A SEMI lineNumber SECOND ORDER ELLIPTIC SYSTEM (*)

by PAUL EGBERTS (in Delft) (**) 

SOMMARIO. - In questa nota si considera un'equazione del tipo

$$\begin{cases} 
L u + \beta(u) \ni f(x,u) & \text{in } \Omega; \\
u = 0 & \text{su } \partial \Omega, 
\end{cases}$$

dove $\Omega \subset \mathbb{R}^n (n \geq 1)$ è un aperto con frontiera regolare, $L = \text{diag}(L_1, \ldots, L_N) (N \geq 1)$ è una matrice diagonale di operatori ellittici, $\beta$ è un grafico massimale monotono in $\mathbb{R}^N$ ed $f : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ è una funzione di tipo Caratheodory soddisfacente ad una condizione di crescita. Per questa equazione si prova un risultato di esistenza.

SUMMARY. - In this note we consider an equation of the form

$$\begin{cases} 
L u + \beta(u) \ni f(x,u) & \text{in } \Omega; \\
u = 0 & \text{on } \partial \Omega, 
\end{cases}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is an open set with smooth boundary $\partial \Omega$, $L = \text{diag}(L_1, \ldots, L_N) (N \geq 1)$ is a diagonal matrix of second order elliptic operators, $\beta$ is an $m$-accretive graph in $\mathbb{R}^N$ and $f : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is a given Caratheodory function satisfying some growth condition. We prove an existence result for this system.

1. Introduction.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with a smooth boundary and let $\mathbb{R}^N$ be equipped with some innerproduct denoted by $\langle \cdot, \cdot \rangle$ and norm $| \cdot | := \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. In this note we consider a system of the type

$$L u + \beta(u) \ni f ,$$  \hspace{1cm} (1)

(*) Pervenuto in Redazione il 27 dicembre 1991.
(**) Indirizzo dell'Autore: Faculty of Mathematics and Informatics – Delft University of Technology – Mekelweg 4, 2628 CD Delft (Netherlands).
where $L$ is an $m$-accretive operator in $L^2(\Omega; \mathbb{R}^N)$ such that $L - cI$ is accretive for some constant $c > 0$, $\beta$ an $m$-accretive (equivalently maximal monotone) graph in $(\mathbb{R}^N, \langle \cdot, \cdot \rangle)$ containing the origin and $f$ a given function in $L^2(\Omega; \mathbb{R}^N)$ or, more generally, $f$ depending on $u$. As usual we shall identify a graph with its corresponding nonlinear, possibly multivalued operator.

As a consequence of a result due to Brezis and Nirenberg [B-N, Theorem III.6', Remark III.5], it follows that if the set

$$\{ u \in D(L) : ||u||_1 \leq 1, ||Lu||_1, ((Lu, u)) \leq 1 \},$$

is relatively compact in $L^1(\Omega; \mathbb{R}^N)$ and $\beta$ single-valued, continuous, everywhere defined satisfying

$$\forall R > 0 \exists C_R > 0 \text{ such that } \langle x, \beta(x) \rangle \geq R|\beta(x)| - C_R \forall x \in \mathbb{R}^N,$$

then, for $f \in L^2(\Omega; \mathbb{R}^N)$, there exists $u \in L^2(\Omega; \mathbb{R}^N)$, such that $\beta(u) \in L^1(\Omega; \mathbb{R}^N)$, satisfying the system

$$\tilde{L}u + \beta(u) = f,$$

where $\tilde{L}$ denotes the closure of $L$ in $L^2(\Omega; \mathbb{R}^N) \times L^1(\Omega; \mathbb{R}^N)$.

It is shown that if $\beta$ is a subdifferential, $\beta = \partial \varphi$, with $\varphi : \mathbb{R}^N \to \mathbb{R}^+$ convex and of class $C^1$ then $\beta$ satisfies the above inequality (see [B-N, Remark III.4]).

We obtain stronger results if the operator $L$ in $L^2(\Omega; \mathbb{R}^N)$ is of the form

$$Lu = (L_1 u_1, L_2 u_2, \ldots, L_N u_n), \quad u = (u_1, u_2, \ldots, u_N),$$

where $L_1, L_2, \ldots, L_N$ are $N$ strictly elliptic $m$-accretive second order differential operators in $L^2(\Omega)$ with smooth coefficients. We show that equation (1), where $\beta$ is not necessarily a subdifferential, has a unique solution $u \in \{ W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \}^N$ for all $f \in L^2(\Omega; \mathbb{R}^N)$. In fact we show an existence result for equation (1), where $f$ depends on $u = (u_1, u_2, \ldots, u_N)$ using a fixed point theorem (see also [E]).
2. An existence result to a semilinear second order elliptic system.

