REGULARITY THEOREMS IN LIMIT CASES FOR SOLUTIONS OF LINEAR AND NONLINEAR ELLIPTIC EQUATIONS (*)

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SOMMARIO. *In questo lavoro otteniamo risultati di regolarità per le soluzioni di equazioni ellittiche lineari e fortemente non lineari con termini di ordine inferiore* $Lu = -(f_i)_{x_i}$ *in un sottoinsieme aperto limitato* Ω *di* R^n. *Dimostriamo che u appartiene allo spazio di Orlicz* $L_\phi(\Omega)$ $(\phi(t) = \exp|t|^{n/(n-1)} - 1)$ *quando* $f_i \in L^{n/(p-1)}(\Omega)$, $i = 1, 2, \ldots, n$, *dove* $p = 2$ *nel caso lineare.*

SUMMARY. *In this paper we obtain regularity results for the solutions u of linear ad strongly nonlinear elliptic equations with lower order terms* $Lu = -(f_i)_{x_i}$ *in a bouded open subset* Ω *of* R^n. *We prove that u belongs to the Orlicz space* $L_\phi(\Omega)$ $(\phi(t) = \exp|t|^{n/(n-1)} - 1)$ *when* $f_i \in L^{n/(p-1)}(\Omega)$, $i = 1, 2, \ldots, n$, *where* $p = 2$ *in the linear case.*

1. INTRODUCTION.

The aim of this paper is to obtain regularity theorems for the weak solutions of the homogeneous Dirichlet problems for the linear uniformly elliptic equations with lower order terms

$$(1.1)\qquad -\left(a_{ij}(x)u_{x_i}\right)_{x_j} + \left(b_i(x)u\right)_{x_i} + c(x)u = -(f_i)_{x_i} \text{ in } \Omega$$

$$(1.2)\qquad -\left(a_{ij}(x)u_{x_i}\right)_{x_j} + b_i(x)u_{x_i} + c(x)u = -(f_i)_{x_i} \text{ in } \Omega$$

and for the weak solutions of the strongly nonlinear elliptic equation

$$(1.3)\qquad -\left(a_i(x, u, Du)\right)_{x_i} + h(x, u) = -(f_i)_{x_i} \text{ in } \Omega$$

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where Ω is a bounded open subset in R^n (see §3 and §4 for the assumptions about the coefficients).

Existence and uniqueness of the solutions of these problems are studied by many authors (see e.g. [16], [21], [7]).

Estimate formulas for the solutions of (1.1) and (1.2) are well-known when $f_i \in L^p(\Omega)$, $i = 1, ..., n$, and $2 \leq p$ (see [15], [16], [21]).

In [6] regularity theorems are proved for solutions u of (1.3) in the case that $f_i \in L^q(\Omega)$, $i = 1, ..., n$, and $q \gtrless n/(p-1)$.

Now, denoted by $L_\phi(\Omega)$ the Orlicz space defined by the function $\phi(t)$ (see [1], [13]) and by $\|u\|_\phi$ the norm of u in $L_\phi(\Omega)$, herein we prove that if u is the weak solution in $W_\circ^{1,2}(\Omega)$ of (1.1) or (1.2) then

$$\|u\|_\phi \leq c\,\|f\|_n \tag{1.4}$$

and if u is a weak solution in $W_\circ^{1,p}(\Omega)$ of (1.3), then

$$\|u\|_\phi \leq c\,\|f\|_{n/(p-1)} \tag{1.5}$$

where $f = \left(\sum_{i=1}^{n} f_i^2\right)^{\frac{1}{2}}$ and $\phi(t) = \exp|t|^{n/(n-1)} - 1$.

The estimate (1.4) improves a result due to Trudinger. In fact in [21] he proves that (1.4) holds with $\phi(t) = e^{|t|} - 1$ (actually in [21] are studied elliptic problems more general than (1.1) or (1.2)).

Moreover some years before Stampacchia ([15]) obtained the same estimate as Trudinger but for the case $b_i(x) = c(x) = 0$, $i = 1, ..., n$, and for uniformly elliptic equations.

Employing a Trudinger's imbedding theorem, which states that $W_\circ^{1,n}(\Omega) \hookrightarrow L_\phi(\Omega)$, in [10] (1.4) is proved with $\phi(t) = \exp|t|^{n/(n-1)} - 1$ and $b_i(x) = c(x) = 0$, $i = 1, ..., n$. In [10] the authors consider equations of degenerate type.

To prove (1.4) and (1.5) we compare u with a solution $v(x)$ of a spherically symmetric problem by isoperimetric inequalities and symmetrization techniques. Since $v(x)$ is explicitely calculated, this comparison allows either to derive (1.4) and (1.5) or to find known estimates for L^p-norm of u (see [16], [21], [6]) making explicit the numerical values of the constants.

Comparison theorems for (1.1) and (1.2) with the right-hand side equal to $f(x)$ are proved in [19] and [4]; in the last one the coefficients are supposed bounded.

Furthermore see [18] for comparison theorems and sharp estimates of the L^p-norm of a solution u of nonlinear elliptic problems with the right-hand side equal to $f(x)$.

2. NOTATIONS AND PRELIMINAR RESULTS.

Throughout this paper we employ the summation convention.

We mostly use the same notations as Talenti in [17], as listed below

Ω = an open bounded subset of R^n,

$|\Omega|$ = the measure of Ω,

$\Omega^\#$ = the ball of R^n centered at the origin and with the same measure as Ω,

$C_n = \pi^{n/2}/\Gamma(1 + n/2)$ = the measure of the unit ball of R^n.

