ALGEBRA OF PSEUDO-DIFFERENTIAL C*-OPERATORS(*)

by NOOR MOHAMMAD (in Islamabad)(**).

SOMMARIO.- In questo lavoro si studia l' algebra degli operatori pseudo-differenziali nel contesto delle C*-algebre. In particolare si fa vedere che lo spazio di tutti gli operatori pseudo-differenziali su una varietà compatta è un algebra involutiva.

SUMMARY.- In this paper we study the algebra of pseudo-differential operators in the context of C*-algebras. We essentially show that the space of all pseudo-differential operators on a compact manifold is an involutive algebra.

1. Introduction

Various functional methods have been intensively applied to topology during the last decade. The technique of C*-algebras is quite useful in a number of problems connected with topological properties of manifolds. In particular, C*-algebras and their representations play an important role in K-theory and problems associated with it. Several authors have obtained interesting results concerning pseudo-differential operators in the framework of C*-algebras. Mishenko [5] and [6] interpreted several versions of the theory of pseudo-differential operators in terms of C*-algebras, that is, elliptic pseudo-differential operators on a compact manifold; pseudo-differential operators on a Euclidian space, operators with almost periodic functions etc. Mishenko and Fomenko [7] derived analogues of the well-known Atiyah-Singer index formulas in this situation.

In this paper we study the algebra of pseudo-differential operators in the framework of C*-algebras. We essentially prove that every pseudo-differential operator of order \(m\) admits an adjoint operator, in this case, which is again a pseudo-differential operator. Consequently, we get that the space of all pseudo-differential operators on a compact manifold is an involutive algebra. This kind of problem has been discussed, in the classical case, by Hörmander [1], Kohn and Nirenberg [2] and Kumano-go [3].

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(**) Indirizzo dell'Autore: Department of Mathematics, Quaid-I-Azam University, Islamabad (Pakistan).
and [4] among many others. The formulation of such a problem naturally arises when one seeks analogues of the Atiyah-Bott formulas for this case.

Regarding the general theory of pseudo-differential operators we refer to [10].

2. Preliminaries

Let $A$ be any $C^*$-algebra with an identity and $K$ a right $A$-module. Recall the definition of a Hilbert $C^*$-module as in ([9] and [5]).

**Definition 2.1.** A pre-Hilbert $C^*$-module is a right $A$-module $K$ equipped with an inner product $\langle \cdot, \cdot \rangle : K \times K \to A$ satisfying $\forall x, y, z \in K, \lambda \in C$ the following conditions:

i) $\langle x, x \rangle \geq 0; \langle x, x \rangle = 0$ only if $x = 0$;
ii) $\langle x, y \rangle = \langle y, x \rangle^*$;
iii) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle; \langle x, \lambda y \rangle = \lambda \langle x, y \rangle$;
iv) $\langle xa, y \rangle = a^* \langle x, y \rangle; \langle x, ya \rangle = \langle x, y \rangle a$.

For all $x \in K$, $\|x\|_K^2 = \|\langle x, x \rangle\|$ defines a norm on a pre-Hilbert $C^*$-module $K$. Moreover, it satisfies $\|\lambda a\|_K \leq \|\lambda\|_K \cdot \|a\|$ and $\|\langle x, y \rangle\| \leq \|x\|_K \cdot \|y\|_K$, $\forall x, y \in K, a \in A$. The inner product gives the homomorphism

$$\varphi : K \to K^* = \text{Hom}_A(K, A)$$

which is not, however, an isomorphism for an arbitrary $C^*$-algebra $A$, in contrast to the case where $A$ is the field of complex numbers.

For this reason, a supplementary condition is imposed.

v) the homomorphism $\varphi : K \to K^* = \text{Hom}_A(K, A)$ is an isomorphism. This guarantees the existence of an adjoint to any bounded homomorphism of Hilbert $C^*$-modules.

