k} \frac{[N(N-2)\mu]^{(N-2)/2}}{[\mu + |x - x_0|^2]^{N-2}} dx \geq$$
$$\geq \mu^{(N-2)/2} \left[\int_{B_\rho \setminus B_{\rho/2}} \frac{[c(x)\phi^2(x) - \tilde{k}][N(N-2)]^{(N-2)/2}}{[\mu + |x - x_0|^2]^{N-2}} dx - \right.$$
$$\left. - \int_{\mathbf{R}^N \setminus B_\rho} \tilde{k} \frac{[N(N-2)]^{(N-2)/2}}{|x - x_0|^{2(N-2)}} dx \right] + \mu \int_{\mathbf{R}^N} \tilde{k} \frac{[N(N-2)]^{(N-2)/2}}{[1 + |x|^2]^{N-2}} dx.$$

Therefore, by (3.9), (3.19) and the above relation

$$\int_\Omega c(x) |\tilde{\psi}_\mu^i|^2 dx = \int_\Omega c(x) ||P_+^i \psi_\mu||^{-2} [\psi_\mu^2 - (P_-^i \psi_\mu)^2] \geq k_3 \mu + O(\mu^{1+\alpha}) .$$

In case $N = 4$, arguing as in [BN] (see verification of (1.13)), we have

$$(3.20) \quad \int_\Omega c(x)\psi_\mu^2 dx = \int_\Omega c(x)\phi^2(x) \frac{8\mu}{(\mu + |x - x_0|^2)^2} dx \geq$$
$$\geq \int_{B_\rho \setminus B_{\rho/2}} (c(x)\phi^2(x) - \tilde{k}) \frac{8\mu}{(\mu + |x - x_0|^2)^2} dx +$$
$$+ \int_{B_\rho} \tilde{k} \frac{8\mu}{(\mu + |x - x_0|^2)^2} dx =$$
$$= O(\mu) + k_5 \mu |\log \mu| .$$

So, from (3.9), (3.19) and (3.20) we obtain the relation (3.11).

Verification of (3.12)
It is analogous to (3.11)'s one.

Verification of (3.13)

Easy calculations prove that (see also [CFP], Remark (2.4))

$$(3.21)\quad \left||P_+^i\psi_\mu|_{2^*}^{2^*} - |\psi_\mu|_{2^*}^{2^*}\right| \le \text{const}.\{|\psi_\mu|_{2^*-1}^{2^*-1}|P_-^i\psi_\mu|_\infty + |P_-^i\psi_\mu|_{2^*}^{2^*}\}\,.$$

In case $N \geq 5$, from (3.19) it follows

$$(3.22)\qquad |P_-^i\psi_\mu|_{2^*}^{2^*} = O(\mu^{1+\alpha}) \quad i = 1,2\,.$$

So, by (3.8), (3.19), (3.21) and (3.22) we get

$$(3.23)\qquad \left||P_+^i\psi_\mu|_{2^*}^{2^*} - |\psi_\mu|_{2^*}^{2^*}\right| = O(\mu^{1+\alpha}) \quad i = 1,2\,.$$

Therefore, combining the relations (3.6) and (3.23) we obtain

$$|P_+^i\psi|_{2^*}^{2^*} = S^{N/2} + O(\mu^{1+\alpha}) \quad i = 1,2\,.$$

Analogously in case $N = 4$ we deduce the relation (3.13) by (3.6), (3.8), (3.19) and (3.21).

