MULTIPLE PERIODIC SOLUTIONS OF SOME SECOND ORDER DYNAMICAL SYSTEM (*)

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RIASSUNTO. *Generalizziamo l'idea del linking locale al caso in cui la condizione di Palais-Smale non è verificata per qualche valore. Come applicazione, studiamo l'esistenza di soluzioni multiple T-periodiche di alcuni sistemi dinamici del secondo ordine.*

SUMMARY. *We generalize the local linking idea to the case in which the Palais and Smale compactness condition fails at some value. As an application, we study the existence of multiple T-periodic solutions of some second order dynamical system.*

In this paper, we study the existence of multiple T-periodic solutions of Hamiltonian system of the form

$$-\ddot{y} = \nabla_y V(t, y) \tag{0.1}$$

where $y \in R^N$, $V \in C^1(R \times R^N, R)$, V is T-periodic in t and ∇ denotes the gradient.

In section 1, we prove an existence theorem of critical point which is more general than the theorem given in [8] or [9]. We generalize the local linking idea to the case in which the Palais and Smale compactness condition fails at some value.

In section 2, we consider the existence of multiple T-periodic solutions of (0.1). We will deal with three cases:

(1) Potentials bounded from above.

(2) Asymptotically quadratic linear case.

(3) Bounded potentials.

The paper which are devoted to study these problem can be found in [1], [2], [3], [4], [5], [6], [7], [11], [13].

(*) Pervenuto in Redazione il 3 giugno 1988.

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1. Local linking and an existence theorem.

Let X be a real Hilbert space and have a decomposition $X = X_1 + X_2$ where $\dim X_2 = m < \infty$. Let $f \in C^1(X, R)$.

DEFINITION 1.1. If there exist $r, b > 0$ such that

$$\begin{array}{ll} f(x) \geq b & x \in S_1 \\ f(x) \geq 0 & x \in B_1 \\ f(x) \leq 0 & x \in B_2 \\ f(x) \leq -b & x \in S_2 \end{array} \tag{1.1}$$

where

$$\begin{aligned} B_1 &= \{x \in X_1 \mid \|x\| \leq r\} \\ B_2 &= \{x \in X_2 \mid \|x\| \leq r\} \\ S_1 &= \partial B_1 \\ S_2 &= \partial B_2 \end{aligned} \tag{1.2}$$

then we say that f has a local linking at 0.

DEFINITION 1.2. We will say that f satisfies the (P.S) (Palais-Smale) condition in S where S is a subset of X, if for every sequence $\{u_n\}$ in S such that $f(u_n)$ is bounded and $\nabla f(u_n) \to 0$ there exists a subsequence converging to an element of S.

Set

$$\begin{aligned} f_b &= \{x \in X \mid f(x) \leq b\} \\ f^b &= \{x \in X \mid f(x) > b\} \\ f^{-1}(b) &= \{x \in X \mid f(x) = b\} \\ K_c &= \{x \in X \mid f(x) = c \quad \nabla f(x) = 0\}. \end{aligned}$$

THEOREM 1.3. *Suppose that $f \in C^1(X, R)$ is bounded below and satisfies the* (P.S) *condition in $X \setminus f^{-1}(0)$, f has a local linking at* 0 *then f has at least three distinct critical points.*

Proof. Let $m < 0$ be the minimal value of f, x_0 be the unique point such that $f(x_0) = m$, we also suppose that f has no critical value in $(m, 0)$, otherwise we get the theorem. There exists a negative pseudogradient flow $\eta(t,x) : [0, 1] \times S_2 \to f_0$, such that $\eta(1,x) = x_0$ (refer [8], [9] for the details). Set

$$\begin{aligned} \Gamma &= B_2 \cup \eta([0, 1] \times S_2) \\ Q &= B_1 \times B_2. \end{aligned}$$

Define a mapping ϕ: ∂ ([0, 1] $\times B_2) \to f_0$ by

$$\phi(t,x) = \begin{cases} x & \forall x \in B_2 \quad t = 0 \\ \eta(t,x) & \forall x \in S_2 \quad t \in (0,1) \\ x_0 & \forall x \in B_2 \quad t = 1. \end{cases}$$

