REMARKS ON SEQUENTIAL ENVELOPES (*)

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SOMMARIO. - *Questo lavoro è dedicato alla teoria degli inviluppi sequenziali. Come è noto, l'inviluppo sequenziale di uno spazio di convergenza è un concetto analogo a quello di compattificazione di Čech - Stone. In questo articolo, dopo aver richiamato i risultati principali di tale teoria, si prova che l'inviluppo sequenziale commuta con i prodotti finiti. Alcuni problemi aperti sono infine proposti per un'ulteriore ricerca.*

SUMMARY. - *This paper is devoted to the theory of sequential envelopes. As known, the sequential envelope of a convergence space is a sequential analogue of the Čech - Stone compactification. In this article we recall the basic notions and facts about such theory and prove that the sequential envelope commutes with finite products. Some open problems are then proposed for a further research.*

The sequential envelope was introduced by J. Novák in [NO1] as a sequential analogue of the Čech-Stone compactification. The construction of the sequential envelope and some of its basic properties (with full proofs) can be found in [NO2]. Further generalization and their properties were subsequently given in [NO3], [KOU], [FR1], [FR2], [FRK], [FMR], [FRH] and [MIS]. Even though the theory of sequential envelopes underwent a remarkable development, no definite results concerning the sequential envelopes of products were achieved. The only paper dealing with products is [FR3], where a sufficient condition for the sequential envelope of the product of two spaces to be the product of their respective envelopes is given and it is shown via an example that the condition is not a necessary one.

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The purpose of the present paper is twofold. First, we recollect basic notions and facts about sequential regularity, sequential completeness, and the sequential envelopes. This will make the paper self-contained and also will serve the reader as an introduction to the last two sections. The second part deals with the sequential envelopes of products. We prove that the sequential envelope of the product of two sequentially regular convergence spaces is the product of their respective sequential envelopes. This extends easily to all finite products. Similar assertion holds also for the $\{0,1\}$-sequential envelope. In the last section we present several problems concerning sequential envelopes and their generalizations. We hope that this will stimulate further research in this area and this is the second purpose of the paper.

1. By a *convergence space* we understand a nonempty set equipped with a sequential convergence satisfying all four axioms of convergence (in particular, we assume unique limits and the Urysohn axiom) and the associated sequential closure operator.

Let X and Y be convergence spaces. Denote by $C(X,Y)$ the set of all sequentially continuous mappings of X into Y. If Y is the real line R, then $C(X,R)$ is condensed to $C(X)$ and $C^*(X)$ denotes the set of all bounded functions in $C(X)$.

Let X and Y be convergence spaces and let C_0 be a subset of $C(X,Y)$. Let X be a subspace of a convergence space $\overline{X}$. If each $f \in C_0$ can be extended over $\overline{X}$ to a continuous mapping $\overline{f}$ into Y (i.e. $\exists \overline{f} \in C(\overline{X},Y)$, $\overline{f} \upharpoonright X = f$), then X is said to be C_0- *embedded* in $\overline{X}$. Observe that if X is *top-dense* in $\overline{X}$, i.e., no proper subset of $\overline{X}$ containing X is sequentially closed in $\overline{X}$, then (due to the uniqueness of limits) each $f \in C(X,Y)$ has at most one continuous extension $\overline{f} \in C(\overline{X},Y)$.

Let X and Y be convergence spaces and let C_0 be a subset of $C(X,Y)$. Then X is said to be C_0-*sequentially regular* if the convergence in X is projectively generated by C_0, i.e., a sequence $<x_n>$ converges in X to $x \in X$ iff for each $f \in C_0$ the sequence $<f(x_n)>$ converges in Y to $f(x)$. Observe that in the C_0- sequentially regular spaces X (due to the uniqueness of limits) C_0 separates points of X. A sequence $<x_n>$ in X is said to be C_0-*fundamental* if for each $f \in C_0$ the sequence $<f(x_n)>$ converges in Y, and X is said to be C_0-*sequentially complete* if each C_0-fundamental sequence converges in X. Usually, C_0- sequential completeness is restricted to C_0-sequentially regular spaces; throughout the paper we stick to this restriction. If E is a subspace of R and $C_0 = C(X,E)$, then instead of C_0- sequential regularity we speak of *E-sequential regularity* and, further, if $E = R$, then we speak of *sequential regularity*. Similar convention is used for *E-sequential completess* and *sequential completeness*.