Verification of (3.14)

It follows immediately from (3.7), (3.9) and (3.19). ◇

Proof of lemma 3.2. Arguing in a similar way than in [CFP] (see (2.10)), by (3.10) we deduce that

$$\begin{aligned}(3.24)\quad & \left||u|_{2^*}^{2^*} - |t\tilde\psi_\mu^i|_{2^*}^{2^*} - |u^-|_{2^*}^{2^*}\right| \le \\ & \le k_6(|u^-|_{2^*}^{2^*-1}|t\tilde\psi_\mu^i|_1 + |t\tilde\psi_\mu^i|_{2^*-1}^{2^*-1}|u^-|_{2^*}) \le \\ (3.24\,a)\quad & \le k_6 t^{2^*-1}|u^-|_{2^*}|\tilde\psi_\mu^i|_{2^*-1}^{2^*-1} + \frac{1}{4}|u^-|_{2^*}^{2^*} + k_7 t^{2^*}|\tilde\psi_\mu^i|_1^{2^*} \le \\ & \le \frac{1}{2}|u^-|_{2^*}^{2^*} + t^{2^*}[k_8|\tilde\psi_\mu|_1^{2N/(N+2)} + k_7|\tilde\psi_\mu|_1^{2^*}]\,.\end{aligned}$$

Then by (3.14) we get immediately (3.15). ◇

Now we are able to show that

$$(3.25)\qquad \sup f_\lambda(Q_\mu^1) < \frac{1}{N}S^{N/2} \quad \begin{array}{ll}\text{if } \nu_n \le \lambda < \nu_{n+1} & \text{and } N \ge 5 \\ \text{if } \nu_n < \lambda < \nu_{n+1} & \text{and } N = 4\end{array}$$

$$(\text{resp. } \sup f_\lambda(Q_\mu^2) < \frac{1}{N} S^{N/2} \quad \begin{array}{ll} \text{if } \mu_{n+1} < \lambda \le \mu_n & \text{and } N \ge 5 \\ \text{if } \mu_{n+1} < \lambda < \mu_n & \text{and } N = 4 \end{array})$$

for μ small enough.

In order to do this we notice that for every $u \in H_0^1(\Omega)$ such that $||u||^2 - \lambda \int_\Omega c(x) u^2 dx > 0$,

$$\sup_{t\in\mathbf{R}} f_\lambda(tu) = \frac{1}{N} \left(\frac{||u||^2 - \lambda \int_\Omega c(x) u^2 dx}{|u|_{2^*}^2} \right)^{N/2} .$$

Therefore (3.25) holds, if we prove that

$$(3.26) \qquad ||u||^2 - \lambda \int_\Omega c(x) u^2 dx < S \quad \text{on } G_1 \ (\text{resp. } G_2) ,$$

where $G_i = \{u \in H_0^1(\Omega) : u = u^- + t\tilde{\psi}_\mu^i, u^- \in H_-^i, t \in \mathbf{R}, |u|_{2^*} = 1\}$, $i = 1,2$.

Since $u \in G_1$ (resp. G_2), by lemma 3.2. it is easily seen that, if μ is sufficiently small

$$t^{2^*} \le \frac{1}{|\tilde{\psi}_\mu^i|_{2^*}^{2^*} - A(\mu)} .$$

Then if $N \ge 5$, using (3.9), (3.11) (resp. (3.12)), and (3.13), for $u \in G_1$ (resp. G_2) we deduce

$$\begin{aligned} ||u||^2 - \lambda \int_\Omega c(x) u^2 dx &= \int_\Omega [|\nabla u^-|^2 - \lambda c(x)(u^-)^2] dx + \\ &\quad + t^2(||\tilde{\psi}_\mu^1||^2 - \lambda \int_\Omega c(x)(\tilde{\psi}_\mu^1)^2 dx) \le \\ &\le \frac{S - \lambda k_9 \mu + O(\mu^{1+\alpha})}{[1 + O(\mu^{1+\alpha})]^{2/2^*}} < S \\ (\text{resp.} &= \int_\Omega [|\nabla u^-|^2 - \lambda c(x)(u^-)^2] dx + \\ &\quad + t^2(||\tilde{\psi}_\mu^2||^2 - \lambda \int_\Omega c(x)(\tilde{\psi}_\mu^2)^2 dx) \le \\ &\le \frac{S + \lambda K_{10} \mu + O(\mu^{1+\alpha})}{[1 + O(\mu^{1+\alpha})]^{2/2^*}} < S) \end{aligned}$$

for μ sufficiently small.