Set

$$F_t = x_1 - \tilde{\phi}\ (t, x_2) \quad x_1 \in B_1, x_2 \in B_2, t \in [0, 1]$$

where $\tilde{\phi}(t, x_2)$ is any extension of $\phi(t, x_2)$ on $[0, 1] \times B_2$. We claim that there exists a $t_0 \in [0, 1], x_2^0 \in B_2, x_1^0 \in \partial B_1$ such that

$$x_1^0 = \tilde{\phi}(t_0, x_2^0). \tag{1.3}$$

If (1.3) is not true then we have

$$F_t(\partial B_1 \times B_2) \neq 0\ \forall t \in [0, 1]. \tag{1.4}$$

From that $f(x_1) \geq 0\ \forall x_1 \in B_1, f(\tilde{\phi}\ (t, x_2)) = f(\phi\ (t, x_2)) < 0\ \forall\ t \in [0, 1], x_2 \in S_2$, we have

$$F_t(B_1 \times \partial B_2) \neq 0 \tag{1.5}$$

combining (1.4) and (1.5) we have that $F_t(\partial Q) \neq 0\ \forall t \in [0, 1]$.

Computing the Leray-Schauder degree for the mappings F_t and by the homotopy invariance property of degree we have that

$$\begin{aligned}(-1)^m &= \deg(P_1 - P_2, Q, 0) = \deg(F_0, Q, 0) = \deg(F_1, Q, 0) \\ &= \deg(P_1 - x_0, Q, 0) = 0 \end{aligned} \tag{1.6}$$

contradiction! where P_1, P_2 are orthogonal projections from X to X_1, X_2, respectively. So (1.3) holds. It means that ∂B_1 and Γ link nontrivially, see [12]. We claim that f must have a critical value which is not less than b. Let $\tilde{\phi}$ be an extension of ϕ. Take b_0 such that $b > b_0 > 0$. Set

$$W = \{\Psi \in C(\tilde{\phi}([0,1] \times B_2, X)) \mid \Psi(x) = \begin{cases} \text{Id} & \text{as } x \in \tilde{\phi}([0,1] \times B_2) \cap f_{b_0} \\ \text{continuous as } & x \in \tilde{\phi}([0,1] \times B_2) \cap f^{b_0} \end{cases} \quad (1.7)$$

Define

$$c = \inf_{\Psi \in W} \sup_{x \in \tilde{\phi}([0,1] \times B_2)} f(\Psi(x)). \quad (1.8)$$

From (1.3) we have $c \geq b$. Take ε small such that $0 < \varepsilon < \frac{1}{2}(c - b_0)$.

Since f satisfies the (P.S) condition in $X \setminus f^{-1}(0)$ by the deformation theorem there exists a flow $\zeta(t, x)$ which satisfies the following properties:

(a) $\zeta(t, x) \in C([0, 1] \times f^{b_0}, f^{b_0})$
(b) $\zeta(0, x) = x \;\; \forall x \in f^{b_0}$
(c) $\zeta(t, x) = x \;\; \forall x \in f^{-1}([b_0, c\text{-}2\varepsilon]) \cup f^{c+2\varepsilon}$
(d) If $K_c = \emptyset$ then $\zeta(1, f_{c+\varepsilon}) \subset f_{c-\varepsilon}$.

By the standard minimax argument we get a contradiction. We have proved the theorem.

REMARK 1.4. In definition 1.1 if we replace the ball B_1, B_2 by any other domain, say, a rectangle theorem 1.3 still holds.

With $M^0(\cdot), M^-(\cdot)$ we denote in the following the zero and negative Morse index, respectively of the symmetric matrix defining it.

DEFINITION 1.5. Let $f \in C^1(X, R)$. We say that f is asymptotically quadratic, if there is a bounded self-adjoint operator A_∞ satisfying

$$\|\nabla f(x) - A_\infty x\| / \|x\| \to 0 \text{ as } \|x\| \to \infty \;. \quad (1.9)$$

We say that f is nonresonance at the infinity, if

$$M^0(A_\infty) = 0.$$

DEFINITION 1.6. Let $\{e_1, e_2,..., e_l,...\}$ be a complete orthonormal system of X, X_l = Span $\{e_1,...e_l\}$. If for any sequence $x_l \in X_l$ $(l = 1, 2,...)$ satisfies $f(x_l) \leq c < +\infty$, $\| \nabla f|_{X_l}(x_l) \|_{X_l} \to 0$ possesses a convergent subsequence in X, then we say that f satisfies (P.S)* condition (where $f|_{X_l}$ is the restriction of f on X_l).

