MULTIPLE PERIODIC SOLUTIONS OF SOME SECOND ORDER DYNAMICAL SYSTEM (*)

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RIASSUNTO. Generalizziamo l'idea del linking locale al caso in cui la condizione di Palais-Smale non è verificata per qualche valore. Come applicazione, studiamo l'esistenza di soluzioni multiple T-periodiche di alcuni sistemi dinamici del secondo ordine.

SUMMARY. We generalize the local linking idea to the case in which the Palais and Smale compactness condition fails at some value. As an application, we study the existence of multiple T-periodic solutions of some second order dynamical system.

In this paper, we study the existence of multiple T-periodic solutions of Hamiltonian system of the form

\[-\ddot{y} = \nabla_y V(t, y)\] (0.1)

where \(y \in \mathbb{R}^N, V \in C^1(R \times \mathbb{R}^N, \mathbb{R}), V\) is T-periodic in \(t\) and \(\nabla\) denotes the gradient.

In section 1, we prove an existence theorem of critical point which is more general than the theorem given in [8] or [9]. We generalize the local linking idea to the case in which the Palais and Smale compactness condition fails at some value.

In section 2, we consider the existence of multiple T-periodic solutions of (0.1). We will deal with three cases:

(1) Potentials bounded from above.
(2) Asymptotically quadratic linear case.
(3) Bounded potentials.

The paper which are devoted to study these problem can be found in [1], [2], [3], [4], [5], [6], [7], [11], [13].

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1. Local linking and an existence theorem.

Let \( X \) be a real Hilbert space and have a decomposition \( X = X_1 + X_2 \) where \( \dim X_2 = m < \infty \). Let \( f \in C^1(X, R) \).

**Definition 1.1.** If there exist \( r, b > 0 \) such that

\[
\begin{align*}
    f(x) &\geq b, & x &\in S_1, \\
    f(x) &\geq 0, & x &\in B_1, \\
    f(x) &\leq 0, & x &\in B_2, \\
    f(x) &\leq -b, & x &\in S_2
\end{align*}
\]  

(1.1)

where

\[
\begin{align*}
    B_1 &= \{ x \in X_1 \mid \|x\| \leq r \} \\
    B_2 &= \{ x \in X_2 \mid \|x\| \leq r \} \\
    S_1 &= \partial B_1 \\
    S_2 &= \partial B_2
\end{align*}
\]  

(1.2)

then we say that \( f \) has a local linking at 0.

**Definition 1.2.** We will say that \( f \) satisfies the (P.S) (Palais-Smale) condition in \( S \) where \( S \) is a subset of \( X \), if for every sequence \( \{u_n\} \) in \( S \) such that \( f(u_n) \) is bounded and \( \nabla f(u_n) \to 0 \) there exists a subsequence converging to an element of \( S \).

Set

\[
\begin{align*}
    f_b &= \{ x \in X \mid f(x) \leq b \} \\
    f^b &= \{ x \in X \mid f(x) > b \} \\
    f^{-1}(b) &= \{ x \in X \mid f(x) = b \} \\
    K_c &= \{ x \in X \mid f(x) = c \quad \nabla f(x) = 0 \}.
\end{align*}
\]

**Theorem 1.3.** Suppose that \( f \in C^1(X, R) \) is bounded below and satisfies the (P.S) condition in \( X \setminus f^{-1}(0) \), then \( f \) has at least three distinct critical points.

**Proof.** Let \( m < 0 \) be the minimal value of \( f \), \( x_0 \) be the unique point such that \( f(x_0) = m \), we also suppose that \( f \) has no critical value in \( (m, 0) \), otherwise we get the theorem. There exists a negative pseudogradient flow \( \eta(t, x) : [0, 1] \times S_2 \to f_0 \), such that \( \eta(1, x) = x_0 \) (refer [8], [9] for the details). Set

\[
\begin{align*}
    \Gamma &= B_2 \cup \eta([0, 1] \times S_2) \\
    Q &= B_1 \times B_2.
\end{align*}
\]
Define a mapping \( \phi: \partial ([0, 1] \times B_2) \to f_0 \) by

\[
\phi (t, x) = \begin{cases} 
x & \forall x \in B_2 \quad t = 0 \\
\eta(t, x) & \forall x \in S_2 \quad t \in (0,1) \\
x_0 & \forall x \in B_2 \quad t = 1.
\end{cases}
\]