Given a measurable real-valued function $u(x)$ in R^n, we denote the *distribution function* of u by

$$\mu(t) = |\{x : |u(x)| > t\}|$$

and by

$$u^*(s) = \inf\{t \geq 0 : \mu(t) < s\} \quad , \quad s \geq 0,$$

and

$$u^\#(x) = u^*\left(C_n |x|^n\right) \quad , \quad x \varepsilon R^n,$$

respectively the *decreasing rearrangement* and the *spherically symmetric rearrangement* of u.

The functions u, u^* and $u^\#$ are equidistribuited (see [5]).

If $\phi(t)$ is a continuous function then (see [5], [12])

$$\int_\Omega \phi\left(|u(x)|\right) dx = \int_{\Omega^\#} \phi\left(u^\#(x)\right) dx. \tag{2.1}$$

For any measurable subset E of R^n

$$\int_E |u(x)|\, dx \leq \int_0^{|E|} u^*(s)\, ds \tag{2.2}$$

(see [5], [12]).

For any function $u \varepsilon W_\circ^{1,2}(\Omega)$

(2.3) $$nC_n^{\,1/n}\mu(t)^{1-1/n} \leq -\frac{d}{dt}\int_{|u|>t} |Du|dx$$

because of a Fleming-Rishel formula and a De Giorgi isoperimetric theorem (see [8], [9], [17], [19]), hence by Schwarz's inequality

(2.4) $$nC_n^{1/n}\mu(t)^{1-1/n} \leq \left(-\mu'(t)\right)^{\frac{1}{2}}\left(-\frac{d}{dt}\int_{|u|>t}|Du|^2\,dx\right)^{\frac{1}{2}}.$$

See also [5], [12], [17] for the main facts about the rearrangement of u.

Now, given a real-valued measurable function u in Ω, for s in $[0, |\Omega|]$ denote by $D(s)$ a subset of Ω such that

- meas $D(s) = s$,
- $s_1 < s_2 \Rightarrow D(s_1) \subset D(s_2)$,
- $D(s) = \{x\varepsilon\Omega : |u(x)| > t\}$ if $s = |\{x : |u(x)| > t\}|$.

If $f\varepsilon L^1(\Omega)$, $\int_{D(s)} f(x)\,dx$ is an absolutely continuous function in $[0, |\Omega|]$. Set

$$F(s) = \frac{d}{ds}\int_{D(s)} f(x)dx,$$

it is known the following

LEMMA 2.1 ([3]) - *If $f(x)\,\varepsilon L^p(\Omega)$, $p \geq 1$, there exists a sequence $\{f_k(s)\}$ of functions which are equidistribuited with $f(x)$ and such that*

$$f_k \rightharpoonup F \ \ in\ L^p\left(0,|\Omega|\right),$$

if $p > 1$, and

$$\lim_k \int_0^{|\Omega|} f_k(s)\,g(s)\,ds = \int_0^{|\Omega|} F(s)\,g(s)\,ds, \quad g\,\varepsilon\,BV([0,|\Omega|]),$$

if $p = 1$.

The following Lemma is a slight modification of Gronwall's Lemma (see [14] for the proof of the classical one)

LEMMA 2.2 - *Given the functions $\lambda, \gamma, \phi, \mu$ in $[\alpha, +\infty)$, suppose that $\lambda \geq 0, \gamma \geq 0$ and that $\lambda\gamma, \lambda\phi$ and $\lambda\mu$ belong to $L^1(\alpha, +\infty)$. If for a.e. $t \geq \alpha$*

$$\phi(t) \leq \mu(t) + \gamma(t) \int_t^\infty \lambda(\tau)\phi(\tau)d\tau,$$

then for a.e. $t \geq \alpha$

$$\phi(t) \leq \mu(t) + \gamma(t) \int_t^\infty \mu(\tau)\lambda(\tau) \exp\left(\int_t^\tau \lambda(r)\gamma(r)\,dr\right) d\tau .$$

3. COMPARISON AND REGULARITY RESULTS FOR SOLUTIONS OF LINEAR ELLIPTIC EQUATIONS.

Consider the following equations

$$(3.1) \qquad -\left(a_{ij}(x)\,u_{x_i}\right)_{x_j} + \left(b_i(x)\,u\right)_{x_i} + c(x)\,u = -(f_i)_{x_i} \text{ in } \Omega$$

$$(3.2) \qquad -\left(a_{ij}(x)\,u_{x_i}\right)_{x_j} + b_i u_{x_i} + c(x)\,u = -(f_i)_{x_i} \text{ in } \Omega$$

and assume

i) $a_{ij} \varepsilon L^\infty(\Omega)$, $i, j = 1,\dots,n$, and $a_{ij}(x)\,\xi_i\xi_j \geq |\xi|^2$, $\forall \xi \varepsilon R^n$, a. e. $x \varepsilon \Omega$;

ii) $b(x) = \left(\sum_{i=1}^n b_i^2\right)^{\frac{1}{2}} \varepsilon L^p(\Omega)$, $p>n$;

iii) $c \varepsilon L^{n/2}(\Omega)$ if $n>2$ or $c \varepsilon L^q(\Omega)$, $q>1$, if $n=2$, and $c(x) \geq 0$ *in* Ω;

iv) $f(x) = \left(\sum_{i=1}^n f_i^2\right)^{\frac{1}{2}} \varepsilon L^2(\Omega)$.