We state the following proposition which was first introduced in [10].

PROPOSITION 1.7. *Suppose that $f \in C^1(X, R)$ is a bounded functional at any bounded set of X, f is asymptotically quadratic and nonresonance at the infinity. If f satisfies the* (P.S) *and* (P.S)* *conditions, f has a local linking at* 0 *and $M^-(A_\infty) \neq m$ ($\dim X_2$) then f has at least one non-trivial critical point.*

2. Applications to the search of periodic solutions for second order Hamiltonian systems.

In this section we study the existence of multiple T-periodic solutions of (0.1).

First we will make the following assumptions on V

(V_1) $V(t, s) \to -\infty$ as $|s| \to \infty$ uniformly in t.

(V_2) There exists a $N \times N$ symmetric matrix V_0 such that

$$|\nabla_s V(t, s) - V_0 s| \,/\, |s| \to 0 \text{ as } |s| \to 0 \text{ uniformly in } t. \tag{2.1}$$

It is easy to see that (V_2) implies that $V(t, 0) = m$, $\nabla_s V(t, 0) = 0$, where m is a function of t.

Set

$$M^0\left(\frac{-d^2}{dt^2} - V_0\right) = \sum_{k=0}^{\infty} M^0\left(\begin{bmatrix} \frac{k^2 4\pi^2}{T^2} & & 0 \\ & \ddots & \\ 0 & & \frac{k^2 4\pi^2}{T^2} \end{bmatrix} - V_0\right)$$

$$M^{-}\left(\frac{-d^2}{dt^2}-V_0\right)=\sum_{k=0}^{\infty}M^{-}\left(\begin{bmatrix}\frac{k^2 4\pi^2}{T} & & 0\\ & \ddots & \\ 0 & & \frac{k^2 4\pi^2}{T}\end{bmatrix}-V_0\right)$$

THEOREM 2.1. *If $V(t, s)$ satisfies* (V_1), (V_2), $M^0\left(-\frac{d^2}{dt^2}-V_0\right)=0$, $M^{-}\left(\frac{-d^2}{dt^2}-V_0\right)\neq 0$ then (0.1) has at least three T-periodic solutions.

Proof. Let $X=H^1(S^1, R^N)$ with the inner product

$$(x,y)_X=\int_0^T <\dot{x},\dot{y}> dt + \int_0^T <x,y> dt$$

where <, > denotes the scalar product in R^N. Define $f: X \to R$ as

$$f(y)=\frac{1}{2}\int_0^T |\dot{y}|^2 - \int_0^T V(t,y).$$

It is well known that the critical point of f on X are T- periodic solutions of (0.1). From lemma 4.2 in [7] we know that f satisfies the (P.S) condition in X. We claim that f has a local linking at 0. From Riesz's representation theorem we get a bounded self-adjoint operator A_0 such that

$$\left(-\frac{d^2}{dt^2}y(t)-V_0 y(t), z(t)\right)=(A_0 y, z)_X \ \forall z\in X \tag{2.3}$$

where $(\cdot,\cdot)$ denotes the duality product. From $M^0\left(-\frac{d^2}{dt^2}-V_0\right)=M^0(A_0)=0$, $M^-(A_0)\neq 0$ we know that X can have a decomposition $X=X_1+X_2$ where X_1 is the positive subspace of A_0 and X_2 is the negative subspace of A_0.

If r is taken small enough $b=\frac{r^2}{4}$ dist $(0,\sigma(A_0))$ where $\sigma(A_0)$ denotes the spectrum of A_0, then it is easy to see that f has a local linking at 0. f is bounded below. From theorem 1.3 we get the theorem.

THEOREM 2.2. *If* $V(t, s)$ *satisfies* (V_1), (V_2), $M^0(-\frac{d^2}{dt^2} - V_0) \neq 0$ *then* (0.1) *has at least three T-periodic solutions provided that either*

(1) $M^-(-\frac{d^2}{dt^2} - V_0) \neq 0$

$m + \frac{1}{2} < V_0 s, s > - V(t, s) > 0$ *as* $|s|$ *small* $s \neq 0$ *or*

(2) $m + \frac{1}{2} < V_0 s, s > - V(t, s) < 0$ *as* $|s|$ *small* $s \neq 0$

holds.