Set

\[ F_t = x_1 - \tilde{\phi} (t, x_2) \quad x_1 \in B_1, x_2 \in B_2, t \in [0, 1] \]

where \( \tilde{\phi} (t, x_2) \) is any extension of \( \phi (t, x_2) \) on \([0, 1] \times B_2\). We claim that there exists a \( t_0 \in [0, 1], x_2^0 \in B_2, x_1^0 \in \partial B_1 \) such that

\[ x_1^0 = \tilde{\phi} (t_0, x_2^0). \] (1.3)

If (1.3) is not true then we have

\[ F_t (\partial B_1 \times B_2) \neq 0 \forall t \in [0, 1]. \] (1.4)

From that \( f (x_1) \geq 0 \forall x_1 \in B_1, f (\tilde{\phi} (t, x_2)) = f (\phi (t, x_2)) < 0 \forall t \in [0, 1], x_2 \in S_2 \), we have

\[ F_t (B_1 \times \partial B_2) \neq 0 \] (1.5)

combining (1.4) and (1.5) we have that \( F_t (\partial Q) \neq 0 \forall t \in [0, 1] \).

Computing the Leray-Schauder degree for the mappings \( F_t \) and by the homotopy invariance property of degree we have that

\[ (-1)^m = \deg (P_1 - P_2, Q, 0) = \deg (F_0, Q, 0) = \deg (F_1, Q, 0) = \deg (P_1 - x_0, Q, 0) = 0 \] (1.6)

contradiction! where \( P_1, P_2 \) are orthogonal projections from \( X \) to \( X_1, X_2 \), respectively. So (1.3) holds. It means that \( \partial B_1 \) and \( \Gamma \) link nontrivially, see [12]. We claim that \( f \) must have a critical value which is not less than \( b \).

Let \( \tilde{\phi} \) be an extension of \( \phi \). Take \( b_0 \) such that \( b > b_0 > 0 \). Set
\[ W = \{ \Psi \in C(\tilde{\phi}([0,1] \times B_2, X)) \mid \Psi(x) = \begin{cases} \text{Id} & \text{as } x \in \tilde{\phi}([0,1] \times B_2) \cap f_{b_0} \\ \text{continuous as } x \in \tilde{\phi}([0,1] \times B_2) \cap f_{b_0} & \end{cases} \} \]

(1.7)

Define
\[ c = \inf_{\Psi \in W} \sup_{x \in \tilde{\phi}([0,1] \times B_2)} f(\Psi(x)). \]

(1.8)

From (1.3) we have \( c \geq b \). Take \( \varepsilon \) small such that \( 0 < \varepsilon < \frac{1}{2} (c - b_0) \).

Since \( f \) satisfies the (P.S) condition in \( X \setminus f^{-1}(0) \) by the deformation theorem there exists a flow \( \zeta(t,x) \) which satisfies the following properties:

\begin{enumerate}
    \item \( \zeta(t,x) \in C([0,1] \times f^{b_0}, f^{b_0}) \)
    \item \( \zeta(0,x) = x \quad \forall x \in f^{b_0} \)
    \item \( \zeta(t,x) = x \quad \forall x \in f^1([b_0, c-2\varepsilon]) \cup f^{c+2\varepsilon} \)
    \item If \( K_c = \emptyset \) then \( \zeta(1,f_{c+\varepsilon}) \subset f_{c-\varepsilon} \).
\end{enumerate}

By the standard minimax argument we get a contradiction. We have proved the theorem.

REMARK 1.4. In definition 1.1 if we replace the ball \( B_1, B_2 \) by any other domain, say, a rectangle theorem 1.3 still holds.

With \( M^0(\cdot), M^-(\cdot) \) we denote in the following the zero and negative Morse index, respectively of the symmetric matrix defining it.

DEFINITION 1.5. Let \( f \in C^1(X, R) \). We say that \( f \) is asymptotically quadratic, if there is a bounded self-adjoint operator \( A_\infty \) satisfying
\[ \| \nabla f(x) - A_\infty x \| / \| x \| \to 0 \text{ as } \| x \| \to \infty. \]

(1.9)

We say that \( f \) is nonresonance at the infinity, if
\[ M^0(A_\infty) = 0. \]
DEFINITION 1.6. Let \( \{e_1, e_2, \ldots, e_l, \ldots\} \) be a complete orthonormal system of \( X, X_l = \text{Span} \{e_1, \ldots, e_l\} \). If for any sequence \( x_l \in X_l \) \( l = 1, 2, \ldots \) satisfies \( f(x_l) \leq c < +\infty \), \( \| \nabla f \|_{X_l} \| x_l \|_{X_l} \to 0 \) possesses a convergent subsequence in \( X \), then we say that \( f \) satisfies \((\text{P.S})^*\) condition (where \( f|_{X_l} \) is the restriction of \( f \) on \( X_l \)).