In case $N = 4$, by (3.9), (3.10), (3.11), (3.13), (3.14) and (3.24a), we obtain

$$(3.27) \qquad \|u\|^2 - \lambda \int_\Omega c(x) u^2 dx \leq$$

$$\leq (\nu_n - \lambda) \int_\Omega c(x)(u^-)^2 dx + \frac{\|\tilde\psi_\mu^1\|^2 - \lambda \int_\Omega c(x)(\tilde\psi_\mu^1)^2 dx}{|\tilde\psi_\mu^1|_4^2} |t\tilde\psi_\mu^1|_4^2 \leq$$

$$\leq (\nu_n - \lambda) \int_\Omega c(x)(u^-)^2 dx + \frac{S - \lambda k_{11}\mu|\log\mu| + O(\mu)}{1 + O(\mu)} (1 - \frac{3}{4}|u^-|_4^4 +$$

$$+ k_{12} t^3 \mu^{1/2} |u^-|_4 + O(\mu^2))^{1/2} \leq$$

$$\leq (\nu_n - \lambda) \int_\Omega c(x)(u^-)^2 dx + k_{12} t^3 \mu^{1/2} |u^-|_4 +$$

$$+ \frac{S - \lambda k_{11}\mu|\log\mu| + O(\mu)}{1 + O(\mu)} (1 + O(\mu^2)) ,$$

$$(\text{resp.} \quad \leq (\mu_n - \lambda) \int_\Omega c(x)(u^-)^2 dx + k_{13} t^3 \mu^{1/2} |u^-|_4 +$$

$$+ \frac{S + \lambda k_{14}\mu|\log\mu| + O(\mu)}{1 + O(\mu)} (1 + O(\mu^2)) .$$

We put

$$B(\mu, u^-) = (\nu_n - \lambda) \int_\Omega c(x)(u^-)^2 dx + k_{12} t^3 \mu^{1/2} |u^-|_4$$

$$(\text{resp.} \quad = (\mu_n - \lambda) \int_\Omega c(x)(u^-)^2 dx + k_{13} t^3 \mu^{1/2} |u^-|_4)$$

and we observe that

$$(3.28) \quad B(\mu, u^-) \leq 0 \text{ or } B(\mu, u^-) \leq \frac{k_{15}\mu}{\lambda - \nu_n} \left(\text{resp.} \leq \frac{k_{16}\mu}{\mu_n - \lambda}\right) .$$

So using (3.27) and (3.28) we get (3.26).

Now set $u = u^- + t\tilde\psi_\mu^1$ with $u^- = \sum_{k=1}^n a_k v_k \in H_-^1$ (resp. $u = u^- + t\tilde\psi_\mu^2$ with $u^- = \sum_{k=1}^n b_k m_k \in H_-^2$).

By lemma 3.2. we infer

$$f_\lambda(u) \le \frac{1}{2}\int_\Omega (|\nabla u^-|^2 - \lambda c(x)(u^-)^2)\,dx + \frac{t^2}{2}\int_\Omega (|\nabla \tilde{\psi}^1_\mu|^2 -$$

$$- \lambda c(x)(\tilde{\psi}^1_\mu)^2\,dx - \frac{1}{2^*}\left[\frac{1}{2}|u^-|^{2^*}_{2^*} + t^{2^*}(|\tilde{\psi}^1_\mu|^{2^*}_{2^*} - A(\mu))\right] \le$$

$$\le \frac{1}{2}\sum_{k=1}^{n}\left(1 - \frac{\lambda}{\nu_k}\right)a_k^2 + \frac{t^2}{2}\int_\Omega (|\nabla\tilde{\psi}^1_\mu|^2 - \lambda c(x)(\tilde{\psi}^1_\mu)^2)\,dx -$$