Proof. We only need to prove that f has a local linking at 0. Let X_+, X_- and X_0 be the positive, the negative and the zero space of A_0, respectively. In case (1) we take $X_1 = X_+ + X_0, X_2 = X_-$ and in case (2) we take $X_1 = X_+, X_2 = X_- + X_0$. Since $H^1(S^1, R^N)$ is continuoulsy embedded in $C(S^1, R^N)$, from (1) or (2) we know that f has a local linking at 0. From theorem 1.3 we get the theorem.

COROLLARY 2.3. *Suppose the assumptions of theorem* 2.1 *or* 2.3 *are satisfied and for any fixed* $s \neq 0$ $\nabla_s V(t, s) \not\equiv 0$ *then* (0.1) *has at least two non-trivial solutions (i.e. non-constant solution).*

Next, we will be concerned with potentials which satisfy the following

(V_3) There exists a $N \times N$ symmetric matrix V_∞ such that

$$|\nabla_s V(t, s) - V_\infty s| / |s| \to 0 \text{ as } |s| \to \infty \text{ uniformly in } t. \tag{2.4}$$

We have the following

THEOREM 2.4. *If* $V(t, s)$ *satisfies* (V_2), (V_3), $M^0(-\frac{d^2}{dt^2} - V_0) = 0$, $M^0(-\frac{d^2}{dt^2} - V_\infty) = 0$, $M^-(-\frac{d^2}{dt^2} - V_0) \neq M^-(-\frac{d^2}{dt^2} - V_\infty)$ *then* (0.1) *has at least two T-periodic solutions.*

Proof. Since $M^0(-\frac{d^2}{dt^2} - V_\infty) = 0$, so for any sequence $\{u_n\}$ in X such that $\nabla f(u_n) \to 0$ we have

$$A_\infty u_n - B(\nabla_y V(t, Ju_n) - V_\infty Ju_n) \to 0 \tag{2.5}$$

where A_∞ and B is the representation of $-\frac{d^2}{dt^2} - V_\infty$ and identity in $H^1(S^1, R^N)$, respectively, J is the compact embedding operator from $H^1(S^1, R^N)$ to $L^2(0, T; R^N)$. If $\|u_n\| \to \infty$ then from (2.4) we have

$$\|A_\infty^{-1} B (\nabla_y V(t, Ju_n) - V_\infty Ju_n)\| / \|u_n\|$$

$$\leq c\,\|\nabla_y V(t, Ju_n) - V_\infty Ju_n\|_{L^2} / \|u_n\|$$

$$\leq c\left(\int_0^T |\nabla_s V(t, u_n(t)) - V_\infty u_n(t)|^2\right)^{\frac{1}{2}} / \|u_n\| \to 0$$

contradiction! So $\{u_n\}$ are bounded in X. It is the same to prove that f satisfies the (P.S)* condition. From proposition 1.7 we get the theorem.

THEOREM 2.5. *If $V(t, s)$ satisfies* (V_2), (V_3), $M^0(-\frac{d^2}{dt^2} - V_\infty) = 0$ *but* $M^0(-\frac{d^2}{dt^2} - V_0) \neq 0$ *then* (0.1) *has at least two T-periodic solutions provided that either*

(1) $M^-(-\frac{d^2}{dt^2} - V_0) \neq M^-(-\frac{d^2}{dt^2} - V_\infty)$

$m + \frac{1}{2} < V_0 s, s > - V(t, s) > 0$ *as* $|s|$ *small* $s \neq 0$ *or*

(2) $M^-(-\frac{d^2}{dt^2} - V_0) + M^0(-\frac{d^2}{dt^2} - V_0) \neq M^-(-\frac{d^2}{dt^2} - V_\infty)$

$m + \frac{1}{2} < V_0 s, s > - V(t, s) < 0$ *as* $|s|$ *small* $s \neq 0$

holds.

COROLLARY 2.6. *Suppose the assumptions of theorem* 2.4 *or* 2.5 *are satisfied and* $\nabla_s V(t, s) \not\equiv 0$ *for any fixed* $s \neq 0$ *then* (0.1) *has at least one non-trivial T-periodic solution.*

The following theorem give some equivalent conditions.