The i), ii), iii), iv) assure the existence of an unique weak solution $u \varepsilon W_\circ^{1,2}(\Omega)$ of the equations (3.1) and (3.2) (see [11], [16], [21]), that is

$$(3.3) \int_\Omega \left(a_{ij} u_{x_i} \phi_{x_j} - b_i u \phi_{x_i} + cu\phi\right) dx = \int_\Omega f_i \phi_{x_i} dx, \ \forall \phi \varepsilon W_\circ^{1,2}(\Omega),$$

(3.4) $\int_\Omega \left(a_{ij} u_{x_i} \phi_{x_j} + b_i u_{x_i} \phi + cu\phi\right) dx = \int_\Omega f_i \phi_{x_i}, \forall \phi \varepsilon W_\circ^{1,2}(\Omega),$

respectively.

We first prove our result about the solution $u(x)$ of (3.1).

THEOREM 3.1 - *It results that*

$$u^\#(x) \leq v(x) \quad a.e. \text{ in } \Omega^\# \tag{3.5}$$

where

$$v(x) = \int_{C_n|x|^n}^{|\Omega|} \frac{F(\sigma)d\sigma}{nC_n^{1/n}\sigma^{1-1/n}} \exp\left(\int_{C_n|x|^n}^{|\sigma|} \frac{B(r)dr}{nC_n^{1/n}r^{1-1/n}}\right) \tag{3.6}$$

is the weak solution in $W_\circ^{1,2}(\Omega^\#)$ *of the equation*

$$-\Delta v - \left(\frac{x_i}{|x|} B\left(C_n|x|^n\right)v\right)_{x_i} = \left(\frac{x_i}{|x|} F(C_n|x|^n)\right)_{x_i} \text{ in } \Omega^\# \tag{3.7}$$

and $B^2(s)$, $F^2(s)$, $s \varepsilon [0, |\Omega|]$ *are the functions related to* b^2 *and* f^2 *respectively in the sense of lemma 2.1.*

Proof- Let be $h > 0$. Insert the function

$$\phi(x) = \begin{cases} \frac{|u|-t}{h} sign\, u & , t<|u(x)|\leq t+h \\ sign\, u & , |u(x)|>t+h \\ 0 & , |u(x)|\leq t \end{cases} \tag{3.8}$$

in (3.3) to see that

$$\frac{1}{h}\int_{t<|u|\leq t+h} a_{ij} u_{x_i} u_{x_j} dx + \int_\Omega cu\phi dx = \frac{1}{h}\int_{t<|u|\leq t+h} b_i u u_{x_i} dx + \tag{3.9}$$

$$+\frac{1}{h}\int_{t<|u|\leq t+h} f_i u_{x_i} dx \leq \frac{t+h}{h}\int_{t<|u|\leq t+h} b|Du|dx + \frac{1}{h}\int_{t<|u|\leq t+h} f|Du|dx.$$

Moreover ellipticity condition i), (3.9), Hölder's inequality and iii) imply

$$\frac{1}{h}\int_{t<|u|\le t+h}|Du|^2\,dx\le$$

$$\le(t+h)\left(\frac{1}{h}\int_{t<|u|\le t+h}b^2dx\right)^{\frac{1}{2}}\left(\frac{1}{h}\int_{t<|u|\le t+h}|Du|^2dx\right)^{\frac{1}{2}}+$$

$$+\left(\frac{1}{h}\int_{t<|u|\le t+h}f^2dx\right)^{\frac{1}{2}}\left(\frac{1}{h}\int_{t<|u|\le t+h}|Du|^2dx\right)^{\frac{1}{2}}.$$

Next send h to 0 to get

$$\left(-\frac{d}{dt}\int_{|u|>t}|Du|^2dx\right)^{\frac{1}{2}}\le t\left(-\frac{d}{dt}\int_{|u|>t}b^2dx\right)^{\frac{1}{2}}+\left(-\frac{d}{dt}\int_{|u|>t}f^2dx\right)^{\frac{1}{2}}.$$

Let be $\mu(t)$ the distribution function of u. Last inequality and (2.4) imply

$$(3.10)\quad 1\le\frac{(-\mu'(t))^{\frac{1}{2}}}{nC_n^{1/n}\mu(t)^{1-1/n}}\left[t\left(-\frac{d}{dt}\int_{|u|>t}b^2dx\right)^{\frac{1}{2}}+\left(-\frac{d}{dt}\int_{|u|>t}f^2dx\right)^{\frac{1}{2}}\right].$$

According to Lemma 2.1, consider the functions $B^2(s)$ and $F^2(s)$ for which

$$(3.11)\qquad \int_{|u|>t}b^2dx=\int_0^{\mu(t)}B^2(s)ds,\ \int_{|u|>t}f^2dx=\int_0^{\mu(t)}F^2(s)ds,$$

and employ (3.11) in (3.10) to obtain

$$(3.12)\qquad 1\le\frac{-\mu'(t)}{nC_n^{1/n}\mu(t)^{1-1/n}}\left[tB(\mu(t))+F(\mu(t))\right].$$

Integrating (3.12) between 0 and t it follows that

$$t\le\int_{\mu(t)}^{|\Omega|}\frac{u^*(\sigma)B(\sigma)+F(\sigma)}{nC_n^{1/n}\,\sigma^{1-1/n}}\,d\sigma,$$

then

$$u^*(s) \leq \int_s^{|\Omega|} \frac{u^*(\sigma)B(\sigma) + F(\sigma)}{nC_n^{1/n}\,\sigma^{1-1/n}}\,d\sigma, \text{ for a.e. } s \geq 0,$$

by definition of $u^*(s)$.