We shall use the following terminology. A pre-Hilbert $C^*$-module $K$ which is complete with respect to the norm $\|\cdot\|_K$, together with an inner product satisfying (i)-(iv), is called a Hilbert $C^*$-module. If (v) is also satisfied, then $K$ is called self-dual. The inner product $\langle \cdot, \cdot \rangle$ will be called a Hermitian product. Denote by $\text{Hom}_A^*(K_1, K_2)$ the space of $A$-homomorphisms $T : K_1 \to K_2$ having adjoints, that is, homomorphisms $T^*$ such that
\[ <Tx, y> = <x, T^*y>, x \in K_1, y \in K_2. \]

It is clear that \( \text{Hom}_A^*(K_1, K_2) \subset \text{Hom}_A(K_1, K_2) \) is a closed subspace in the operator norm. Also, the space \( \text{End}_A^*(K) = \text{Hom}_A^*(K, K) \) is a \( C^* \)-algebra. If we denote by \( K^\# = \text{Hom}_A^*(K, A) \), then the homomorphism \( \phi : K \to K^\# \) induced by the inner product \( <x, y> \) realizes the isomorphism \( \phi : K \to K^\# \subset K^*. \)

Thus we have a natural category \( \mathcal{J} \) whose objects are Hilbert \( C^* \)-modules and whose morphisms are operators in \( \text{Hom}_A^*(K_1, K_2) \).

Examples of such objects are numerous. For instance, the \( C^* \)-algebra \( A \) itself as well as the direct sum \( A^k \) of \( k \) copies of \( A \), with the Hermitian product given by \( <x, y> = \sum_{i=1}^{k} x_i^* y_i \), is a Hilbert \( C^* \)-module. Also, these modules are self-dual. Furthermore, \( A^k \) is projective relative to epimorphisms.

Denote by \( l_2(A) \) the space of sequences \( x = (x_1, \ldots, x_n, \ldots) \forall x_n \in A \), which satisfy the condition that \( \sum_{n=1}^{\infty} x_n^* x_n \) converges in \( A \). We can define a Hermitian product in \( l_2(A) \) by putting

\[ <x, y> = \sum_{n=1}^{\infty} x_n^* y_n \quad (2.1) \]

and hence the norm by

\[ \|x\|^2 = \| \sum_{n=1}^{\infty} x_n^* x_n \|. \quad (2.2) \]

The convergence of the series (2.1) follows from an analogue of Cauchy's inequality for \( C^* \)-algebras (see [7]):

\[ \| \sum_{n=1}^{\infty} x_n^* y_n \|^2 < \| \sum_{n=1}^{\infty} x_n^* x_n \| \cdot \| \sum_{n=1}^{\infty} y_n^* y_n \|. \quad (2.3) \]

Then \( l_2(A) \) is a Hilbert \( C^* \)-module ([7]).
For the sake of simplicity we also assume that each Hilbert C*-module $K$ has at most a countable set of generators, that is, a countable subset whose $A$-linear span is dense in $K$.

**THEOREM 2.1 ([6]).** Every countably generated Hilbert C*-module is a projective object in the category $\mathcal{M}$ relative to epimorphisms.

If the Hilbert C*-module $K$ is finitely generated, then $K$ is self-dual, for there is an epimorphism $f : A^k \to K$ having an adjoint. Then by Theorem 2.1, $K$ is a direct summand in the module $A^k$ and is, therefore, self-dual.

3. Vector $A$-bundles and pseudo-differential operators over C*-algebras

**DEFINITION 3.1.** Let $A$ be a C*-algebra. By a vector $A$-bundle we mean a locally trivial fibre bundle $\xi = (E, p, M, F, G)$ where $E$ is a total space, $M$ is the base of the fibre bundle, $p : E \to M$ is a projection, $F$ is a fibre of the fibre bundle which is finitely generated projective Hilbert C*-module and $G$ is the structural group equal to $\text{Aut}_A(F)$, the group of $A$-automorphisms of the Hilbert C*-module $F$. If $F$ is a free, $k$-dimensional Hilbert C*-module, then $\text{Aut}_A(F)$ is the same as the group of invertible $k^{\text{th}}$-order matrices with coefficients in $A$.

