AN INVARIANCE PROPERTY OF THE CRITICAL GROUPS (*)

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SOMMARIO. - *Il risultato principale di questo articolo è il seguente: sia* (φ_λ) *una famiglia di funzioni a valori reali definita su uno spazio di Hilbert reale e sufficientemente regolare per poter applicare la Teoria di Morse classica. Le applicazioni* $\lambda \to \varphi_\lambda(u)$ *e* $\lambda \to \nabla\varphi_\lambda(u)$ *siano continue in* λ, *uniformemente in* $u \in B[0, r]$. *Se per ogni* λ, u_λ *è il solo punto critico di* φ_λ *in* $B[0, r]$, *allora i gruppi critici* $C_n(\varphi_\lambda, u_\lambda)$ *non dipendono da* λ.
Il risultato è applicato ad un problema nonlineare ellitico.

SUMMARY. - *The main result in this article is as follow: suppose that* (φ_λ) *is a family of real-valued functions defined on real Hilbert space and sufficiently smooth such that the classical Morse Theory can be applied. Suppose that the mappings* $\lambda \to \varphi_\lambda(u)$ *and* $\lambda \to \nabla\varphi_\lambda(u)$ *are continuous in* λ, *uniformly in* $u \in B[0, r]$. *If for each* λ, u_λ *is the only critical point of* φ_λ *in* $B[0, r]$, *then the critical groups* $C_n(\varphi_\lambda, u_\lambda)$ *are independent of* λ.
An application to a nonlinear elliptic problem is given.

0. Introduction.

Let H be a real Hibert space and $(\varphi_\lambda) \subset C^1(H, R)$, $\lambda \in [0, 1]$ such that for every $\lambda \in [0, 1]$ $\nabla\varphi_\lambda(u)$ is locally lipschitzean. If there exists a closed ball $B[0, r]$, $r > 0$, such that for every λ, φ_λ satisfies the Palais-Smale condition over $B[0, r]$ and the mappings $\lambda \to \varphi_\lambda(u)$ and $\lambda \to \nabla\varphi_\lambda(u)$ are continuous in λ, uniformly in $u \in B[0, r]$, we prove that the critical groups $C_n(\varphi_\lambda, u_\lambda)$, $n \in N$, are independent of λ, provide u_λ the only critical point of φ_λ in $B[0, r]$. The proof is essentially based on continuity of critical groups [4].

As application of this property we give a proof of the following Dancer's result [1]: consider the boundary value problem

$$\text{(P)} \qquad -\Delta u = g(u) + h \text{ in } \Omega,\ u/\partial\Omega = 0,$$

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where Ω is bounded domain with smooth boundary, $g \in C^1(R, R)$ and such that $\sup_{t \in R} | g'(t) | < \infty$, $h \in L^2(\Omega)$.

If the interior of the range of g' contains an eigenvalue of the problem

$$-\Delta u = \lambda u \text{ in } \Omega, u/\partial\Omega = 0,$$

then there exists an $h \in L^2(\Omega)$ such that the problem (P) has at least two different solutions.

Other proofs of this result are given by LAZER [3], with the restrictive condition: the eigenvalue in the interior of the range of g' is of odd multiplicity, and in the general case by M. WILLEM [6] by using Bifurcation Theory and Morse Theory.

Our approach will be applied to second order Hamiltonian Systems in a subsequent paper.

1. Morse theory.

Let H be a Hilbert space, U an open neighbourhood of $u \in H$ and let $\varphi \in C^2(U, R)$. Let $L : H \to H$ be the linear operator defined implicitely by

$$(Lv, \omega) = \varphi''(u)(v.\omega)$$

we shall identify L and $\varphi''(u)$.

