SUBHARMONIC SOLUTIONS OF LARGE NORM FOR SOME ASYMPTOTICALLY QUADRATIC PROBLEMS(*)

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\[
\begin{align*}
H(t, z) &= \frac{1}{2} (A(t) z, z) + \hat{H}(t, z) \\
\text{where} \\
\lim_{|z| \to +\infty} \frac{\hat{H}(t, z)}{|z|^2} &= 0 \quad \text{uniformly in } t.
\end{align*}
\]

Consider the linearized equation at infinity
\[
\dot{z} = JA(t)z
\]  \hspace{1cm} (1.2)

and let \( X(t) \) the solution matrix of \( \dot{X}(t) = JA(t)X \) and \( X(0) = \text{Id} \). It is known that \( X(t) \) is a symplectic matrix (cf. e.g. [9], § 4). We shall make the following assumptions:

\[
(H_2) \quad \begin{cases} 
\text{every symplectic matrix sufficiently close to } X(t) \text{ has an} \\
\text{eigenvalue } e^{j\omega} \text{ on the unit circle; }
\end{cases}
\]

\[
(H_3) \quad \begin{cases} 
\text{i) } \hat{H}(t, z) \geq a_1 |z|^2 - a_2 \text{ for any } t \in R, z \in R^n, \\
\text{ii) } 0 < (\hat{H}_2(t, z), z) \leq p \hat{H}(t, z) \text{ for any } z \in R^n, |z| > R.
\end{cases}
\]

We shall state the following theorem:

**Theorem 1.3.** Suppose that (H_1) – (H_3) hold. Then for \( k \in N \) (1.1) possesses a \( kT \)-periodic solutions \( z_k \). Moreover \( \sup_{t} |x(t)| \to \infty \) as \( k \to \infty \).

**Remark 1.4.** We do not know if \( kT \) is the minimal period of \( z_k \). Subharmonic solutions having arbitrarily long minimal periods have been found in [2], [3], [4], [11], [14], [15]. Moreover in [2], [3], [4], [11] the subharmonic solutions have been localized near the origin.

In the second part we consider the forced semilinear wave equation
\[
\begin{align*}
u_{tt} - u_{xx} &= f(x, t, u) \quad t \in R, x \in [0, \pi] \\
u(0, t) &= u(\pi, t) = 0 \quad t \in R
\end{align*}
\]  \hspace{1cm} (1.5)
where \( f \) is \( C^1([0,\pi] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and \( T \)-periodic in \( t \), \( T \) is a rational multiple of \( \pi \). We are concerned with the existence of subharmonic solutions for (1.5), i.e. we search for \( kT \)-periodic solutions \( (k \in \mathbb{N}) \) \( u_k(x,t) \) of (1.5).

This problem has been studied in [12], when the function

\[
F(x,t,u) = \int_0^u f(x,t,s) \, ds
\]

is either sub- or super-quadratic at infinity (i.e. \( F(x,t,u)/|u|^2 \to 0 \) or \( \infty \) as \( |u| \to \infty \)). Moreover, to overcome the difficulties arising from the infinite dimensional kernel of \( u_{xx} - u_x \), in [12] \( f \) is assumed strictly monotone in \( u \). More recently, in [7], the monotonicity assumption has been avoided using a Coron's trick.

Here we assume that \( F \) is asymptotically quadratic at infinity, more precisely there exists a real number \( f'(\infty) \) s.t.

\[
\begin{cases}
  f(x,t,u) = f'(\infty)u + g(x,t,u) \\
  \text{where} \\
  g(x,t,u) \to 0 \quad \text{as} \quad |u| \to +\infty \quad \text{uniformly in} \quad (x,t) \in [0,\pi] \times \mathbb{R}.
\end{cases}
\]

We set \( G(x,t,u) = \int_0^u g(x,t,s) \, ds \). The following theorem holds:

**Theorem 1.6.** Let \( \frac{T}{2\pi} = \frac{b}{a} \), \( a, b \in \mathbb{N}, b \) odd. Assume (W₁),

(W₂) \( f(x,t,u) \) is \( T/2 \)-periodic in \( t \), \( f(\pi-x,t,u) = f(x,t,u) \);

(W₃) there exist some positive constants \( a_i, i = 3, \ldots, 6, s \) and \( p \) s.t.

\[
\begin{align*}
  i) & \quad G(x,t,u) \geq a_3 |u|^s - a_4 \quad 1 < s < 2, \\
  ii) & \quad g(x,t,u) \leq a_5 |u|^p + a_6 \quad 0 < p < 1.
\end{align*}
\]

Then for \( k \in \mathbb{N}, k \) odd, (1.5) possesses a \( kT \)-periodic solution \( u_k \). Moreover \( \sup_{x,t} |u_k(x,t)| \to \infty \) as \( k \to \infty \).