Suppose that the base $M$ is a compact smooth manifold (of class $C^\infty$). Denote by $\Gamma(\xi)$ the space of continuous sections of the bundle $\xi$. The space $\Gamma(\xi)$ is endowed with the natural structure of Hilbert C*-module, coinciding with the structure of the Hilbert C*-module in each fibre $F$ when $\Gamma(\xi)$ is restricted to $F$. Each $A$-bundle $\xi$ admits a fibre Hermitian product with values in $A$. Thus if $u_1, u_2 \in \Gamma(\xi)$ are two sections, then a continuous function $\langle u_1, u_2 \rangle \in C(M, A)$ is defined, $C(M, A)$ being the algebra of continuous functions on $M$ with values in $A$. Without loss of generality we can assume that the gluing functions of the vector bundle are smooth sections (of class $C^\infty$), and denote by $\Gamma^\infty(\xi)$ the space of smooth sections (of class $C^\infty$) of the bundle $\xi$. We can then choose the fibre Hermitian product to be smooth, i.e. for any two sections $u_1, u_2 \in \Gamma^\infty(\xi)$, the function $\langle u_1, u_2 \rangle \in C^\infty(M, A)$, where $C^\infty(M, A)$ is the space of smooth functions (of class $C^\infty$) with values in $A$.

We now want to introduce Sobolev norms in $\Gamma^\infty(\xi)$. First we consider the local situation.

Let $X \subset \mathbb{R}^n$ be a bounded open set, and $C^\infty_0(X, A)$ the ring of smooth functions (of class $C^\infty$) with compact supports and with values in $A$. Let
$S(\mathbb{R}^n, A)$ denote the space of $C^\infty$-functions whose derivatives decrease faster than any power of $\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\nu_2}$ as $\|x\| \to \infty$.

For $u \in S(\mathbb{R}^n, A)$ we define the Fourier transform $\hat{u}(\xi)$ by

$$\hat{u}(\xi) = \int e^{-i x \cdot \xi} u(x) dx,$$

(3.1)

where $\xi \in \mathbb{R}^n$, $x \cdot \xi = x_1 \xi_1 + \ldots + x_n \xi_n$ and $dx$ is the Lebesgue measure on $\mathbb{R}^n$.

The inverse Fourier transform can be defined as

$$u(x) = \int e^{i x \cdot \xi} \hat{u}(\xi) d\xi,$$

(3.2)

where $d\xi = (2\pi)^{-n} d\xi$.

We denote by $\Delta$ the operator

$$\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{(\partial x_i)^2}.$$

For $u \in C_0^\infty(X, A)$ we put

$$\|u\|_s^2 = \| \int_X ((1 + \Delta)^s u^*(x)) u(x) dx \|.$$

(3.3)

where $s$ is any real number, and denote by $H_0^s(X, A)$ the completion of $C_0^\infty(X, A)$ relative to the Sobolev norm (3.3). Then

**Lemma 3.1 ([7]).** The space $H_0^s(X, A)$ is isomorphic to $l_2(A)$ as a Hilbert $C^*$-module.

Note that a Hermitian product in $H_0^s(X, A)$ is given by

$$\langle u, v \rangle_s = \int_X ((1 + \Delta)^s u^*(x)) v(x) dx.$$

(3.4)

For $s = 0$, $H_0^0(X, A)$ equals $L_2(X, A)$ where $L_2(X, A)$ denotes the space of such measurable functions (i.e. classes) $f$, for which the integral $\int_X f^*(x)$
\( f(x) \, dx \) converges. Likewise, in \( L_2(X, A) \) we have a Hermitian product defined by

\[
<f, g> = \int_X f^*(x) g(x) \, dx,
\]

and denote the induced norm by \( \| \cdot \|_{L_2} \).

Let \( E \) be trivial \( A \)-bundle on the domain \( X \) with fibre \( P \), where \( P \) is a finitely generated projective Hilbert \( C^* \)-module with a nondegenerate, positive definite inner product with values in \( A \). For the sake of simplicity, we assume that \( P \) is \( A^k \), where \( A^k \) is a direct sum of \( k \)-copies of \( A \). Analogous to (3.3) we define Sobolev norms in the space \( \Gamma_0^s(X, A^k) \) of sections with compact support by

\[
\| u \|_s^2 = \int_X < (1 + \Delta)^s u(x), u(x) > \, dx.
\]

The completion of the space relative to the Sobolev norm is denoted by \( \mathcal{H}_0^s(X, A^k) \). Then it follows trivially from Lemma 3.1 that the Hilbert \( C^* \)-module \( \mathcal{H}_0^s(X, A^k) \) is isomorphic to the module \( l_2(A^k) \), the direct sum in the module \( l_2(A) \times \cdots \times l_2(A) \).