$$- \frac{t^{2^*}}{2^*}(|\tilde{\psi}^1_\mu|^{2^*}_{2^*} - A(\mu)) - \vartheta\left(\sum_{k=1}^{n} a_k^2\right)^{2^*/2}$$

where

$$\vartheta = \frac{N-2}{4N}\left(\frac{1}{|c|_{N/2}}\frac{1}{\nu_n}\right)^{2^*/2}$$

$$\Bigg(\text{resp.}\quad f_\lambda(u) \le \frac{1}{2}\sum_{k=1}^{n}\left(1 - \frac{\lambda}{\mu_k}\right)b_k^2 + \frac{t^2}{2}\int_\Omega (|\nabla\tilde{\psi}^2_\mu|^2 -$$

$$- \lambda c(x)(\tilde{\psi}^2_\mu)^2)\,dx - \frac{t^{2^*}}{2^*}(|\tilde{\psi}^2_\mu|^{2^*}_{2^*} - A(\mu)) - \vartheta' \cdot \left(\sum_{k=1}^{n} b_k^2\right)^{2^*/2}$$

where

$$\vartheta' = \frac{N-2}{4N}\left(\frac{1}{|c|_{N/2}}\frac{1}{|\mu_n|}\right)^{2^*/2}\Bigg).$$

This easily implies that

$$(3.29)\qquad f_\lambda(u) \le 0 \quad \text{on} \quad \partial Q^1_\mu\ (\text{resp.}\ \partial Q^2_\mu)\ ,$$

for R_1 and R_2 suitable large.

The final step is to show that

$$(3.30)\qquad f_\lambda(u) \ge \alpha > 0 \quad \text{if } u \in H^1_+\ (\text{resp.}\ H^2_+),\ \|u\| = R_3\ .$$

Set $u = \sum_{i=m+1}^{\infty} d_i v_i + \sum_{j=1}^{\infty} h_j m_j$ (resp. $u = \sum_{i=1}^{\infty} d_i v_i + \sum_{j=m+1}^{\infty} h_j m_j$).

We have

$$f_\lambda(u) = \frac{1}{2}||u||^2 - \frac{\lambda}{2}\left[\sum_{i=n+1}^{\infty} d_i^2 \int_\Omega c(x) v_i^2\, dx + \right.$$

$$\left. + \sum_{j=1}^{\infty} h_j^2 \int_\Omega c(x) m_j^2\, dx\right] - \frac{1}{2^*}|u|_{2^*}^{2^*} =$$

$$= \frac{1}{2}||u||^2 - \frac{\lambda}{2}\left[\sum_{i=n+1}^{\infty} d_i^2 \frac{1}{\nu_i} + \sum_{j=1}^{\infty} h_j^2 \frac{1}{\mu_j}\right] - \frac{1}{2^*}|u|_{2^*}^{2^*} \geq$$

$$\geq \frac{1}{2}||u||^2 - \frac{\lambda}{2\nu_{n+1}}\left(\sum_{i=n+1}^{\infty} d_i^2 + \sum_{j=1}^{\infty} h_j^2\right) - k_{17}||u||^{2^*} =$$

$$= \frac{1}{2}||u||^2\left(1 - \frac{\lambda}{\nu_{n+1}}\right) - k_{18}||u||^{2^*}$$

(resp. $f_\lambda(u) \geq \ldots \geq \frac{1}{2}||u||^2(1 - \frac{\lambda}{\mu_{n+1}}) - k_{19}||u||^{2^*})$.

Then (3.30) follows for $||u|| = R_3$ small enough.

Thus we can apply theorem 2.2. to the functional f_λ. By this and by lemma 2.1., we deduce that there exists a non trivial solution u of (1.1) such that

$$Z(u) \leq n+1 \ . \qquad \diamond$$

4. Proof of Theorem 1.2.

Since $2 < p < 2^*$, standard computations show that the functional f_λ fulfils the (P.S.) condition.