Now apply Lemma 2.2 with $\phi(s) = u^*(s)$ and hence obtain

$$u^*(s) \leq \int_s^{|\Omega|} \frac{F(r)}{nC_n^{1/n}\,r^{1-1/n}}\,dr + \tag{3.13}$$

$$+ \int_s^{|\Omega|} d\sigma \left(\int_\sigma^{|\Omega|} \frac{F(r)dr}{nC_n^{1/n}\,r^{1-1/n}}\right) \frac{d}{d\sigma}\left[\exp\left(\int_s^\sigma \frac{B(r)dr}{nC_n^{1/n}\,r^{1-1/n}}\right)\right].$$

Since

$$\int_\sigma^{|\Omega|} F(r)\left(nC_n^{1/n}\right)^{-1} r^{1/n-1}dr \text{ and } \exp\left(\int_s^\sigma B(r)\left(nC_n^{1/n}\right)^{-1} r^{1/n-1}dr\right)$$

are absolutely continuous functions in $[s, |\Omega|]$, $\forall\, s > 0$, integrating by parts the last term of the right-hand side in (3.13), one gets

$$u^*(s) \leq \int_s^{|\Omega|} \frac{F(\sigma)d\sigma}{nC_n^{1/n}\,\sigma^{1-1/n}} \exp\left(\int_s^\sigma \frac{B(r)dr}{nC_n^{1/n}\,r^{1-1/n}}\right) = w(s).$$

Setting $v(x) = w(C_n |x|^n)$ the conclusion follows.

Denote, now, by $L_\phi(\Omega)$ the Orlicz space defined by $\phi(t) = e^{|t|^{n/(n-1)}} - 1$ (see [1], [13]) and by

$$\|u\|_{L_\phi(\Omega)} = \|u\|_\phi = \inf\left\{k>0: \int_\Omega \phi\left(\frac{u}{k}\right)dx \leq 1\right\}. \tag{3.14}$$

Trudinger in [20] proved that $W_\circ^{1,n}(\Omega)$ is continuously imbedded in $L_\phi(\Omega)$ and a sharp version of this result, due to Alvino [2], shows that

$$\|v\|_\phi \leq \frac{(1+|\Omega|)^{1-1/n}}{nC_n^{1/n}} \|Dv\|_n. \tag{3.15}$$

Using (3.5) and the above inequality we shall prove the following

THEOREM 3.2 - *If $f_i \varepsilon L^n(\Omega)$, $i = 1, \ldots, n$, than $u \varepsilon L_\phi(\Omega)$ and*

(3.16) $$\|u\|_\phi \le c\left(n,p,|\Omega|,\|b\|_p\right)\cdot\|f\|_n,$$

where $c(n,p,|\Omega|,\|b\|_p)$ *is given by* (3.19).

Proof - From (3.6) we derive that

$$|Dv(x)| = F\left(C_n|x|^n\right) + B\left(C_n|x|^n\right)v(x)$$

then

$$\|Dv\|_n \le \|F\|_n + \|B\cdot v\|_n .$$

Moreover (3.6) and Hölder's inequality give

$$\|B\cdot v\|_n \le \bar{c}\cdot\|F\|_n,$$

where

$$\bar{c}^n = \int_{\Omega^\#} B^n\left(C_n|x|^n\right)\cdot$$

$$\cdot\left\{\int_{C_n|x|^n}^{|\Omega|}\left[\left(nC_n^{1/n}\right)^{-1}\sigma^{1/n-1}\exp\left(\int_{C_n|x|^n}^{\sigma}\left(nC_n^{1/n}\right)^{-1}r^{1/n-1}B(r)dr\right)\right]^{\frac{n}{n-1}}d\sigma\right\}^{n-1}dx.$$

Note that $\bar{c}$ depends on the level sets of $u(x)$ because of the presence of B. Now because of Lemma 2.1

(3.17) $$\|B\|_p \le \|b\|_p\ ,\quad \|F\|_n \le \|f\|_n\ ,$$

hence it is easy to check that

$$\bar{c} \le c_0 = c_1\cdot\left(nC_n^{1/n}\right)^{-1}\|b\|_p\left\{\int_0^{|\Omega|}\left(\log\frac{|\Omega|}{s}\right)^{p(n-1)/(p-n)}ds\right\}^{(p-n)/np},$$

where

(3.18) $$c_1 = \exp\left\{\left(nC_n^{1/n}\right)^{-1}\|b\|_p\,|\Omega|^{(p-n)/np}\left[\frac{n(p-1)}{p-n}\right]^{(p-1)/p}\right\},$$

by Hölder's inequality.

Combine the previous inequalities and recall (3.5), (3.15) to obtain (3.16) with

$$c\left(n,p,|\Omega|,\|b\|_p\right) = \left(1+c_0\right)\ \frac{(1+|\Omega|)^{1-1/n}}{nC_n^{1/n}}. \tag{3.19}$$

REMARK 3.1 - Let u be the weak solution in $W_\circ^{1,2}(\Omega)$ of the equation

$$-\left(a_{ij}(x)u_{x_i}\right)_{x_j} + \left(b_i(x)u\right)_{x_i} + c(x)u = g(x) - \left(f_i\right)_{x_i} \text{ in } \Omega$$

under the assumptions i), ...iv) and the additional one

v) $g\varepsilon L^{2n/(n+2)}(\Omega)$ if $n>2$ or $g\varepsilon L^s(\Omega), s>1,$ if $n=2$.