Let X be a convergence space and let E be a subspace of R. Then the relationship between various types of E-sequential regularity, resp. between various types of E-sequential completeness, can be summarized as follows (cf. [FR4], [MIS]):

(SR1) If X is C_0-sequentially regular for some $C_0 \subset C(X)$ and $C_0 \subset C_1 \subset C(X)$, then X is C_1- sequentially regular;

(SR2) If X is E-sequentially regular and E does not contain any interval, then X is $\{0,1\}$-sequentially regular and also F- sequentially regular for each $F \subset R, \| F \| >1$;

(SR3) If X is E-sequentially regular and E contains an interval, then X is F-sequentially regular for each $F \subset R$, F contains an interval;

(SR4) Every interval $[a,b]$, $a \neq b$, is sequentially regular but fails to be $\{0,1\}$-sequentially regular;

(SC1) If X is C_0-sequentially complete for some $C_0 \subset C(X)$ and $C_0 \subset C_1 \subset C(X)$, then X is C_1- sequentially complete;

(SC2) If X is E-sequentially complete and E does not contain any interval, then X is $\{0,1\}$-sequentially complete;

(SC3) If X is E-sequentially complete and E contains an interval, then X is F-sequentially complete for each $F \subset R$, F contains an interval;

(SC4) There is a $\{0,1\}$-sequentially regular space X which is sequentially complete but fails to be $\{0,1\}$-sequentially complete (the space X is contructed in [MIS] under a set-theoretic assumption weaker than the continuum hypothesis).

Recall that the class of all E-sequentially regular spaces is closed with respect to products and subspaces. The class of all E-sequentially complete spaces is closed with respect to products and closed subspaces. Additional information about C_0-sequentially regular and C_0-sequentially complete spaces can be found in [NO3], [FR4], [FRK], [FMR], [FRH] and [MIS].

DEFINITION. Let X be a convergence space and let C_0 be a subset of $C(X)$. Let X be C_0-sequentially regular. Let $\overline{X}$ be a convergence space such that:

(E1) X is a top-dense C_0-embedded subspace of $\overline{X}$;

(E2) $\overline{X}$ is $\overline{C}_0$-sequentially regular, where $\overline{C}_0 = \{\bar{f} \in C(\overline{X},Y);\ \bar{f} \restriction X \in C_0\}$;

(E3) $\overline{X}$ is $\overline{C}_0$-sequentially complete.

Then $\overline{X}$ is said to be a C_0-sequential envelope of X; we use to denote this fact by putting $\overline{X} = \sigma_{C_0}X$ and for $C_0 = C(X)$ we condense $\sigma_{C_0}X$ to σX.

Let X be a C_0-sequentially regular space, $C_0 \subset C(X)$. The properties of C_0-sequential envelopes can be summarized ad follows:

(SE1) $\sigma_{C_0}X$ always exists and it is uniquely determined up to a homeomorphism pointwise fixed on X (cf. Theorem 5 in [NO3] and Theorem 2.2 in [FRK]);

(SE2) X is C_0-sequentially complete iff $\sigma_{C_0}X = X$ (cf. Corollary 2.3 in [FRK]);

(SE3) $\sigma_{C_0}X$ is a C_1-sequential envelope of X for each C_1 such that $C_0 \subset C_1 \subset C(\sigma_{C_0}X) \restriction X$ (cf. Corollary 2.4 in [FRK]);

(SE4) Let φ be a sequentially continuous mapping of (the C_0-sequentially regular space) X into a C_1-sequentially regular space Y, where $C_1 \subset C(Y)$. If for each $f \in C_1$ we have $f \circ \varphi \in C_0$, then φ can be uniquely extended to a continuous mapping $\overline{\varphi}$ of $\sigma_{C_0}X$ into $\sigma_{C_1}Y$ (i.e. $\exists\,!\ \overline{\varphi} \in C(\sigma_{C_0}X, \sigma_{C_1}Y), \overline{\varphi} \restriction X = \varphi$) (cf. Theorem 3.1 in [FRK]).

In particular, property (SE4) implies that the sequential envelope yields an epireflector of the category of all sequentially regular spaces into its subcategory of sequentially complete spaces.