Assume that u is a critical point of φ. The *Morse index* of u is defined as the supremum of the dimensions of the vector subspaces of H on which $\varphi''(u)$ is negative definite. The critical point u will be said to be *non-degenerate* if $\varphi''(u)$ is invertible. The *critical groups* (over a field F) of an isolated critical point u of φ are defined by

$$C_n(\varphi, u) = H_n(\varphi^c \cap V, \varphi^c \cap V \setminus \{u\}), n = 0, 1, \ldots$$

where $c = \varphi(u)$, $H_n(A, B)$ denotes for each n, the n^{the} singular homology group of the pair (A, B) over a field F, V is a closed neighbourhood of u, $\varphi^c = \{x \in H \mid \varphi(x) \leq c\}$. By excision the critical groups are independent of V. If u is a non-degenerate critical point of φ with Morse index k, then

$$C_n(\varphi, u) = \delta_{n,k} F$$

$\varphi \in C^1(H, R)$ satisfies *the Palais-Smale condition over a closed subset* S of H if every sequence $(u_j) \subset S$ such that $(\varphi(u_j))$ is bounded and $\|\nabla \varphi(u_j)\| \to 0$ contains a convergent subsequence.

$\varphi \in C^{2\text{-}0}(H, R)$ if $\varphi \in C^1(H, R)$ and $\nabla \varphi$ is locally lipschitzean.

THEOREM 1. (Continuity of critical groups). *Let U be an open neighbourhood of v in a Hilbert space H and let $\varphi, \psi \in C^{2\text{-}0}(U, R)$. Assume that φ and ψ have v as the only critical point and satisfy the Palais-Smale condition over a closed ball $B[v, r] \subset U$. Then there exists $\eta > 0$ depending only upon φ, such that condition*

$$\sup_{u \in U} \left(|\psi(u) - \varphi(u)| + \|\nabla \psi(u) - \nabla \varphi(u)\| \right) \leq \eta$$

implies

$$\dim C_n(\psi, v) = \dim C_n(\varphi, v), n \in N.$$

Proof [4].

Now let $(\varphi_\lambda)_\lambda \subset C^{2\text{-}0}(H, R)$, $\lambda \in [0, 1]$. Assume that there exists a closed ball $B[0, s] \subset H$, $s > 0$ such that

(H_1) the mappings $\lambda \to \varphi_\lambda(u)$, $\lambda \to \nabla \varphi_\lambda(u)$ are continuous in λ, uniformly in $u \in B[0, s]$;

(H_2) for every $\lambda \in [0, 1]$, φ_λ satisfies (P.S.) over $B[0, s]$.

THEOREM 2. *Under assumptions (H_1) - (H_2), if for every $\lambda \in [0, 1]$ φ_λ has $u_\lambda \in B(0, s)$ as the only critical point and*

(H_3) *the mapping $\lambda \to u_\lambda \in B(0,s)$ is continuous, then the critical groups*

$$C_n(\varphi_\lambda, u_\lambda), n \in N$$

are independent of λ.

Proof. Let $\lambda_0 \in [0, 1]$, and let $\rho > 0$ be such that

$$B[u_{\lambda_0}, \rho] \subset B(0, s). \tag{1}$$

From (H_3), there exists $\delta > 0$ such that for every $\lambda \in [0, 1]$,

$$|\lambda - \lambda_0| < \delta \Rightarrow u + u_\lambda - u_{\lambda_0}, \in B\,[u_{\lambda_0}, \rho] \qquad (2)$$

for every $u \in B\,[u_{\lambda_0}, \rho/2]$. Define the mappings ψ and φ by

$$\varphi(u) = \varphi_{\lambda_0}(u);\ \psi(u) = \varphi_\lambda(u + u_\lambda - u_{\lambda_0}).$$

It follows that u_{λ_0} is the unique critical point of φ and ψ. Now let $\eta > 0$ be such in theorem 1. Since $\nabla\varphi_{\lambda_0}$ and φ_{λ_0} are continuous in u_{λ_0}, there exists $\delta_1 > 0$ such that

$$\|u_\lambda - u_{\lambda_0}\| \leq \delta_1 \Rightarrow |\varphi_{\lambda_0}(u) - \varphi_{\lambda_0}(u_{\lambda_0})| + \|\nabla\varphi_{\lambda_0}(u) - \nabla\varphi_{\lambda_0}(u_{\lambda_0})\| \leq n/2. \qquad (3)$$