Then

$$u^{\#}(x) \leq v(x) \text{ a.e. in } \Omega^{\#} \tag{3.20}$$

where

$$v(x) = \int_{C_n|x|^n}^{|\Omega|} \left(nC_n^{1/n}\right)^{-1} \sigma^{1/n-1} \cdot$$

$$\cdot\left[\frac{\int_0^\sigma g*(r)dr}{nC_n^{1/n}\sigma^{1-1/n}} + F(\sigma)\right] \exp\left[\int_{C_n|x|^n}^{\sigma} \left(nC_n^{1/n}\right)^{-1} r^{1/n-1}B(r)dr\right]$$

is the weak solution in $W_\circ^{1,2}(\Omega^{\#})$ of the equation

$$-\Delta v - \left(\frac{x_i}{|x|}B\left(C_n\,|x|^n\right)\right)_{x_i} = g^{\#}(x) + \left(\frac{x_i}{|x|}F\left(C_n\,|x|^n\right)\right)_{x_i} \text{ in } \Omega^{\#}.$$

The comparison result (3.20) is a straight generalization of (3.5) and Theorem 1a of [19].

Now let $u(x)$ be the weak solution in $W_\circ^{1,2}(\Omega)$ of the equation (3.2).

THEOREM 3.3 - *It results that*

$$u^{\#}(x) \leq v(x) \textit{ a.e. in } \Omega^{\#}, \tag{3.21}$$

where

$$(3.22)\qquad v(x)=\int_{C_n|x|^n}^{|\Omega|}\left(nC_n^{1/n}\right)^{-1}\sigma^{1/n-1}F(\sigma)d\sigma+$$

$$+\int_{C_n|x|^n}^{|\Omega|}\left(nC_n^{1/n}\right)^{-2}\sigma^{2/n-2}d\sigma\int_0^\sigma B(\rho)F(\rho)\exp\left[\int_\rho^\sigma\left(nC_n^{1/n}\right)^{-1}r^{1/n-1}B(r)dr\right]d\rho$$

is the weak solution in $W_o^{1,2}(\Omega^\#)$ *of the equation*

$$(3.23)\qquad -\Delta v+\left(\frac{x_i}{|x|}B\left(C_n|x|^n\right)\right)v_{x_i}=\left(\frac{x_i}{|x|}F\left(C_n|x|^n\right)\right)_{x_i}\ \text{in } \Omega^\#$$

and $B^2(s)$, $F^2(s)$, $s\in[0,|\Omega|]$ *are related to* $b^2(x)$ *and* $f^2(x)$ *respectively, according to Lemma* 2.1.

Proof - Taking the function ϕ defined by (3.8) as test function in (3.4), it is easy to get

$$\frac{1}{h}\int_{t<|u|\le t+h}a_{ij}u_{x_i}u_{x_j}dx+\int_\Omega cu\phi dx=-\int_{t<|u|\le t+h}b_iu_{x_i}\frac{|u|-t}{h}\operatorname{sign}u\,dx+$$

$$+\int_{|u|>t+h}b_iu_{x_i}\operatorname{sign}u\,dx+\frac{1}{h}\int_{t<|u|\le t+h}f_iu_{x_i}dx.$$

Hence by i) and iii) it follows that

$$\frac{1}{h}\int_{t<|u|\le t+h}|Du|^2dx\le\int_{t<|u|\le t+h}b\,|Du|\,dx+\int_{|u|>t+h}b\,|Du|\,dx+$$

$$+\frac{1}{h}\int_{t<|u|\le t+h}f\,|Du|\,dx\le\int_{t<|u|\le t+h}b\,|Du|\,dx+\int_{|u|>t+h}b\,|Du|\,dx+$$

$$+\left(\frac{1}{h}\int_{t<|u|\le t+h}f^2dx\right)^{\frac{1}{2}}\left(\frac{1}{h}\int_{t<|u|\le t+h}|Du|^2dx\right)^{\frac{1}{2}},$$

by Hölder's inequality. So, when $h\to 0$,

$$-\frac{d}{dt}\int_{|u|>t}|Du|^2dx\le\int_{|u|>t}b\,|Du|\,dx+$$

$$+\left(-\frac{d}{dt}\int_{|u|>t} f^2\, dx\right)^{\frac{1}{2}} \left(-\frac{d}{dt}\int_{|u|>t} |Du|^2\, dx\right)^{\frac{1}{2}}.$$

Now $\int_{|u|>t} b|Du|dx$ is absolutely continuous (cfr.[19],p.941), then

$$\int_{|u|>t} b|Du|dx = \int_t^\infty -\frac{d}{d\tau}\left[\int_{|u|>\tau} b|Du|dx\right] d\tau \leq$$

$$\leq \int_t^\infty \left(-\frac{d}{d\tau}\int_{|u|>\tau} b^2 dx\right)^{\frac{1}{2}} \left(-\frac{d}{d\tau}\int_{|u|>\tau} |Du|^2 dx\right)^{\frac{1}{2}} d\tau,$$

hence

$$-\frac{d}{dt}\int_{|u|>t} |Du|^2 dx \leq \int_t^\infty \left(-\frac{d}{d\tau}\int_{|u|>\tau} b^2 dx\right)^{\frac{1}{2}} \left(-\frac{d}{d\tau}\int_{|u|>\tau} |Du|^2 dx\right)^{\frac{1}{2}} d\tau +$$

$$+\left(-\frac{d}{dt}\int_{|u|>t} f^2 dx\right)^{\frac{1}{2}} \left(-\frac{d}{dt}\int_{|u|>t} |Du|^2 dx\right)^{\frac{1}{2}} \leq$$

$$\leq \frac{(-\mu'(t))^{\frac{1}{2}}}{nC_n^{1/n}\mu(t)^{1-1/n}} \left(-\frac{d}{dt}\int_{|u|>t} |Du|^2 dx\right)^{\frac{1}{2}} \int_t^\infty (-\mu'(\tau))^{\frac{1}{2}} B(\mu(\tau))\cdot$$

$$\cdot\left(-\frac{d}{d\tau}\int_{|u|>\tau} |Du|^2 dx\right)^{\frac{1}{2}} d\tau + (-\mu'(t))^{\frac{1}{2}} F(\mu(t)) \left(-\frac{d}{dt}\int_{|u|>t} |Du|^2 dx\right)^{\frac{1}{2}}$$

by Lemma 2.1. and (2.4).