Similarly one can verify that, \( L_2(X, A^k) \) is the same as \( \mathcal{H}_0^0(x, A^k) \) for \( s = 0 \), in which the Hermitian product is given by

\[
<f, g> = \int f(x), g(x) \, dx; \quad <f(x), g(x)> = : \sum_{i=1}^{k} f_i^*(x) g_i(x).
\]

We now define pseudo-differential \( A \)-operators in spaces of sections of \( A \)-bundles \( E_1 \) and \( E_2 \). Let \( \pi : T^* X \to X \) be the natural projection of a cotangent bundle. Consider the pre-images of the bundles \( \pi^*(E_i) \), \( i = 1, 2 \), taking account of the fact that the inner product in each fibre is induced by the inner product in the fibres of \( E_i \). Without loss of generality, we assume that \( E_i = E, i = 1, 2 \), with fibre \( A^k \). We consider the \( A \)-homomorphisms of the bundles \( a : \pi^*(E) \to \pi^*(E) \) as a family of \( A \)-homomorphisms

\[
a(x, \xi) : A^k \to A^k
\]

parametrized by points of the cotangent bundle \( (x, \xi) \in T^* X \). Suppose that \( a(x, \xi) \) satisfy the following conditions:
(a) \[ \| \partial_{x}^{\alpha} \partial_{\xi}^{\beta} a(x, \xi) \| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\alpha|} \] (3.8)

where \( \langle \xi \rangle \) stands for \( (1 + \sum_{i=1}^{n} \xi_{i}^{2})^{1/2} \), \( \alpha, \beta \) are multi-indices of non-negative integers and \( m \) being any real number.

(b) \( a(x, \xi) \) have compact support in the variable \( x \), that is, \( \pi(\text{supp } a) \subset X \) is a compact set.

We shall call the homomorphism satisfying conditions (a) and (b) the symbol of the pseudo-differential \( A \)-operator \( T \), which is defined by:

\[ T u(x) = \int e^{ix \cdot \xi} a(x, \xi) \hat{u}(y) \, d\xi, \] (3.9)

where \( u \in \Gamma_{0}^{\infty}(X, A^{k}) \).

The number \( m \) is called the order of the operator \( T \). We shall denote by \( S^{m} \) the class of symbols satisfying the properties (a) and (b) as given above.

THEOREM 3.1. The operator \( T \) defined by (3.9) is a continuous map from \( \Gamma_{0}^{\infty}(X, A^{k}) \) into itself.

The proof of Theorem 3.1 is an easy computation (see [8]).

By means of Theorem 3.1, we can extend the operator \( T \) as a continuous map of \( S(R^{n}, A^{k}) \) into \( S(R^{n}, A^{k}) \).

THEOREM 3.2. ([7]). Every pseudo-differential \( A \)-operator \( T \) of order \( m \):

\[ T : H_{0}^{s}(X, A^{k}) \to H_{0}^{s-m}(X, A^{k}) \]

is a bounded operator in the Sobolev norms.

It now follows trivially from Theorem 3.2 the following:

THEOREM 3.3. \((L_{2}\text{-continuity})\). A pseudo-differential \( A \)-operator \( T \) of order zero can be extended to a bounded map of \( L_{2}(R^{n}, A^{k}) \) into \( L_{2}(R^{n}, A^{k}) \), i.e., there exists a constant \( C > 0 \) such that

\[ \| Tu \|_{L_{2}} \leq C \| u \|_{L_{2}}, \quad u \in \Gamma_{0}^{\infty}(X, A^{k}) \].
Next we consider the adjoint of a pseudo-differential $A$-operator. Note that

$$< u, v > = \int < u(x), v(x) > \, dx,$$

where $u, v \in \Gamma_0^\infty(X, A^k)$ and $< u(x), v(x) >$ equals $\sum_{i=1}^{k} u_i^*(x) \, v_i(x)$.