We define, for \( k = 1, \ldots, N \), the following elliptic differential operators

\[
L_k u = - \sum_{i,j}^n \frac{\partial}{\partial x_i} (a_{ij}^{(k)} \frac{\partial u}{\partial x_j}) + \sum_i^n \frac{\partial}{\partial x_i} (a_i^{(k)} u) + a^{(k)} u,
\]

for \( u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \), where

\[
\begin{align*}
& a_{ij}^{(k)}, a_i^{(k)} \in C^1(\Omega), a^{(k)} \in L^\infty(\Omega), \quad i, j = 1, \ldots, n; \\
& \sum_{i,j}^n a_{ij}^{(k)} \xi_i \xi_j \geq \alpha \sum_i^n \xi_i^2 \text{ on } \Omega, \quad \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \text{ for some } \alpha > 0; \\
& a^{(k)} \geq 0, \quad a^{(k)} + \frac{1}{2} \sum_i \frac{\partial a_i^{(k)}}{\partial x_i} + \alpha \lambda_0 \geq \delta \text{ a.e. for some } \delta > 0.
\end{align*}
\]

Here \( \lambda_0 \) denotes the first eigenvalue of \(-\Delta\) with Dirichlet boundary condition. Now we define the operator \( L \) in \( L^2(\Omega; \mathbb{R}^N) \) by

\[
\begin{align*}
\{ D(L) &= \{ W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \}^N \subset L^2(\Omega; \mathbb{R}^N); \\
L u = (L_1 u_1, L_2 u_2, \ldots, L_N u_N) \text{ for } u = (u_1, u_2, \ldots, u_N) \in D(L).
\end{align*}
\]

The function \( f = (f_1, f_2, \ldots, f_N) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) is assumed to satisfy the following two conditions

\[
(H) \begin{cases}
(i) & f(\cdot, x) \text{ is measurable for all } x \in \mathbb{R}^N \text{ and } f(\omega, \cdot) \text{ is} \\
& \text{continuous, } \omega \in \Omega \text{ almost everywhere (the Caratheodory condition)}.

(ii) & |f(\omega, x)| \leq h(\omega) + c|x|^\beta \text{ for some } h \in L^2(\Omega), \ c \in \mathbb{R}^+ \\
& \text{and } 0 \leq \beta < 1, \text{ for all } x \in \mathbb{R}^N, \ \omega \in \Omega \text{ almost} \\
& \text{everywhere.}
\end{cases}
\]

The Niemytski operator \( F \) is defined by \( F(u)(\omega) = f(\omega, u(\omega)) \), for \( u \in L^2(\Omega; \mathbb{R}^N), \ \omega \in \Omega \) almost everywhere. It is well-known that the operator \( F \) defines a bounded and continuous operator in \( L^2(\Omega; \mathbb{R}^N) \) (see for example [BD]).

Let us denote the inner product in \( L^2(\Omega; \mathbb{R}^N) \) by \((\cdot, \cdot)\) and \( \| \cdot \| := ((\cdot, \cdot))^{\frac{1}{2}} \). The standard inner product in \( L^2(\Omega) \) is denoted by \((\cdot, \cdot)\) and \( \| \cdot \|_2 := (\cdot, \cdot)^{\frac{1}{2}} \).
Now we can state the existence result.

**THEOREM 1.** Let the operator $L$, the function $f$ and the real number $\delta > 0$ be as above and let $\beta$ be an $m$-accretive graph in $\mathbb{R}^N$ such that $0 \in D(\beta)$. Then there exists $u = (u_1, u_2, \ldots, u_N) \in \{W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)\}^N$ satisfying equation (1). Moreover if the function $g = (g_1, g_2, \ldots, g_N) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies (H) and if $u$, $v$ are solutions of (1) with right-hand side $g$ respectively $f$ then $||u - v|| \leq \frac{1}{\delta}||Gu - Fv||$, where $G$ is the Niemytski operator induced by $g$.

An important tool in the proof of this theorem is the following inequality due to Sobolevskii [SO]:

Let $M$ and $N$ be two second order strictly elliptic operators with bounded measurable coefficients and leading coefficients belonging to $C^1(\bar{\Omega})$. Then there exist constants $a > 0, b \geq 0$ such that

$$ (Mu, Nu) \geq a||u||_{W^{2,2}(\Omega)}^2 - b||u||_2^2 \text{ for all } u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega). $$

(2)

See also [L-U, page 182], [B-E] and [SK].