We state the following proposition which was first introduced in [10].

PROPOSITION 1.7. Suppose that \( f \in C^1(X, R) \) is a bounded functional at any bounded set of \( X, f \) is asymptotically quadratic and nonresonance at the infinity. If \( f \) satisfies the (P.S) and (P.S)* conditions, \( f \) has a local linking at 0 and \( M^- (A_\infty) \neq m (\dim X_2) \) then \( f \) has at least one non-trivial critical point.

2. Applications to the search of periodic solutions for second order Hamiltonian systems.

In this section we study the existence of multiple \( T \)-periodic solutions of (0.1).

First we will make the following assumptions on \( V \)

\((V_1)\) \( V(t, s) \to -\infty \) as \( |s| \to \infty \) uniformly in \( t \).

\((V_2)\) There exists a \( N \times N \) symmetric matrix \( V_0 \) such that
\[
|\nabla_s V(t, s) - V_0 s| / |s| \to 0 \text{ as } |s| \to 0 \text{ uniformly in } t.
\] (2.1)

It is easy to see that \((V_2)\) implies that \( V(t, 0) = m, \nabla_s V(t, 0) = 0 \), where \( m \) is a function of \( t \).

Set
\[
M^0 \left( \begin{array}{c} \frac{-d^2}{dt^2} - V_0 \\ \end{array} \right) = \sum_{k=0}^{\infty} M^0 \left( \begin{array}{ccc} k^2 4\pi^2 & 0 \\ \frac{T^2}{T^2} & \ddots & \frac{k^2 4\pi^2}{T^2} \\ 0 & \frac{T^2}{T^2} & \ddots & \end{array} \right) - V_0
\]
\[ M^- \left( \frac{-d^2}{dt^2} - V_0 \right) = \sum_{k=0}^{\infty} M^- \begin{bmatrix} \frac{k^2 4\pi^2}{T} & 0 \\ 0 & \frac{k^2 4\pi^2}{T} \end{bmatrix} - V_0 \] 

**Theorem 2.1.** If \( V(t, s) \) satisfies \( (V_1), (V_2), M^0 \left( -\frac{d^2}{dt^2} - V_0 \right) = 0, \)

\[ M^- \left( -\frac{d^2}{dt^2} - V_0 \right) \neq 0 \] then (0.1) has at least three \( T \)-periodic solutions.

**Proof.** Let \( X = H^1(S^1, R^N) \) with the inner product

\[ (x, y)_X = \int_0^T \langle \dot{x}, \dot{y} \rangle dt + \int_0^T \langle x, y \rangle dt \]

where \( \langle, \rangle \) denotes the scalar product in \( R^N \). Define \( f : X \to R \) as

\[ f(y) = \frac{1}{2} \int_0^T |\dot{y}|^2 - \int_0^T V(t, y). \]

It is well known that the critical point of \( f \) on \( X \) are \( T \)-periodic solutions of (0.1). From lemma 4.2 in [7] we know that \( f \) satisfies the (P.S) condition in \( X \). We claim that \( f \) has a local linking at 0. From Riesz's representation theorem we get a bounded self-adjoint operator \( A_0 \) such that

\[ \left( -\frac{d^2}{dt^2} y(t) - V_0 y(t), z(t) \right) = (A_0 y, z)_X \forall z \in X \quad (2.3) \]

where \( (\cdot, \cdot) \) denotes the duality product. From \( M^0 \left( -\frac{d^2}{dt^2} - V_0 \right) = M^0 \left( A_0 \right) = 0, M^- \left( A_0 \right) \neq 0 \) we know that \( X \) can have a decomposition \( X = X_1 + X_2 \) where \( X_1 \) is the positive subspace of \( A_0 \) and \( X_2 \) is the negative subspace of \( A_0 \).

If \( r \) is taken small enough \( b = \frac{r^2}{4} \text{dist} (0, \sigma (A_0)) \) where \( \sigma (A_0) \) denotes the spectrum of \( A_0 \), then it is easy to see that \( f \) has a local linking at 0. \( f \) is bounded below. From theorem 1.3 we get the theorem.
THEOREM 2.2. If $V(t, s)$ satisfies (V1), (V2), $M^0 \left( -\frac{d^2}{dt^2} - V_0 \right) \neq 0$ then (0.1) has at least three $T$-periodic solutions provided that either

1. $M^* \left( -\frac{d^2}{dt^2} - V_0 \right) \neq 0$
   
   \[
   m + \frac{1}{2} < V_0 s, s > - V(t, s) > 0 \text{ as } |s| \text{ small } s \neq 0 \text{ or }
   \]

2. $m + \frac{1}{2} < V_0 s, s > - V(t, s) < 0 \text{ as } |s| \text{ small } s \neq 0$

holds.