Let X be a convergence space and E a subspace of R containing at least two points. Assume that X is E-sequentially regular. Then there are two possibilites:

1. E contains no interval. Then, according to (SR2), X is $\{0,1\}$-sequentially regular and also F-sequentially regular for each $F \subset R$, $\|F\| > 1$. Then (cf. Corollary 2.4 in [FR4]), either $\sigma_F X = \sigma_{\{0,1\}}X$, or $\sigma_F X = \sigma X$ (here $\sigma_E X = \sigma_F X$ means that both sequential envelopes exist and that they are homeomorphic with the homeomorphism pointwise fixed on X).

2. E contains an interval. Then X is either $\{0,1\}$-sequentially regular and hence F-sequentially regular for each $F \subset R$, $\|F\| > 1$, or X is not $\{0,1\}$-sequentially regular and hence, by (SR2), X fails to be F-sequentially regular for each $F \subset R$ which does not contain an interval. In the first case X can have at most two different F-sequential envelopes, viz., $\sigma_{\{0,1\}}X$ and σX, in the second case X has the only F-sequential envelope, viz., σX.

The previous consideration can be summarized as follows:

(SE5) Let X be a sequentially regular convergence space and let F be a subspace of R, $\|F\| > 1$. If X is F- sequentially regular, then either $\sigma_F X = \sigma_{\{0,1\}} X$, or $\sigma_F X = \sigma X$.

(SE6) Let X be a sequentially regular convergence space. Then $\sigma_{C^*(X)} X = \sigma X$.

(SE7) There is a $\{0,1\}$-sequentially regular convergence space X such that $\sigma_{\{0,1\}} X \neq \sigma X$ (cf. [FRH]).

EXAMPLE 1. Let (X,m) be a metric space and let U be the set of all uniformly continuous functions on X. Then X equipped with the usual metric convergence is U-sequentially regular. Let $(\overline{X,m})$ be the completion of (X,m). Then $\overline{X}$ is the U-sequentially envelope of X. E.g., R is the U-sequential envelope of Q.

EXAMPLE 2 ([FR2]). Let $B(R)$ be the set of all Borel measurable real function equipped with the pointwise convergence. For $r \in R$, denote by ev_r the mapping of $B(R)$ into R defined by $ev_r(f) = f(r)$, and put $C_0 = \{ev_r; r \in R\}$. Then each ev_r is a sequentially continuous function on $B(R)$, $B(R)$ is C_0-sequentially regular and C_0- sequentially complete. Further, $B(R)$ is the C_0-sequential envelope of its top-dense subspace $C(R)$.

EXAMPLE 3 ([NO3]). Let X be an infinite set. Consider an algebra A of subsets of X equipped with the usual convergence $A_n \to A$ iff $A = \lim\sup A_n = \lim\inf A_n$. Denote by P the set of all probability measures on A. Then each $p \in P$ is a sequentially continuous function on A and A is P- sequentially regular. Let $\sigma(A)$ be the generated σ-algebra of subsets of X. Then $\sigma(A)$ is the P- sequential envelope of A.

EXAMPLE 4 ([FRK]). Let X be a sequentially regular convergence space. Then $C^*(X)$ separates points of X. Denote by $\tilde{X}$ the set $|X|$ equipped with the weak topology with respect to $C^*(X)$. Then $\tilde{X}$ is completely regular and $C^*(X) = C^*(\tilde{X})$. Let $\beta\tilde{X}$ be the Čech-Stone compactification of $\tilde{X}$ and $\nu\tilde{X}$ the Hewitt realcompactification of $\tilde{X}$. Then $|\tilde{X}| \subset |\nu\tilde{X}| \subset |\beta\tilde{X}|$. Let $s(\beta\tilde{X})$ be the set $|\beta\tilde{X}|$ equipped with the sequential convergence in $\beta\tilde{X}$ and let $\overline{X}$ be the smallest sequentially closed subset of $s(\beta\tilde{X})$ containing $|X|$ and equipped with the inherited sequential convergence. Then $|X| = |\tilde{X}| \subset |\overline{X}| \subset |\nu\tilde{X}| \subset |\beta\tilde{X}|$ and $\overline{X}$ is the sequential envelope (by (SE6) also the $C^*(X)$-sequential envelope) of X.

Similarly, if X is $\{0,1\}$-sequentially regular, then $\sigma_{\{0,1\}} X$ can be obtained as the smallest sequentially closed set in the Banaschewski 0-dimensional compactification $\beta_0\tilde{X}$ of $\tilde{X}$ containing $|X|$ and equipped with the corresponding sequential convergence and closure.