Another crucial inequality we use is contained in

**LEMMA 2.** Let $L_0$ be the operator in $L^2(\Omega; \mathbb{R}^N)$ defined by

$$
\begin{align*}
D(L_0) &= \{W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)\}^N; \\
L_0 &= (-\Delta, \ldots, -\Delta).
\end{align*}
$$

Let $\beta$ be an $m$-accretive graph in $(\mathbb{R}^N, \langle \cdot, \cdot \rangle)$ satisfying $0 \in \beta(0)$. Then

$$
\int_{\Omega} \langle L_0 u(\omega), \beta_\lambda(u(\omega)) \rangle d\omega \geq 0, \text{ for all } u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega), \lambda > 0,
$$

where $\beta_\lambda, \lambda > 0$, denotes the Yosida-approximation of $\beta$, that is, $\beta_\lambda = \frac{1}{\lambda}(I - (I + \lambda\beta)^{-1})$.

**Proof.** We note that this lemma is a special case of [C-E, Theorem 1.1]. However, the inequality can be proven directly by partial integration as we will indicate.

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(1) The author thanks Patrick Fitzpatrick for pointing out these references.
Observe that \( 0 = \beta_0(0) \) since \( 0 \in \beta(0) \) and recall that \( \beta_\lambda = (\beta_{\lambda,1}, \ldots, \beta_{\lambda,N}) \) is Lipschitz continuous. Let \( C = (c_{ij})_{i,j=1}^N \) be the positive definite \( N \times N \) matrix such that \( \langle x, y \rangle = x^T C y, \ x, y \in \mathbb{R}^N \). Since the Yosida-approximation \( \beta_\lambda \) is \( m \)-accretive in \( (\mathbb{R}^N, \langle \cdot, \cdot \rangle) \) as well, we have

\[
\sum_{i,j=1}^N c_{ij} \sum_{l=1}^N \frac{\partial \beta_{\lambda,i}}{\partial x_l}(u) \xi_i \xi_j, \quad \text{for all } u, \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N.
\]

Then for \( u = (u_1, \ldots, u_N) \in D(L_0) \) we obtain

\[
\int_\Omega \langle L_0 u(\omega), \beta_\lambda(u(\omega)) \rangle \, dx = \int_\Omega \sum_{i,j=1}^N c_{ij}(\Delta) u_j(x) \beta_{\lambda,i}(u(x)) \, dx
\]

\[
= -\sum_{k=1}^n \sum_{i,j=1}^N c_{ij} \int_\Omega \frac{\partial^2 u_j}{\partial x_k^2}(x) \beta_{\lambda,i}(u(x)) \, dx
\]

\[
= \sum_{k=1}^n \sum_{i,j=1}^N c_{ij} \int_\Omega \frac{\partial u_j}{\partial x_k}(x) \frac{\partial}{\partial x_k} \beta_{\lambda,i}(u(x)) \, dx
\]

\[
= \sum_{k=1}^n \sum_{i,j=1}^N c_{ij} \int_\Omega \frac{\partial u_j}{\partial x_k}(x) \sum_{l=1}^N \frac{\partial \beta_{\lambda,i}}{\partial x_l}(u(x)) \frac{\partial u_l}{\partial x_k}(x) \, dx
\]

\[
= \sum_{k=1}^n \sum_{i,j=1}^N c_{ij} \sum_{l=1}^N \frac{\partial \beta_{\lambda,i}}{\partial x_l}(u(x)) \frac{\partial u_l}{\partial x_k}(x) \frac{\partial u_j}{\partial x_k}(x) \, dx \geq 0.
\]

**Proof of Theorem 1.** Since \( \Omega \) is bounded we may assume that \( 0 \in \beta(0) \), otherwise consider the \( m \)-accretive operator \( \beta_0 := \beta - x_0 \), where \( x_0 \in \beta(0) \). It is well-known that the operators \( L_k, k = 1, \ldots, N \) with domain \( D(L_k) = W^{2,2} \cap W^{1,2}_0(\Omega) \) are \( m \)-accretive in \( L^2(\Omega) \), see [B-S]. Thus \( L \) is an \( m \)-accretive operator in \( L^2(\Omega; \mathbb{R}^N) \). Let the operator \( L_0 \) be as in Lemma 2 and define the operator \( B \) in \( L^2(\Omega; \mathbb{R}^N) \) by

\[
u \in D(B), \ u \in Bu \text{ if and only if } u, v \in L^2(\Omega; \mathbb{R}^N)
\]

and \( v(\cdot) \in \beta(u(\cdot)) \) a.e.