**Theorem 3.4.** Every pseudo-differential $A$-operator $T$ of order $m$ admits an adjoint operator, $T^*$, given by

$$< Tu, v > = < u, T^* v >, \, u, v \in \Gamma_0^\infty(X, A^k), \quad (3.10)$$

which is again a pseudo-differential $A$-operator.

**Proof** (see also [8]): We can write for $u, v \in \Gamma_0^\infty(X, A^k)$,

$$< Tu, v > = \iint e^{-i(x-y) \cdot \xi} \, u^*(y) \, a^#(x, \xi) \, v(x) \, dyd\xi dx.$$

$$= \int u^*(y) \left\{ \iint e^{i(y-x) \cdot \xi} \, a^#(x, \xi) \, v(x) \, dxd\xi \right\} \, dy,$$

where $a^#(x, \xi) = (a_{ij}^*(x, \xi))$, the matrix of the symbol $a(x, \xi)$ being $a_{ij}(x, \xi)$. Hence it follows that

$$T^* v(x) = \iint e^{i(x-y) \cdot \xi} \, a^#(y, \xi) \, v(y) \, dyd\xi,$$

which is evidently a pseudo-differential $A$-operator. One can see easily that (3.10) determines $T^*$ uniquely.

Thus a pseudo-differential $A$-operator is a morphism in the category $\mathcal{A}$, described in the preceding section.

**Theorem 3.5.** Let $T$ be a pseudo-differential $A$-operator with symbol $a(x, \xi)$ and let $T^*$ be its adjoint operator. Then the symbol of $T^*$ has the following expansion:

$$a^*(x, \xi) = \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha a^#(x, \xi)$$

(3.12)

where the asymptotic sum runs over all multi-indices $\alpha$. 
THEOREM 3.6. (Composition). Let $S$ and $T$ be pseudo-differential $A$-
operators with symbols $a(x, y) \in S^{m_1}$ and $b(x, \xi) \in S^{m_2}$ respectively. Then $R = ST$ is a pseudo-differential $A$-operator with symbol $r(x, \xi) \in S^{m_1 + m_2}$, and one has the analogue of Leibniz's formula:

$$r(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^\alpha b(x, \xi) \cdot D_x^\alpha a(x, \xi). \quad (3.13)$$

Proof of Theorem 3.6.: One can write

$$R u(x) = \iint e^{i(x \cdot y + \xi \cdot \eta)} b(x, \xi) \int e^{i \eta \cdot \eta} a(y, \eta) \, dy \, d\eta \, d\xi$$

$$= \int r(x, \eta) e^{i x \cdot \eta} a(\eta) \, d\eta,$$

where

$$r(x, \eta) = \iint e^{i(x \cdot y - \eta \cdot \xi)} b(x, \xi) a(y, \eta) \, dy \, d\eta \, d\xi.$$ 

It can be shown easily, as in the classical case, that $r(x, \xi) \in S^{m_1 + m_2}$, and by Taylor's formula

$$r(x, \eta) = \iint \left( \sum_{\alpha | \leq N} \frac{1}{\alpha!} \partial_\eta^\alpha b(x, \eta)(\xi - \eta)^\alpha \right) \cdot a(y, \eta) e^{i(x \cdot y)(\xi \cdot \eta)} \, dy \, d\eta \, d\xi + r_N.$$ 

Integrating first with respect to $y$ and then with respect to $\xi$, one verifies that

$$\iint e^{i(x \cdot y)(\xi \cdot \eta)} a(y, \eta)(\xi, \eta)^\alpha \, dy \, d\xi = D_x^\alpha a(x, \eta).$$

Hence

$$r(x, \eta) = \sum_{|\alpha| \leq N} \frac{1}{\alpha!} \partial_\eta^\alpha b(x, \eta) \cdot D_x^\alpha a(x, \eta) + r_N.$$ 

It remains to show that $r_N \in S^{m_1 + m_2 - N}$. This can be done in a similar way as the corresponding assertion in the case of classical pseudo-differential operators, provided we take into consideration the standard estimates while applying to a $C^*$-algebra.
The proof of Theorem 3.5. uses Taylor's formula as above, and can be proved as in the classical case. Therefore, we omit the proof.