Proof. We only need to prove that $f$ has a local linking at 0. Let $X_+, X$, and $X_0$ be the positive, the negative and the zero space of $A_0$, respectively. In case (1) we take $X_1 = X_+ + X_0, X_2 = X$, and in case (2) we take $X_1 = X_+, X_2 = X + X_0$. Since $H^1(S^1, \mathbb{R}^N)$ is continuously embedded in $C(S^1, \mathbb{R}^N)$, from (1) or (2) we know that $f$ has a local linking at 0. From theorem 1.3 we get the theorem.

COROLLARY 2.3. Suppose the assumptions of theorem 2.1 or 2.3 are satisfied and for any fixed $s \neq 0 \nabla_s V(t, s) \neq 0$ then (0.1) has at least two non-trivial solutions (i.e. non-constant solution).

Next, we will be concerned with potentials which satisfy the following

(V3) There exists a $N \times N$ symmetric matrix $V_\infty$ such that

\[
|\nabla_s V(t, s) - V_\infty s| / |s| \to 0 \text{ as } |s| \to \infty \text{ uniformly in } t. \tag{2.4}
\]

We have the following

THEOREM 2.4. If $V(t, s)$ satisfies (V2), (V3), $M^0 \left( -\frac{d^2}{dt^2} - V_0 \right) = 0,$

\[
M^0 \left( -\frac{d^2}{dt^2} - V_\infty \right) = 0, M^* \left( -\frac{d^2}{dt^2} - V_0 \right) \neq M^* \left( -\frac{d^2}{dt^2} - V_\infty \right) \text{ then (0.1) has at least two } T\text{-periodic solutions.}
\]

Proof. Since $M^0 \left( -\frac{d^2}{dt^2} - V_\infty \right) = 0$, so for any sequence $\{u_n\}$ in $X$ such that $\nabla f(u_n) \to 0$ we have

\[
A_\infty u_n - B (\nabla V(t, Ju_n) - V_\infty Ju_n) \to 0 \tag{2.5}
\]
where $A_\infty$ and $B$ is the representation of $-\frac{d^2}{dt^2} - V_\infty$ and identity in $H^1(S^1, R^N)$, respectively, $J$ is the compact embedding operator from $H^1(S^1, R^N)$ to $L^2(0, T; R^N)$. If $\|u_n\| \to \infty$ then from (2.4) we have

$$\|A_\infty^{-1} B (\nabla_y V(t, Ju_n) - V_\infty Ju_n) \| / \|u_n\|$$

$$\leq c \|\nabla_y V(t, Ju_n) - V_\infty Ju_n \|_{L^2} / \|u_n\|$$

$$\leq c(\int_0^T |\nabla_y V(t, u_n(t)) - V_\infty u_n(t)|^2)^{1/2} / \|u_n\| \to 0$$

contradiction! So $\{u_n\}$ are bounded in $X$. It is the same to prove that $f$ satisfies the (P.S)* condition. From proposition 1.7 we get the theorem.

**Theorem 2.5.** If $V(t, s)$ satisfies $(V_2)$, $(V_3)$, $M^0 ( -\frac{d^2}{dt^2} - V_\infty) = 0$ but $M^0 ( -\frac{d^2}{dt^2} - V_0) \neq 0$ then (0.1) has at least two T-periodic solutions provided that either

1. $M^\ast ( -\frac{d^2}{dt^2} - V_0) \neq M^\ast ( -\frac{d^2}{dt^2} - V_\infty)$

$$m + \frac{1}{2} < V_0, s, s > - V(t, s) > 0 \text{ as } |s| \text{ small } s \neq 0 \text{ or}$$

2. $M^\ast ( -\frac{d^2}{dt^2} - V_0) + M^0 ( -\frac{d^2}{dt^2} - V_0) \neq M^\ast ( -\frac{d^2}{dt^2} - V_\infty)$

$$m + \frac{1}{2} < V_0, s, s > - V(t, s) < 0 \text{ as } |s| \text{ small } s \neq 0$$

holds.

**Corollary 2.6.** Suppose the assumptions of theorem 2.4 or 2.5 are satisfied and $\nabla_y V(t, s) \neq 0$ for any fixed $s \neq 0$ then (0.1) has at least one non-trivial T-periodic solution.

The following theorem give some equivalent conditions.

**Theorem 2.7.** Suppose that $V$ satisfies $(V_3)$, $M^0 ( -\frac{d^2}{dt^2} - V_\infty) = 0$ then the following conditions are equivalent:

1. $f(y) = \frac{1}{2} \int |\dot{y}|^2 - V(t, s)$ is bounded below.
(2) \( V_\infty \) is the negative definite.