Now use Lemma 2.2 with $\phi(t) = \left(-\frac{d}{dt}\int_{|u|>t} |Du|^2 dx\right)^{\frac{1}{2}}$ to obtain

$$(3.24) \qquad \left(-\frac{d}{dt}\int_{|u|>t} |Du|^2 dx\right)^{\frac{1}{2}} \leq (-\mu'(t))^{\frac{1}{2}} F(\mu(t)) +$$

$$+\frac{(-\mu'(t))^{\frac{1}{2}}}{nC_n^{1/n}\mu(t)^{1-1/n}} \int_t^\infty -\mu'(\tau)\, B(\mu(\tau))F(\mu(\tau)) \exp\left[\int_t^\tau \frac{-\mu'(\sigma)B(\mu(\sigma))}{nC_n^{1/n}\mu(\sigma)^{1-1/n}} d\sigma\right] d\tau.$$

Again by (2.4) and by (3.24) it follows that

$$1 \leq \frac{-\mu'(t)}{nC_n^{1/n}\mu(t)^{1-1/n}} F(\mu(t)) + \frac{-\mu'(t)}{n^2 C_n^{2/n}\mu(t)^{2-2/n}} \cdot \tag{3.25}$$

$$\cdot \int_t^\infty B(\mu(\tau))F(\mu(\tau)) \exp\left[\int_t^\tau \frac{-\mu'(\sigma)B(\mu(\sigma))}{nC_n^{1/n}\mu(\sigma)^{1-1/n}} d\sigma\right](-\mu'(\tau))\, d\tau \,.$$

Integrating (3.25) between 0 and t one gets

$$u^*(s) \leq w(s) = \int_s^{|\Omega|} \frac{F(\sigma)}{nC_n^{1/n}\sigma^{1-1/n}} d\sigma +$$

$$+\int_s^{|\Omega|} \frac{d\sigma}{n^2 C_n^{2/n}\sigma^{2-2/n}} \int_0^\sigma B(\rho)F(\rho) \exp\left[\int_\rho^\sigma \frac{B(r)}{nC_n^{1/n}r^{1-1/n}} dr\right] d\rho \,,$$

and $v(x) = w(C_n|x|^n)$ is the solution of (3.23) belonging to $W_\circ^{1,2}(\Omega^\#)$.

THEOREM 3.4. - *If* $f_i \varepsilon L^n(\Omega)$, $i = 1, \ldots, n$, *then* $u \varepsilon L_\phi(\Omega)$ *and moreover*

$$\|u\|_\phi \leq c(n,p,|\Omega|,\|b\|_p) \cdot \|f\|_n \tag{3.26}$$

where $c(n,p,|\Omega|,\|b\|_p)$ *is given by* (3.27).

Proof - By (3.22) it follows

$$|Dv(x)| = F(C_n|x|^n) +$$

$$+\frac{1}{nC_n|x|^{n-1}} \int_0^{C_n|x|^n} F(r)B(r) \exp\left[\int_r^{C_n|x|^n} \frac{B(t)}{nC_n^{1/n}t^{1-1/n}} dt\right] dr \,.$$

On the other hand the last term on the right-hand side is less or equal to

$$\frac{\bar{c}(x)}{nC_n} \cdot \|F\|_n \,,$$

where

$$\bar{c}(x) = \frac{1}{|x|^{n-1}}\left\{\int_0^{C_n|x|^n}\left[B(r)\exp\left(\int_r^{C_n|x|^n}\frac{B(t)}{nC_n^{1/n}t^{1-1/n}}dt\right)\right]^{n/(n-1)}dr\right\}^{(n-1)/n}.$$

Recall (3.17) to see that

$$\bar{c}(x) \le c_1 \cdot C_n^{1-1/n-1/p}\|b\|_p |x|^{-n/p}$$

by Hölder's inequality, where c_1 is given by (3.18), and so

$$\|\bar{c}\|_n \le c_1 C_n^{1-1/n}\|b\|_p\,(p/(p-n))^{1/n}|\Omega|^{1/n-1/p}.$$

Hence by Theorem 3.3 and by (3.15)

$$\|u\|_\phi \le \frac{(1+|\Omega|)^{1-1/n}}{nC_n^{1/n}}\|Dv\|_n,$$

therefore from the above it follows (3.26), where

$$(3.27)\qquad c\left(n,p,|\Omega|,\|b\|_p\right) =$$

$$= \frac{(1+|\Omega|)^{1-1/n}}{nC_n^{1/n}}\left[1+\frac{c_1}{nC_n^{1/n}}\|b\|_p\left(\frac{p}{p-n}\right)^{1/n}|\Omega|^{1/n-1/p}\right].$$