EXAMPLE 5 ([BER]). Let X be a completely regular Fréchet space. Then X is also sequentially regular. If βX is a Fréchet space, then βX is also the sequential envelope of X. Under the set-theoretic assumption that all *MAD*-families on ω have the same cardinality, there is a completely regular Fréchet space X such that βX is also Fréchet and $\beta X \setminus X$ is a singleton.

EXAMPLE 6 ([FMR]). Let X be a convergence space. We say that in $C(X)$ a sequence $<f_n>$ *converges continuously* to $f \in C(X)$, in symbols $f_n \xrightarrow{c} f$, if the sequence $<f_n(x_n)>$ converges to $f(x)$ for each sequence $<x_n>$ converging in X to x. This defines the so-called sequential *continuous convergence* for $C(X)$ which satisfies all four axioms of convergence. The set $C(X)$ equipped with the continuous convergence and the associated sequential closure will be denoted by $C_c(X)$. Then $C_c(X)$ is sequentially regular and also sequentially complete. Consider the dual space $C_c(C_c(X))$. For $x \in X$, denote by ev_x the mapping of $C(X)$ into R defined by $ev_x(f) = f(x)$. Then ev is a continuous mapping of X into $C_c(C_c(X))$. Assume that X is sequentially regular. Then ev is a homeomorphism of X onto the subspace $ev(X)$ of $C_c(C_c(X))$. Denote by δX the smallest sequentially closed subspace of $C_c(C_c(X))$ containing $|ev(X)|$. Then δX is sequentially complete and it is the sequential envelope of X. Further, if $f_n \xrightarrow{c} f$ in $C_c(X)$, then $\bar{f}_n \xrightarrow{c} \bar{f}$ in $C_c(\delta X)$, where for $g \in C(X)$ we denote by $\bar{g}$ its unique continuous extension over δX.

EXAMPLE 7. Let X be a convergence space. Denote by $D(X)$ the set $C(X,\{0,1\}) \subset C(X)$ and by $D_c(X)$ the corresponding subspace of $C_c(X)$. It is easy to verify that $D(X)$ is a closed subset of $C_c(X)$. Further, modifying the proofs of the corresponding proposition in [FMR], it can be shown that:

(D1) $D_c(X)$ is $\{0,1\}$-sequentially regular;
(D2) $D_c(X)$ is $\{0,1\}$-sequentially complete;
(D3) X is $\{0,1\}$-sequentially regular iff it is homeomorphic with the corresponding subspace of $D_c(D_c(X))$.

Let X be $\{0,1\}$-sequentially regular. Then the evaluation mapping $ev : X \to D_c(D_c(X))$ is a homeomorphism of X onto the subspace $ev(X)$ of $D_c(D_c(X))$. Denote by $\delta_D X$ the smallest sequentially closed subspace of $D_c(D_c(X))$ containing $|ev(X)|$. Then $\delta_D X$ is $\{0,1\}$-sequentially complete and it is the $\{0,1\}$-sequential envelope of X. Finally, if $f_n \xrightarrow{c} f$ in $D_c(X)$, then $\bar{f}_n \xrightarrow{c} \bar{f}$ in $D_c(\delta_D X)$, where for $g \in D(X)$ we denote by $\bar{g}$ its unique continuous extension over $\delta_D X$.

2. It is known (cf. [GLI]) that $\beta(X \times Y) = (\beta X) \times (\beta Y)$ iff $X \times Y$ is pseudocompact. Based on a lemma in [FRO], Ch. Todd in [TOD] gave a short proof of the sufficiency. Our proof of Theorem 1 is motivated by [TOD].

THEOREM 1. *Let X and Y be sequentially regular convergence spaces. Then $\sigma(X \times Y) = (\sigma X) \times (\sigma Y)$.*

Proof. Since X is top-dense in σX and Y is top-dense in σY, $X \times Y$ is top dense in $(\sigma X) \times (\sigma Y)$. The product $(\sigma X) \times (\sigma Y)$ of two sequentially complete spaces is sequentially complete. According to Theorem 2.2 in [FRK], if each continuous function on $X \times Y$ can be extended to a continuous function on $(\sigma X) \times (\sigma Y)$, then $(\sigma X) \times (\sigma Y)$ is the sequential envelope of $X \times Y$.