2. Proof of theorems 1.3 and 1.6

**Proof of theorem** 1.3. By a suitable change of variable, a \( kT \)-periodic solution of (1.1) corresponds to a \( T \)-periodic solution of
\[ \dot{z} = kJH_z(kt, z). \] (2.1)

If we denote by \( L_k \) the self-adjoint realization in \( L^2(0, T) \) of the operator \( z \rightarrow -J\dot{z} - kA(t)z \) with \( T \)-periodicity conditions, it is easy to see that the functional

\[ f_k(z) = -\frac{1}{2} < L_k z, z > + k \int_0^T \hat{H}(kt, z) dt \] (2.2)

is \( C^1 \) in \( H^{1/2}(\mathbb{R}/k\mathbb{Z}) \) and its critical points are \( kT \)-periodic solutions of (1.1). Moreover standard perturbation results (cf. e.g. [10], pag. 28) permit to deduce that the spectrum \( \sigma(L_k) \) of \( L_k \) consists (like the spectrum of the operator \( z \rightarrow -J\dot{z} \)) of infinitely many positive and infinitely many negative isolated eigenvalues of finite multiplicity. We denote by \( \{\lambda_j\} \) the sequence of the eigenvalues of \( L_k \) and by \( \{M_j\} \) the sequence of the corresponding eigenspaces. We set

\[ H^+ = \bigoplus_{\lambda_j > 0} M_j, \quad H^- = \bigoplus_{\lambda_j < 0} M_j, \quad H_0 = \ker L_k, \]

where the closure is taken in \( H^{1/2} \) and \( H_0 \) can be also the space \{0\}.

Denote by \( \|\cdot\| \) and \( |\cdot|_p \) the norms in \( H^{1/2} \) and in \( L^p(0, T) \); for \( R > 0 \) let be \( B_R = \{ u \in H^{1/2} \ | \ |u| < R \} \) and \( S_R = \partial B_R \). Moreover denote by \( |\cdot|_\infty \) the norm in \( L^\infty \).

Then standard calculations (cf. e.g. [13]) show that there exist \( \alpha, \beta, R > 0 \) and \( \nu \in H^+ \) (\( |\nu| < R \)) such that

\[ f_k(z) \geq \beta \quad \text{on} \ S, \quad \text{(2.3)} \]
\[ f_k(z) \leq \alpha \quad \text{on} \ \partial Q, \quad \text{(2.4)} \]
\[ \sup_Q f_k(z) < + \infty, \quad \text{(2.5)} \]

where \( S = \{ \tilde{z} + \nu \ | \ \tilde{z} \in H^+ \oplus H^0 \} \) and \( Q = B_R \cap H^+ \).

Moreover \( f_k \) satisfies the Palais-Smale condition (even if \( H_0 \neq \{0\} \)), then by theorem 1.18 in [5] \( f_k \) possesses a critical point \( z_k \) such that

\[ f_k(z_k) \geq \beta. \]
Since \( z_1(kt) \) is also a \( kT \)-periodic solution of (1.1), we prove that infinitely many solutions \( z_k \) are distinct. Suppose first (cf. [11])

\[
z_k(t) = z_1(kt) \quad \text{for any } k \in \mathbb{N}, \quad (2.6)
\]

then

\[
c_k = f_k(z_k) = k f_1(z_1) = k c_1. \quad (2.7)
\]

We shall show in fact that, for large \( k \), there exists \( \bar{\beta} > 0 \) such that

\[
c_k \geq \bar{\beta} k^\gamma \quad \gamma > 1 \quad (2.8)
\]

and therefore the solutions \( z_k \) are not reparametrizations of \( z_1 \) for \( k \) large. Moreover, since

\[
c_k = f_k(z_k) = k \int_0^T \left[ \hat{H}(kt, z_k) - \frac{1}{2} \left( \hat{H}_z(kt, z_k) \right) \right] dt,
\]

if \( \{|z_k|_{\infty}\} \) were bounded, we would have

\[
c_k \leq Mk, \quad M > 0,
\]

contrary to (2.8). Then \( |z_k|_{\infty} \to +\infty \) as \( k \to +\infty \).

It remains to verify (2.8). In the sequel we denote by \( d_i \) some positive constants. We shall need the following lemma.

**Lemma 2.9.** Suppose that \( H \) satisfies \( (H_1), (H_2) \). Then, for \( k \) sufficiently large, there exists \( \lambda(k) \in \sigma(L_k) \) such that

\[
d_1 \leq \lambda(k) \leq d_2.
\]

**Proof.** Observe that \( \mu \) is an eigenvalue of the operator \( z \to -J\dot{z} - A(t)z \) in \( L^2(0,kT) \) iff \( k\mu \) is an eigenvalue of the operator \( z \to -J\dot{z} - k A(t)z \) in \( L^2(0,T) \). Then the proof follows by lemma 2.1 of [2].