Let H_+^i, H_-^i, $i = 1,2$ be the sets introduced in (3.1) and in (3.2) respectively, and set

$$Q^1 = \{u^- + tv_{n+1} \ : u^- \in H_-^1, \ ||u^-|| \leq R_2, \ t \in [O, R_1]\}$$

$$Q^2 = \{u^- + tm_{n+1} \ : u^- \in H_-^2, \ ||u^-|| \leq R_2, \ t \in [O, R_1]\} \ .$$

Then, if $\nu_n \leq \lambda < \nu_{n+1}$ (resp. $\mu_{n+1} < \lambda \leq \mu_n$), it is easily seen that for R_3 small enough

$$f_\lambda(u) \geq \alpha > 0, \quad \forall\, u \in H_+^1 \ (\text{resp.}\ H_+^2), \ ||u|| = R_3 \ .$$

Moreover, since $c(x) \in L^{p/(p-2)}(\Omega)$, if R_1 and R_2 are sufficiently large, we obtain

$$f_\lambda(u) \leq 0, \quad \forall\, u \in \partial Q^1 \,(\text{resp.}\, \partial Q^2)\,.$$

Then by theorem 2.2. and by lemma 2.1. the conclusion follows. ◇

REMARK 4.1. We notice that, if $\lambda \in]0, \nu_1[$ (resp. $\lambda \in]\mu_1, 0[$), the operator $(-\Delta - \lambda c(x))$ is coercive. Then, the existence of a solution $u \geq 0$ of (1.1) is well known for $2 < p \leq 2^*$ (if $p \in (2, 2^*)$ see for instance [AR], if $p = 2^*$ see [BN]).

If $c(x)$ is a smooth function and $c(x) \geq 0$ (resp. $c(x) \leq 0$), u is a classical solution and, by the strong maximum principle, $u > 0$ (i.e. $Z(u) = 1$).

REFERENCES

[A] AMBROSETTI A., *Differential equations with multiple solutions and nonlinear functional analysis*, Proc. Equadiff. *82*, Springer Lec. notes (1983), 10-37.

[AR] AMBROSETTI A. and RABINOWITZ P.H., *Dual variational methods in critical points theory and applications*, J. Funct. Analysis *14*, (1973), 349-381.

[BF] BENCI V. and FORTUNATO D., *A remark on the nodal regions of the solutions of some superlinear elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A *111* (1989), 123-128.

[BK] BREZIS H. and KATO T., *Remarks on the Schrödinger operator with singular complex potential*, J. Math. Pures et Appl. *58* (1979), 137-151.

[BL] BAHRY A. and LIONS P.L., *Solutions of superlinear elliptic equations and their Morse indices*, preprint.

[BN] BREZIS H. and NIRENBERG L., *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. *36* (1983), 437-477.

[C] COFFMAN C.V., *Lyusternik-Schnirelman theory: complementary principles and the Morse index*, Nonlinear An., Theory Methods & Applications, *12* n. 5 (1988), 507-529.

[CFP] CAPOZZI A., FORTUNATO D. and PALMIERI G., *An existence result for nonlinear elliptic problems involving critical Sobolev exponent*, Ann. Inst. H. Poincaré, Anal. Nonlineaire, *2* (1985), 463-470.

[CFS] CERAMI G., FORTUNATO D. and STRUWE M., *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. H. Poincaré, Anal. Nonlinaire, *1* (1984), 341-350.

[CH] COURANT R. and HILBERT D., *Methods of mathematical physics*, Vol. I, Interscience, New York, 1953.

[F] FORTUNATO D., *Problemi ellittici con termine non lineare a crescita critica,* Proceedings of the meeting "Problemi differenziali e teoria dei punti critici" (Bari, marzo 1984).

[MM] MANES A. and MICHELETTI A.M., *Un'estensione della teoria variazionale classica degli autovalori per operatori ellittici del secondo ordine,* Boll. U.M.I. (4) 7, (1973), 285-301.

[P] POHOZAEV S.J., *Eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Soviet Math. Doklady *6*, (1965), 1408-1411.