REMARK 3.2. - Also the following inequality holds

$$u^{\#}(x) \le v(x) \quad \text{a.e. in } \Omega^{\#},$$

where $u\varepsilon W_\circ^{1,2}(\Omega)$ is the solution of the equation

$$-\left(a_{ij}(x)u_{x_i}\right)_{x_j} + b_i(x)u_{x_i} + c(x)u = g(x) - \left(f_i\right)_{x_i} \text{ in } \Omega$$

under the assumptions i), ii), iii), iv) and v), and

$$v(x) = \int_{C_n|x|^n}^{|\Omega|}\left[\left(nC_n^{1/n}\right)^{-1}\sigma^{1/n-1}F(\sigma)+\left(nC_n^{1/n}\right)^{-2}\sigma^{2/n-2}\int_0^\sigma g_*(t)dt\right]d\sigma+$$

$$+\int_{C_n|x|^n}^{|\Omega|}\left(nC_n^{1/n}\right)^{-2}\sigma^{2/n-2}\,d\sigma\int_0^\sigma B(\rho)\left[F(\rho)+\left(nC_n^{1/n}\right)^{-1}\rho^{1/n-1}\int_0^\rho g*(t)dt\right]\cdot$$

$$\cdot\exp\left(\int_\rho^\sigma\left(nC_n^{1/n}\right)^{-1}r^{1/n-1}B(r)dr\right)d\rho$$

is the solution in $W_\circ^{1,2}(\Omega^\#)$ of the equation

$$-\Delta v+\frac{x_i}{|x|}B\left(C_n|x|^n\right)v_{x_i}=g^\#(x)+\left(\frac{x_i}{|x|}F\left(C_n|x|^n\right)\right)_{x_i}\quad\text{in }\Omega^\#.$$

4. COMPARISON AND REGULARITY RESULTS FOR STRONGLY NON-LINEAR ELLIPTIC EQUATIONS.

Consider the equation

(4.1) $$-(a_i(x,u,Du))_{x_i}+h(x,u)=-(f_i)_{x_i}\text{ in }\Omega$$

where $a_i(x,\eta,\xi)$ and $h(x,\eta)$ are Caratheodory functions for $x\varepsilon\Omega$, $\eta\varepsilon R$, $\xi\varepsilon R^n$, and assume

j) $a_i(\cdot,u(\cdot),Du(\cdot))\,\varepsilon L^{p'}(\Omega), i=1\ldots n$, $u\varepsilon W^{1,p}(\Omega)(1=1/p+1/p')$ and $a_i(x,\eta,\xi)\,\xi_i\geq\alpha\,|\xi|^p$, $\forall\,\xi\varepsilon R^n$, $\forall\,\eta\varepsilon R$, a.e. $x\varepsilon\Omega$, and $\alpha>0$.

jj) $h(x,\eta)\cdot\eta\geq 0$, $\forall\,\eta\varepsilon R$, a.e. $x\varepsilon\Omega$.

jjj) $f=(\sum_{i=1}^{n}f_i^{\,2})^{1/2}\,\varepsilon L^{p'}(\Omega)$.

We deal with weak solutions $u\varepsilon W_\circ^{1,p}(\Omega)$ of (4.1), that is

$$h(\cdot,u(\cdot))\,\varepsilon L^1(\Omega),\ u(\cdot)\,h(\cdot,u(\cdot))\,\varepsilon L^1(\Omega)$$

and

(4.2) $$\int_\Omega(a_i(x,u,Du)\,\phi_{x_i}+h(x,u)\,\phi)\,dx=$$

$$= \int_\Omega f_i \phi_{x_i} dx \quad \forall \phi \varepsilon W_\circ^{1,p}(\Omega) \cap L^\infty(\Omega)$$

and for $\phi = u$.

Existence theorems for this kind of problems can be found e.g. in [7]. See also [6] for the regularity theorems when $f_i \in L^q(\Omega)$ with $q \gtrless n/(p-1)$.

We prove now the following comparison theorem:

THEOREM 4.1. - *It results that*

$$u^\# (x) \leq v(x) \ \text{a.e. in } \Omega^\#, \tag{4.3}$$

where

$$v(x) = \alpha^{-p'/p} \int_{C_n|x|^n}^{|\Omega|} \left(n C_n^{1/n}\right)^{-1} \sigma^{1/n-1} F^{p'/p}(\sigma)\, d\sigma, \tag{4.4}$$

is weak solution in $W_\circ^{1,p}(\Omega^\#)$ *of the equation*

$$-\left(v_{x_i} |Dv|^{p-2}\right)_{x_i} = \left(\frac{x_i}{|x|} F\left(C_n |x|^n\right)\right)_{x_i} \ \text{in } \Omega^\#, \tag{4.5}$$

and $F^{p'}(s)$ *is the function related with* $f^{p'}(x)$ *according to Lemma 2.1.*

Proof - Insert ϕ defined by (3.8) in (4.2) to get

$$\frac{1}{h} \int_{t<|u|\leq t+h} a_i(x,u,Du,)\, u_{x_i} dx + \int_\Omega h(x,u)\, \phi dx =$$

$$= \frac{1}{h} \int_{t<|u|\leq t+h} f_i\, u_{x_i} dx \leq \frac{1}{h} \int_{t<|u|\leq t+h} f\, |Du| dx.$$

Since $h(x,u)\,\phi \geq 0$ (see jj)), j) and the Hölder inequality imply

$$\alpha \left(\frac{1}{h} \int_{t<|u|\leq t+h} |Du|^p dx\right)^{1/p'} \leq \left(\frac{1}{h} \int_{t<|u|\leq t+h} f^{p'} dx\right)^{1/p'} \quad (1 = 1/p + 1/p')$$

which gives, when $h \to 0$

$$\alpha\left(-\frac{d}{dt}\int_{|u|>t}|Du|^p dx\right)^{1/p'} \leq \left(-\frac{d}{dt}\int_{|u|>t} f^{p'} dx\right)^{1/p'} \quad \text{a.e. } t>0\,.$$