(3) \( V(t, s) \to -\infty \text{ as } |s| \to \infty \text{ uniformly in } t. \)

Proof. (1) \( \Rightarrow \) (2): From \( f(y) \geq m \forall y \in X \) we have \( -m \geq \int_0^T V(t, s) \, dt \) \( \forall s \in R^N. \)
From (V3) we have \( V(t, s) = \frac{1}{2} \langle V_\infty s, s \rangle > + \varepsilon (|s|^2) \) as \( |s| \to \infty \)
so
\[
-m \geq \int_0^T \frac{1}{2} \langle V_\infty s, s \rangle > + \varepsilon (|s|^2).
\]
This implies that \( V_\infty \) is non-positive, but from \( M^0 \left( -\frac{d^2}{dt^2} - V_\infty \right) = 0 \), \( V_\infty \)
must be negative definite.
(2) \( \Rightarrow \) (3): It is clear from \( V(t, s) = \frac{1}{2} \langle V_\infty s, s \rangle > + \varepsilon (|s|^2). \)
(3) \( \Rightarrow \) (1): It is trivial.

Finally, we will be concerned with bounded potentials which satisfies the following

(V4) \( \nabla_s V(t, s) \to 0 \text{ as } |s| \to \infty \text{ uniformly in } t. \)

(V4) means that \( V_\infty = 0. \) Since \( M^0 \left( -\frac{d^2}{dt^2} \right) \neq 0 \) so \( f \) does not satisfy the (P.S) condition in \( X, \) in general. But we can use the following proposition given in [7].

**Proposition 2.8.** If \( 0 \leq V(t, s) \leq M \forall (t, s) \in R \times R^N, V(t, s) \to m \)
as \( |s| \to \infty \text{ uniformly in } t, V \) satisfies (V4), then (P.S) condition holds for \( f \)
in \( X \setminus f^{-1} (-Tm). \)

Using proposition 2.8 and theorem 1.3 we can prove

**Theorem 2.9.** If \( V \) satisfies (V2) and the assumptions of proposition 2.8, \( M^0 \left( -\frac{d^2}{dt^2} - V_0 \right) = 0 \) and \( M^* \left( -\frac{d^2}{dt^2} - V_0 \right) \neq 0 \) then (0.1) has at least three solutions.
Corollary 2.10. If \( 0 \leq V(t,s) \leq M \) \( \forall \ (t,s) \in R \times R^N, V(t,s) \to m \) as \( |s| \to \infty \) uniformly in \( t \) and (V4) holds, \( V_0 = \mu I \ \forall \ t \in [0,1], \mu > \frac{4\pi^2}{T^2} \)
\( \mu \neq k^2 \frac{4\pi^2}{T^2} , \ k = 2, 3, ... \) then theorem 2.9 holds.

Theorem 2.11. If \( V \) satisfies the assumptions of theorem 2.9 but \( M^0 (-\frac{d^2}{dt^2} - V_0) \neq 0 \) then \( (0,1) \) has at least three \( T \)-periodic solutions provided that either

\[
(1) \ M^0 (-\frac{d^2}{dt^2} - V_0) \neq 0, \ m + \frac{1}{2} \ <V_0s,s>-V(t,s) > 0 \ \text{as} \ |s| \ \text{small} \\
\ \ s \neq 0 \ \text{or} \\

(2) \ m + \frac{1}{2} \ <V_0s,s>-V(t,s) < 0 \ \text{as} \ |s| \ \text{small} \ s \neq 0
\]

holds.

Corollary 2.12. If \( 0 \leq V(t,s) \leq M, \forall \ (t,s) \in R \times R^N, V(t,s) \to m \) as \( |s| \to \infty \) uniformly in \( t \) and (V4) holds, \( V_0 = \mu I \ \forall \ t \in [0,T] \) then theorem 2.11 holds provided that either

\[
(1) \ \mu > \frac{4\pi^2}{T^2} , \ m + \frac{1}{2} \ <\mu s,s>-V(t,s) > 0 \ \text{as} \ |s| \ \text{small} \ s \neq 0 \ \text{or} \\

(2) \ m + \frac{1}{2} \ <\mu s,s>-V(t,s) < 0 \ \text{as} \ |s| \ \text{small} \ s \neq 0
\]

holds.

Remark 2.13. If \( \nabla_s V(t,s) \neq 0 \) for any fixed \( s \neq 0 \) we can get at least two non-trivial \( T \)-periodic solutions.

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