As a consequence of Theorems 3.2., 3.4., 3.5. and 3.6., the class \( S^\infty = \bigcup_{m \in \mathbb{R}} S^m \) makes an involutive algebra in the sense that if \( T_i \in S^m_i, i = 1, 2, \) then \( T_1 + T_2 \in S^m \) for \( m = \max(m_1, m_2) \) and \( T_1 \cdot T_2 \in S^{m_1 + m_2} \).

In order to define a pseudo-differential \( A \)-operator in sections of the \( A \)-bundles \( E_1 \) and \( E_2 \) on a compact manifold \( M \), we consider the atlas \( \{ X_\alpha \} \) of charts of the manifold \( M \), in each of which the bundles \( E_1 \) and \( E_2 \) are trivial. Let \( \alpha : \pi^*(E_1) \to \pi^*(E_2) \) be a \( A \)-homomorphism satisfying (3.8). Let \( \{ \varphi_\alpha \} \) be a partition of unity subordinated to the covering \( \{ X_\alpha \} \), and let \( \psi_\alpha \) be functions such that \( \psi_\alpha \| \text{supp } \varphi_\alpha \equiv 1 \) and \( \text{supp } \varphi_\alpha \subset X_\alpha \). We then put

\[
Tu(x) = \sum_\alpha [T_\alpha(\varphi_\alpha u)](x); \tag{3.14}
\]

where \( u \in \Gamma^\infty(M, E_1) \) and \( T_\alpha \) is the pseudo-differential \( A \)-operator defined by (3.9) in the chart \( X_\alpha \) by means of the symbol \( a_\alpha(x, \xi) = a(x, \xi) \psi_\alpha(x) \).

Using the partition of unity \( \varphi_\alpha \) and (3.4), we define Sobolev norms in the space of sections \( \Gamma^\infty(M, E_1) \). Let \( u_1, u_2 \in \Gamma^\infty(M, E_1) \) be arbitrary sections. We put

\[
<u_1, u_2>_s = \sum_\alpha \int (1 + \Delta_\alpha)^s \varphi_\alpha(x) u_1(x), \varphi_\alpha(x) u_2(x)) \, dx,
\]

\[
\|u\|_s^2 = \|<u, u>_s\|. \tag{3.15}
\]

The completion, relative to the Sobolev norm, of the space of sections \( \Gamma^\infty(M, E_1) \) will be denoted by \( H^s(M, E_1) \). In general, if an operator \( T : H^s(M, E_1) \to H^{s-m}(M, E_2) \) is bounded for all \( s \), then we shall say that the operator \( T \) is of order \( m \). We now formulate some necessary propositions for later use.

**Theorem 3.7.** The pseudo-differential \( A \)-operator \( T : H^s(M, E_1) \to H^{s-m}(M, E_2) \) of order \( m \) defined by (3.14) is bounded in the Sobolev norms.
THEOREM 3.8. The definition of pseudo-differential $A$-operator (3.14) does not depend on the partition of unity $\varphi_\alpha$, the functions $\psi_\alpha$, or the local coordinates system to within operators of smaller order.

THEOREM 3.9. Let $a_1$ and $a_2$ be symbols of the pseudo-differential $A$-operator $T_1$ and $T_2$ respectively. Then the operator $T_3$ and $T_2 T_1$, where $a_3 = a_2 a_1$ (i.e., composition of symbols), differ by an operator of lower order.

The proof of Theorems 3.7.-3.9. is entirely analogous to that of the same propositions for the case of classical pseudo-differential operators, provided account is taken of the special features in applying the standard estimates in case of a $C^*$-algebra, for instance, which were introduced in the proof of Lemma 3.2. [7].

Likewise, with obvious modifications one can extend Theorem 3.4. to a compact manifold:

THEOREM 3.10. Every pseudo-differential $A$-operator $T$, $T : H^s(M, E_1) \rightarrow H^{s-m}(M, E_2)$ of order $m$ has an adjoint, $T^*$, which is also a pseudo-differential $A$-operator.

Summarizing all this, we have

THEOREM 3.11. The class $S^\infty = \bigcup S^m$ of pseudo-differential $A$-operators on a compact manifold $M$, is an involutive algebra. Here $S^m$ denotes the class of pseudo-differential $A$-operators $T : H^s(M, E_1) \rightarrow H^{s-m}(M, E_2)$ of order $m$. 
REFERENCES