THEOREM 2.7. *Suppose that V satisfies* (V_3), $M^0(-\frac{d^2}{dt^2} - V_\infty) = 0$ *then the following conditions are equivalent:*

(1) $f(y) = \frac{1}{2} \int |\dot{y}|^2 - V(t, s)$ *is bounded below.*

(2) *V_∞ is the negative definite.*

(3) *$V(t, s) \to -\infty$ as $|s| \to \infty$ uniformly in t.*

Proof. (1) $\Rightarrow$ (2): From $f(y) \geq m$ $\forall y \in X$ we have $-m \geq \int_0^T V(t, s)\, dt$ $\forall s \in R^N$.

From (V_3) we have $V(t, s) = \frac{1}{2} < V_\infty s, s > + \varepsilon(|s|^2)$ as $|s| \to \infty$

so

$$- m \geq \int \frac{1}{2} <V_\infty s, s> + \varepsilon(|s|^2).$$

This implies that V_∞ is non-positive, but from $M^0(-\frac{d^2}{dt^2} - V_\infty) = 0$, V_∞ must be negative definite.

(2) $\Rightarrow$ (3): It is clear from $V(t, s) = \frac{1}{2} < V_\infty s, s > + \varepsilon(|s|^2)$.

(3) $\Rightarrow$ (1): It is trivial.

Finally, we will be concerned with bounded potentials which satisfies the following

(V_4) $\nabla_s V(t, s) \to 0$ as $|s| \to \infty$ uniformly in t.

(V_4) means that $V_\infty = 0$. Since $M^0(-\frac{d^2}{dt^2}) \neq 0$ so f does not satisfy the (P.S) condition in X, in general. But we can use the following proposition given in [7].

PROPOSITION 2.8. *If $0 \leq V(t, s) \leq M$ $\forall (t, s) \in R \times R^N$, $V(t, s) \to m$ as $|s| \to \infty$ uniformly in t, V satisfies* (V_4), *then* (P.S) *condition holds for f in $X \backslash f^{-1}(-Tm)$.*

Using proposition 2.8 and theorem 1.3 we can prove

THEOREM 2.9. *If V satisfies* (V_2) *and the assumptions of proposition 2.8, $M^0(-\frac{d^2}{dt^2} - V_0) = 0$ and $M^-(-\frac{d^2}{dt^2} - V_0) \neq 0$ then* (0.1) *has at least three solutions.*

COROLLARY 2.10. *If* $0 \leq V(t,s) \leq M \ \forall (t,s) \in R \times R^N$, $V(t,s) \to m$ *as* $|s| \to \infty$ *uniformly in t and* (V_4) *holds*, $V_0 = \mu I \ \forall t \in [0,1]$, $\mu > \frac{4\pi^2}{T^2}$ $\mu \neq k^2 \frac{4\pi^2}{T^2}$, $k = 2, 3, \ldots$ *then theorem* 2.9 *holds.*

THEOREM 2.11. *If V satisfies the assumptions of theorem* 2.9 *but* $M^0(-\frac{d^2}{dt^2} - V_0) \neq 0$ *then* (0, 1) *has at least three T-periodic solutions provided that either*

(1) $M^-(-\frac{d^2}{dt^2} - V_0) \neq 0$, $m + \frac{1}{2} <V_0 s, s> - V(t,s) > 0$ *as* $|s|$ *small* $s \neq 0$ *or*

(2) $m + \frac{1}{2} <V_0 s, s> - V(t,s) < 0$ *as* $|s|$ *small* $s \neq 0$

holds.

COROLLARY 2.12. *If* $0 \leq V(t,s) \leq M$, $\forall (t,s) \in R \times R^N$, $V(t,s) \to m$ *as* $|s| \to \infty$ *uniformly in t and* (V_4) *holds*, $V_0 = \mu I \, \forall t \in [0,T]$ *then theorem* 2.11 *holds provided that either*

(1) $\mu > \frac{4\pi^2}{T^2}$, $m + \frac{1}{2} <\mu s, s> - V(t,s) > 0$ *as* $|s|$ *small* $s \neq 0$ *or*

(2) $m + \frac{1}{2} <\mu s, s> - V(t,s) < 0$ *as* $|s|$ *small* $s \neq 0$

holds.

REMARK 2.13. If $\nabla_s V(t,s)) \not\equiv 0$ for any fixed $s \neq 0$ we can get at least two non-trivial T-periodic solutions.

This article was written while I was visiting Scuola Normale Superiore, Pisa. It is my pleasure to thank Professor A. Ambrosetti for his hospitality.

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