Since $\delta_1 > 0$, one has from (H_1) and (H_3) the existence of $\varepsilon_1 > 0$ such that for every $\lambda \in [0, 1]$, $|\lambda - \lambda_0| < \varepsilon_1$ implies

$$|u_\lambda - u_{\lambda_0}| \leq \frac{\delta_1}{2} \qquad (4)$$

and

$$|\varphi_\lambda(u) - \varphi_{\lambda_0}(u)| + \|\nabla\varphi_\lambda(u) - \nabla\varphi_{\lambda_0}(u)\| \leq \frac{n}{2} \qquad (5)$$

for $u \in B\,[0, s]$.

With $r = \min(\delta_1, \frac{\rho}{2})$, $\varepsilon = \min(\varepsilon, \delta)$ and $U \equiv B\,(u_{\lambda_0}, \rho)$, if $|\lambda - \lambda_0| < \varepsilon$ then by (1) - (5) one has

$$\sup_{u \in U}(|\psi(u) - \varphi(u)| + \|\nabla\psi(u) - \nabla\varphi(u)\|) \leq \eta$$

and from theorem 1, one has

$$\dim C_n(\varphi_{\lambda_0}, u_{\lambda_0}) = \dim C_n(\varphi_\lambda, u_{\lambda_0}). \qquad (6)$$

On the other hand, the translation $h : H \rightarrow H$ defined by

$$h(u) = u + u_\lambda - u_{\lambda_0}$$

is an homeomorphisme so that

$$C_n(\varphi_\lambda, u_{\lambda_0}) \approx C_n(\varphi_\lambda, u_\lambda). \tag{7}$$

Thus, by (6) and (7), for every $\lambda_0 \in [0, 1]$ there exists $\varepsilon > 0$ such that $\lambda \in [0, 1]$ and $|\lambda - \lambda_0| \leq \varepsilon$ implies

$$\dim C_n(\psi_{\lambda_0}, u_{\lambda_0}) = \dim C_n(\psi_\lambda, u_\lambda).$$

Since [0, 1] is connected and compact, the theorem follows.

2. Application.

Consider the boundary value problem

(P) $$-\Delta u = g(u) + h, \Omega$$
$$u = 0 \qquad , \partial\Omega$$

where $\Omega \subset R^N$ is a bounded domain, $g \in C^1(R, R)$ and $h \in L^2(\Omega)$.

(H4) $$\sup_{t \in R} |g'(t)| < K.$$

Let $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots$ be the distinct eigenvalues of the problem

$$-\Delta u = \lambda u, \Omega$$
$$u = 0 \ , \partial\Omega.$$

Let H be the real Hilbert space $W_0^{1,2}(\Omega)$ with the inner product given by $(u, v) = \int_\Omega (\nabla u, \nabla v)\, dx$, and let $\|.\|$ be the corresponding norm.

Let $f : H \to R$ be defined by

$$f(u) = \int_\Omega [\frac{1}{2} |\nabla u|^2 - G(u)]\, dx, \tag{8}$$

where $G(s) = \int_0^s g(t)\,dt$. By (H_1), it is proved in [5] that $f \in C^2(H, R)$ and satisfies (P.S.) over any closed ball.

THEOREM 3. *Assume that the interior of the range of g' contains λ_k for some k. Then there exists an $h \in L^2(\Omega)$ such that (P) has at least two different solutions.*

Proof. Since critical points of $\varphi(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 - G(u) - hu\, dx$ are solutions of (P), whe shall proof the existence of $h \in L^2(\Omega)$ such that φ has at least two different critical points.