Now, by Lemma 2.1, there exists a function $F^{p'}\varepsilon L^1(0, |\Omega|)$ such that $\int_{|u|>t} f^{p'}\, dx = \int_0^{\mu(t)} F^{p'} ds$, therefore it follows that

$$\text{(4.6)} \qquad \alpha\left(-\frac{d}{dt}\int_{|u|>t}|Du|^p dx\right)^{1/p'} \leq (-\mu'(t))^{1/p'} F(\mu(t)).$$

On the other hand, (2.3) and Hölder's inequality imply

$$nC_n^{1/n}\mu(t)^{1-1/n} \leq \left(-\frac{d}{dt}\int_{|u|>t}|Du|^p dx\right)^{1/p'} (-\mu'(t))^{1/p'},$$

hence by (4.6) obtain

$$\alpha^{p'/p} \leq \frac{-\mu'(t)F^{p'/p}(\mu(t))}{nC_n^{1/n}\mu(t)^{1-1/n}} \quad \text{a.e. } t>0.$$

Now integrate the above inequality between 0 and $t<\sup|u|$ and recall the definition of u^* to get

$$u^*(s) \leq \alpha^{-p'/p}\int_s^{|\Omega|}\frac{F^{p'/p}(\sigma)}{nC^{1/n}\sigma^{1-1/n}}\, d\sigma,$$

which implies (4.3).

REMARK 4.1 - The inequality (4.3) allow to make explicit the numerical values of the constants in known estimates (cfr. [6]), via the Hölder's and the Bliss inequalities (cfr. [17]), as follows

$$\|u\|_\infty \leq \alpha^{-p'/p}K_1(n,p,q,|\Omega|)\|f\|_q^{1/(p-1)}\,, \text{ when } f\varepsilon L^q(\Omega), p\leq n, q>\frac{n}{p-1};$$

$$\|u\|_{(q(p-1))^*} \leq \alpha^{-p'/p}K_2(n,p,q)\|f\|_q^{1/(p-1)}, \text{ when } f\varepsilon L^q(\Omega), p\leq n, 1<q<n/(p-1),$$

where

$$K_1(n,p,q,|\Omega|) = \left(nC_n^{1/n}\right)^{-1} |\Omega|^{\frac{q(p-1)-n}{nq(p-1)}} \left\{\frac{n(q(p-1)-1)}{q(p-1)-n}\right\}^{\frac{q(p-1)-1}{q(p-1)}},$$

$$K_2(n,p,q,) = \left(nC_n^{1/n}\right)^{-1} \left\{\frac{\Gamma\left(\frac{q\cdot s}{s-q}\right)}{\Gamma\left(\frac{s}{s-q}\right)\Gamma\left(\frac{q(s-1)}{s-q}\right)}\right\}^{\frac{1}{q}-\frac{1}{s}} \left(s\left(\frac{q-1}{q}\right)\right)^{\frac{q-1}{q}}, s=(q(p-1))^*.$$

Now we employ (4.3) to prove the following regularity result

THEOREM 4.2 - *If* $f_i \varepsilon L^{n/(p-1)}(\Omega)$, $i = 1, \ldots, n$, $n \geq p$, *then* $u \varepsilon L_\phi(\Omega)$ *and moreover*

$$\|u\|_\phi \leq \alpha^{-p'/p}\left(nC_n^{1/n}\right)^{-1}(1+|\Omega|)^{1-1/n}\|f\|_{n/(p-1)}^{p'/p}.$$

Proof - By (4.4) we obtain

$$\|Dv\|_n \leq \alpha^{-p'/p}\|F\|_{n/(p-1)}^{p'/p} \leq \alpha^{-p'/p}\|f\|_{n/(p-1)}^{p'/p} \text{ by Lemma 2.1.}$$

Hence the conclusion follows from (4.3) and (3.15) .

REMARK 4.2 - By similar arguments to the ones used for the Th. 4.1, we have

$$u^\# (x) \leq v(x) \text{ in } \Omega^\#$$

where $u \varepsilon W_\circ^{1,p}(\Omega)$ is a weak solution of the equation

$$-(a_i(x,u,Du))_{x_i} + h(x,u) = g(x) - \left(f_i\right)_{x_i} \text{ in } \Omega,$$

with $g \varepsilon L^{p^{*\prime}}(\Omega)$, and

$$v(x) = \alpha^{-p'/p}\int_{C_n|x|^n}^{|\Omega|}\left(nC_n^{1/n}\right)^{-1}\sigma^{1/n-1}\left[\left(nC_n^{1/n}\right)^{-1}\sigma^{1/n-1}\int_0^\sigma g^*(t)dt + F(\sigma)\right]^{p'/p} d\sigma$$

solves the equation

$$-\alpha\left(v_{x_i}|Dv|^{p-2}\right)_{x_i}=g^{\#}(x)+\frac{x_i}{|x|}F\left(C_n|x|^n\right)\text{ in }\Omega^{\#},$$

in $W_{\circ}^{1,p}$ $(\Omega^{\#})$.

Such inequality can be used to find sharp estimates for $\|u\|$.

When $f_i=0$, $i=1,\ldots,n$, this comparison result was already proved in [18], where the author considered problems for other aspects more general.

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