The operator \( B \) is an \( m \)-accretive operator in \( L^2(\Omega; \mathbb{R}^N) \).
We will show that the operator \((L + B)^{-1} F : L^2(\Omega; \mathbb{R}^N) \rightarrow L^2(\Omega; \mathbb{R}^N)\) is compact and then, using a fixed point theorem, the result will follow. We prove first that the operator \(L + B\) is surjective. For that, consider the equation
\[
\varepsilon u_\lambda + Lu_\lambda + Bu_\lambda = h, \quad \varepsilon, \lambda > 0,
\]
where \(h \in L^2(\Omega; \mathbb{R}^N)\). By a contraction argument one shows that this approximate equation has a unique solution \(u_\lambda \in D(L)\) (see [B, page 34]). By taking the innerproduct in \(L^2(\Omega; \mathbb{R}^N)\) with \(L_0 u_\lambda\) we get
\[
\varepsilon ((u_\lambda, L_0 u_\lambda)) + ((Lu_\lambda, L_0 u_\lambda)) + ((Bu_\lambda, L_0 u_\lambda)) = ((h, L_0 u_\lambda)) .
\]
By Lemma 2,
\[
((Bu_\lambda, L_0 u_\lambda)) \geq 0, \quad \text{for all } \lambda > 0 .
\]
Hence \(((Lu_\lambda, L_0 u_\lambda)) \leq \|f\| \|L_0 u_\lambda\|\). Using inequality (2) we obtain that there exist constants \(a > 0, b \geq 0\) such that
\[
((Lu_\lambda, L_0 u_\lambda)) \geq a \|L_0 u_\lambda\|^2 - b \|u_\lambda\|^2, \quad \text{for all } \lambda > 0 . \quad (3)
\]
It follows that \(\|L u_\lambda\|\) remains bounded if \(\lambda \downarrow 0\). Hence by [B, Théorème 2.4], the operator \(L + B\) is \(m\)-accretive in \(L^2(\Omega; \mathbb{R}^N)\). Thus, there exists a unique \(u_\varepsilon \in D(L) \cap D(B)\) satisfying
\[
\varepsilon u_\varepsilon + Lu_\varepsilon + Bu_\varepsilon \ni h \quad (3)
\]
for all \(\varepsilon > 0\). By the assumptions on the coefficients of the operators \(L_k\) we have that \(\delta \|u\|^2 \leq (L_k u, u)\) for all \(u \in D(L_k), k = 1, \ldots, N\) and therefore
\[
\delta \|u\|^2 \leq ((Lu, u)) \quad \text{for all } u \in D(L) . \quad (4)
\]
Using this estimate we get
\[
\|u_\varepsilon\| \leq \frac{1}{\delta} \|h\|, \quad \text{for all } \varepsilon > 0 .
\]
By passing to the limit it follows that there exists a unique \(u \in D(L) \cap D(B)\) such that \(Lu + Bu \ni h\). Since the function \(h \in L^2(\Omega; \mathbb{R}^N)\) is arbitrary, the surjectivity of \(L + B\) is proved.
Observe, using (3) and (4), that \( ||Lu|| \leq c||h|| \) for some constant \( c > 0 \). Therefore the operator

\[
(L + B)^{-1} : L^2(\Omega; \mathbb{R}^N) \to D(L)
\]

is bounded (\( D(L) \) equipped with the graph norm of \( L \)). By a well-known embedding theorem, \( D(L) \) is compactly embedded in \( L^2(\Omega; \mathbb{R}^N) \). Thus we may conclude that the operator

\[
(L + B)^{-1}F : L^2(\Omega; \mathbb{R}^N) \to L^2(\Omega; \mathbb{R}^N)
\]

is compact. If we can show that the set

\[
S = \{ v \in L^2(\Omega; \mathbb{R}^N) : v = \sigma(L + B)^{-1}Fv, \sigma \in [0, 1] \}
\]

is bounded in \( L^2(\Omega; \mathbb{R}^N) \) it follows that \( (L + B)^{-1}F \) has a fixed point (see for example [G-T]). Taking the innerproduct in \( L^2(\Omega; \mathbb{R}^N) \) of \( Lv + Bv \equiv \sigma F(v) \) with \( v \) and using (4), (H) and that \( ((Bu, u)) \geq 0 \) for all \( u \in D(B) \), we obtain that for all \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that

\[
\delta ||v||^2 \leq \sigma((F(v), v)) \leq \sigma(\varepsilon ||v||^2 + C_\varepsilon),
\]

and the boundedness of the set \( S \) is proved. Since \( (L + B)^{-1} \) is Lipschitz continuous with constant \( \frac{1}{\delta} \) the last statement of the theorem follows.

**REFERENCES**


[C-E] Clément Ph. and Egberts P., *On the sum of two maximal monotone operators*, Diff. and Int. Eq. 6 (1990), 1113-1124.