According to our assumption on g, there exist numbers t_0 and t_1 such that

$$\lambda_{k-1} < g'(t_0) < \lambda_\kappa < g'(t_1) < \lambda_{k+1}. \tag{9}$$

Let now $\{K_n\}_1^\infty$ be a sequence of compact subsets of Ω such that $K_m \subseteq K_{m+1}$ and $\Omega = \bigcup_{m=1}^{\infty} K_m$; for each $m \geq 1$, let f_m be a C^∞ function such that $f_m(x) = 1$ for all $x \in K_m$, $0 \leq f_m(x) \leq 1$ everywhere, the support of f_m is contained in Ω. For each $m \geq 1$, set

$$\omega_m^i(x) = f_m(x) \,.\, t_i\,,\; i=0,1,$$

and consider the eigenvalue problem

$$\begin{aligned} -\Delta u &= \mu g'(\omega_m^i)\,u, && \Omega \\ u &= 0 && ,\partial\Omega\,. \end{aligned}$$

By (H_1) and dominated convergence theorem

$$g'\left(\omega_m^i\right) \to g'(t_i)$$

in the $L^{N/2}$ - norm. Hence we obtain by [2, p. 16]

$$\mu_k\left(g'\left(\omega_m^i\right)\right) \to \mu_k\left(g'(t_i)\right) = \frac{\lambda_k}{g'\left(t_i\right)}. \tag{10}$$

By (9) and (10), let m_0 be sufficiently large so that the quadratic forms

$$\phi_i(u,u) = \int_\Omega |\nabla u|^2 - g'\left(\omega_{m_0}^i\right) u^2 dx$$

are non-degenerate and the Morse index of ϕ_i is the Morse index of

$$\overline{\Phi}_i(u,u) = \int_\Omega |\nabla u|^2 - g'\left(t_i\right) u^2 dx.$$

For every $\lambda \in [0, 1]$, set

$$u_\lambda \equiv \left(1-\lambda\right) \omega_{m_0}^0 + \lambda\omega_{m_0}^1 ; \; \overline{h}_\lambda \equiv \nabla f\left(u_\lambda\right) ;$$

and let $\varphi_\lambda : H \to R$ be defined by

$$\varphi_\lambda\left(u\right) = f\left(u\right) - (u, \overline{h}_\lambda) .$$

By (8), $\varphi_\lambda \in C^2(H, R)$ and satisfies (P.S.) over any closed ball. Let set

$$s = \max\left(2\|\omega_{m_0}^0\| + 1, 2\|\omega_{m_0}^1\| + 1\right).$$

Then

$$u_\lambda \in B\left(0, s\right) \text{ and } \nabla\varphi\left(u_\lambda\right) = 0$$

by definition of $\overline{h}_\lambda$ and φ_λ. Obviously, the mapping

$$\lambda \to u_\lambda$$

is continuous. On the other hand, since the mapping $\lambda \to \overline{h}_\lambda$ is continuous, it follows clearly from the definition of φ_λ that the mappings

$$\lambda \to \varphi_\lambda\left(u\right) \; ; \; \lambda \to \nabla\varphi_\lambda\left(u\right)$$

are continuous in λ, uniformly in $u \in B\,[0, s]$ and then it follows from theorem 1 that there exists $\lambda_0 \in [0, 1]$ such that φ_{λ_0} has at least two different critical points since by (9) $\omega_{m_0}^0$ and $\omega_{m_0}^1$ are respectively non-degenerate critical points of φ_0 and φ_1 with respectively Morse index $j_0(k)$ and $j_1(k)$ and such that $j_1(k) - j_0(k)$ is the multiplicity of λ_k. By Green formula one has

$$\left(\bar{h}_{\lambda_0}, v\right) = \left(\nabla f(u_{\lambda_0}), v\right) = \int_\Omega \nabla u_{\lambda_0} \nabla v - g(u_{\lambda_0})\, v\, dx$$

$$= \int_\Omega (-\Delta u_{\lambda_0} - g(u_{\lambda_0}))\, v\, dx$$

$h_{\lambda_0} \equiv \Delta u_{\lambda_0} - g(u_{\lambda_0}) \in L^2(\Omega)$, since u_{λ_0} is C^∞ and $g\left(u_{\lambda_0}\right) \in L^2(\Omega)$, and the theorem follows with $h = h_{\lambda_0}$.

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