Let $f \in C(X \times Y)$. For fixed $x \in X$, define $f_1(x)(y) = f(x,y)$. Then $f_1(x) \in C(y)$. Denote by $C_c(Y)$ the set $C(Y)$ equipped with the continuous convergence (i.e. $g_n \xrightarrow{c} g$ in $C_c(Y)$ iff $y_n \to y$ implies $g_n(y_n) \to g(y)$). Then $f_1 : X \to C_c(Y)$ is a continuous mapping. Each $g \in C(Y)$ can be uniquely extended to $\bar{g} \in C(\sigma Y)$. Denote by $C_c(\sigma Y)$ the set $C(\sigma Y)$ equipped with the continuous convergence. By Theorem 15 in [FMR], the mapping sending each $g \in C(Y)$ to its unique extension $\bar{g} \in C(\sigma Y)$ is a homeomorphism of $C_c(Y)$ onto $C_c(\sigma Y)$. Hence we can consider f_1 as a continuous mapping of X into $C_c(\sigma Y)$. By Corollary 10 in [FMR], the space $C_c(\sigma Y)$ is sequentially complete. Since σX has the universal extension property (cf. Theorem 3.1 in [FRK]), f_1 can be extended to a continuous mapping $\bar{f}_1$ of σX into $C_c(\sigma Y)$. For $(x,y) \in (\sigma X) \times (\sigma Y)$ define $\bar{f}(x,y) = \bar{f}_1(x)(y)$. Clearly, $\bar{f} \upharpoonright (X \times Y) = f$. If $(x_n,y_n) \to (x,y)$ in $(\sigma X) \times (\sigma Y)$, then $x_n \to x$ in σX and $\bar{f}_1(x_n) \xrightarrow{c} \bar{f}_1(x)$ in $C_c(\sigma Y)$. Thus $\bar{f}(x_n, y_n) = \bar{f}_1(x)(y) \to \bar{f}_1(x)(y) = \bar{f}(x, y)$ and hence $\bar{f} \in C((\sigma X) \times (\sigma Y))$. This completes the proof.

COROLLARY 1. *The sequential envelope commutes with finite products.*

THEOREM 2. *Let X and Y be $\{0,1\}$-sequentially regural convergence spaces. Then $\sigma_{\{0,1\}}(X \times Y) = (\sigma_{\{0,1\}}X) \times (\sigma_{\{0,1\}}Y)$.*

Proof. The assertion can be proved virtually in the same way as Theorem 1, but instead of $C_c(X)$ we use $D_c(X)$ and its properties (see Example 7).

COROLLARY 2. *The* $\{0,1\}$*-sequential envelope commutes with finite products.*

3. Let X be a nonempty set, let A be an algebra of subsets of X and let $\sigma(A)$ be the generated σ-algebra of subsets of X. Then, cf. Example 3, $\sigma(A)$ can be considered as the sequential envelope of A with respect to the set P of all probability measures on A. A function f on A is said to be (sequentially) *uniformly continuous* if $(f(A_n) - f(B_n)) \to 0$ whenever $(A_n \div B_n) \to \emptyset$. Denote by U the set of all uniformly continuous functions on A. Clearly $P \subset U \subset C(A)$, and hence A is U- sequentially regular.

PROBLEM 1 (J.Novák). Is $\sigma(A)$ the U-sequential envelope of A ?

Observe that if each $f \in U$ can be continuously extended over $\sigma(A)$, then the answer is YES.

PROBLEM 2. Does the sequential envelope commute with infinite products?

In [KER], D.D. Kent and G.D. Richardson generalized the notion of a sequential envelope to certain categories of filter convergence spaces. In fact, they obtained two generalizations, viz., one via embedding the space in question to the Tychonoff cube (following the usual contruction of the Čech- Stone compactification and Novák's original construction of the sequential envelope) and the other one via embedding the space in question into the second dual $C_c(C_c(X))$ (following Example 6). They showed that, unlikely in the case of sequential envelopes, the two generalizations may differ.

PROBLEM 3. Does any of the generalized envelopes commute with finite or infinite products?(1)

(1) Recently, D.C. Kent and the author proved that under a natural condition the "little t-envelope" commutes with finite products while the "big T-envelope" does not commute with finite products even for the identity functor. The paper entitled "The finite product theorem for certain epireflections" will appear in Math. Nachr.

Let G be a convergence group. A convergence group H is said to be *G-sequentially regular*, if $x_n \to x$ in H iff for each continuous homomorphism $f{:}H \to G$ we have $f(x_n) \to f(x)$ in G.

PROBLEM 4. Is it possible to develop a theory of G-completions of G-sequentially regular groups similar to E-sequential envelopes?

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