In order to verify (2.8), it suffices to prove that

\[
f_k(z) \geq \bar{\beta} k^\gamma \quad \text{on } S,
\]
where \( S = \{ z + \delta \Phi_k \mid z \in H^-, \Phi_k \text{ a normalized eigenvector of } L_k \} \), \( \Phi_k \) being a normalized eigenvector of \( L_k \) corresponding to \( \lambda(k) \), and \( \delta \) a suitable constant.

Let be \( z = \tilde{z} + \delta \Phi_k \), and \( \delta \) is free for the moment. Then by (H_3) i) and (2.2)

\[
f_k(z) \geq -\frac{1}{2} \lambda(k) \delta^2 + k \int_0^T \hat{H}(kt, z) dt \geq -\frac{1}{2} d_2 \delta^2 + a_4 k |z|^s - a_2 k. \tag{2.10}
\]

Arguing as in [11], it is easy to prove that

\[
|z|^s \geq d_3 \delta. \tag{2.11}
\]

By (2.10) and (2.11) it follows

\[
f_k(z) \geq -\frac{1}{2} d_2 \delta^2 + d_4 \delta^s k - a_2 k \geq d_5 k(\delta^s - 1) - d_6 \delta^2. \tag{2.12}
\]

If we choose \( \delta = k^{-\eta} \), with \( 1 + s \eta > 2 \eta > 1 \), there exists \( \overline{\beta} > 0 \) s.t. for \( k \) large, we have

\[
f_k(z) \geq \overline{\beta} k^\gamma \quad \gamma = 1 + s \eta > 1.
\]

**Proof of theorem 1.6.** By a suitable change of variable, a \( kT \)-periodic solution of (1.5) corresponds to a \( T \)-periodic solution of

\[
\begin{cases}
  u_{tt} - k^2 u_{xx} = k^2 f(x,kt,u) & t \in \mathbb{R}, \ x \in [0,\pi] \\
  u(0,t) = u(\pi,t) = 0 & t \in \mathbb{R}.
\end{cases} \tag{2.13}
\]

We set \( \Omega = [0,\pi] \times [0,T] \). Let us denote by \( L_k \) (resp. \( A_k \)) the selfadjoint realization in \( L^2(\Omega) \) of the operator \( u_{tt} - k^2 u_{xx} - k^2 f'(\infty) u \) (resp. \( u_{tt} - k^2 u_{xx} \)) with boundary and periodicity conditions.

It is easy to see that the functional

\[
g_k(u) = -\frac{1}{2} < L_k u, u > + k^2 \int_\Omega G(x,kt,u) \, dx \, dt \tag{2.14}
\]

is \( C^1 \) on a suitable Hilbert space \( E \) (cf. e.g. [1] for its definition) and its critical points are \( kT \)-periodic solutions of (1.5). But in this case we cannot apply theorem 1.18 of [5] to the functional \( g_k \), 0 being an eigenvalue of \( A_k \).
of infinite multiplicity. Therefore, as in [8] (cf. also [1], [7]), we restrict $g_k$ to the subspace

$$
\overline{E} = \{ u \in E \mid u(x,t) = u(x,t + T/2), u(\pi - x,t) = u(x,t) \}.
$$

It is known that $\overline{E}$ is a closed subspace of $E$ and verifies the following properties

$$
\begin{align*}
&i) \quad \overline{E} \cap \ker A_k = \{ 0 \}, \\
&ii) \quad \overline{E} \text{ is invariant under } A_k \text{ and } f. 
\end{align*}
$$

(2.15)

Condition (2.15) ii) assures that the critical points of $g_k|_{\overline{E}}$ are critical points of $g_k$. In the sequel we still denote by $g_k$ (resp. $L_k$) the restriction $g_k|_{\overline{E}}$ (resp. $L_k|_{\overline{E}}$).

Arguing as in the hamiltonian case, it is possible to prove that for any $k \in N$, $k$ odd, $g_k$ has a critical point $u_k$. In order to prove that infinitely many solutions $u_k$ are distinct, suppose first

$$
u_k(x,t) = u_1(x,kt) \quad \text{for any } k \in N, k \text{ odd},
$$

then (cf. [12]):

$$
g_k(u_k) = k^2 g_1(u_1).
$$

(2.16)

Since $f' (\infty) \in R$, it is easy to prove (cf. e. g. [6]) the following lemma

**LEMMA 2.17.** - For $k \in N$, $k$ odd, there exists $\lambda(k) \in \sigma(L_k)$ s.t.

$$
d_1 k \leq \lambda(k) \leq d_2 k, \quad \text{with} \quad d_1, d_2 > 0.
$$

Then by lemma 2.17 and by arguments similar to those used to prove (2.8), it is possible to show that, for $k$ large, $k$ odd, it results

$$
g_k(u_k) \geq \tilde{d} k^\gamma \quad \text{with} \quad \tilde{d} > 0 \text{ and } \gamma > 2.
$$

(2.17)

The conclusion of theorem 1.6 follows as in the hamiltonian case.
REFERENCES