SUBHARMONIC SOLUTIONS OF LARGE NORM FOR SOME ASYMPTOTICALLY QUADRATIC PROBLEMS(*)

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SOMMARIO.- *In questo lavoro si prova l'esistenza di infinite subarmoniche distinte per alcuni problemi asintoticamente quadratici.*

SUMMARY.- *In this paper we shall prove the existence of infinitely many distinct subharmonic solutions for some asymptotically quadratic problems.*

1. Introduction and statement of the results

This paper is divided in two parts. In the first part we study the nonautonomous Hamiltonian system of $2n$ differential equations

$$-J\dot{z} = H_z(t,z) \tag{1.1}$$

where $z \in \mathbf{R}^{2n}$, J is the symplectic matrix in $\mathbf{R}^{2n}$, $H(t,z)$ is T-periodic in t and $H_z(t,z)$ is the gradient of H with respect to z.

We are concerned with the existence of subharmonic solutions for (1.1), i.e. we search for kT-periodic solutions ($k \in \mathbf{N}$) z_k of (1.1). This problem has been studied by Rabinowitz in [11]; he obtains the existence of subharmonic solutions when $H(t,z)$ is sub- or super-quadratic at infinity (i.e. $H(t,z) / |z|^2 \to 0$ or ∞ as $|z| \to +\infty$). Now we shall assume that $H(t,z)$ is asymptotically quadratic, i.e. for any $t \in [0,T]$ there exists a $2n \times 2n$ symmetric matrix $A(t)$ such that

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$$(\mathrm{H}_1)\qquad \begin{cases} H(t,z) = \dfrac{1}{2}(A(t)\,z,z) + \hat{H}(t,z) \\ \text{where} \\ \lim\limits_{|z|\to+\infty} \dfrac{\hat{H}(t,z)}{|z|^2} = 0 \text{ uniformly in } t. \end{cases}$$

Consider the linearized equation at infinity

$$\dot{z} = JA(t)z \tag{1.2}$$

and let $X(t)$ the solution matrix of $\dot{X}(t) = JA(t)X$ and $X(0) = \mathrm{Id}$. It is known that $X(t)$ is a symplectic matrix (cf. e.g. [9], § 4). We shall make the following assumptions:

$$(\mathrm{H}_2)\qquad \begin{cases} \text{every symplectic matrix sufficiently close to } X(t) \text{ has an} \\ \text{eigenvalue } e^{iw} \text{ on the unit circle;} \end{cases}$$

(H_3) there exist some positive constants a_1, a_2, R, s and p, $1 < s < p < 2$, s.t.
i) $\hat{H}(t,z) \geq a_1\,|z|^s - a_2$ for any $t \in \boldsymbol{R}, z \in \boldsymbol{R}^n$,
ii) $0 < (\hat{H}_z(t,z), z) \leq p\,\hat{H}(t,z)$ for any $z \in \boldsymbol{R}^n$, $|z| > R$.

We shall state the following theorem:

THEOREM 1.3.- Suppose that (H_1) – (H_3) hold. Then for $k \in N$ (1.1) possesses a kT-periodic solutions z_k. Moreover $\sup\limits_t |x(t)| \to \infty$ as $k \to \infty$.

REMARK 1.4.- We do not know if kT is the minimal period of z_k. Subharmonic solutions having arbitrarily long minimal periods have been found in [2], [3], [4], [11], [14], [15]. Moreover in [2], [3], [4], [11] the subharmonic solutions have been localized near the origin.

In the second part we consider the forced semilinear wave equation

$$\begin{cases} u_{tt} - u_{xx} = f(x,t,u) & t \in \boldsymbol{R}, x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0 & t \in \boldsymbol{R} \end{cases} \tag{1.5}$$

where f is $C^1([0,\pi] \times R \times R, R)$ and T-periodic in t, T is a rational multiple of π. We are concerned with the existence of subharmonic solutions for (1.5), i.e. we search for kT-periodic solutions ($k \in N$) $u_k(x,t)$ of (1.5).

This problem has been studied in [12], when the function

$$F(x,t,u) = \int_0^u f(x,t,s)ds$$

is either sub- or super-quadratic at infinity (i.e. $F(x,t,u)/|u|^2 \to 0$ or ∞ as $|u| \to \infty$). Moreover, to overcome the difficulties arising from the infinite dimensional kernel of $u_{tt} - u_{xx}$, in [12] f is assumed strictly monotone in u. More recently, in [7], the monotonicity assumption has been avoided using a Coron's trik.

Here we assume that F is asymptotically quadratic at infinity, more precisely there exists a real number $f'(\infty)$ s.t.

$$(\mathrm{W}_1) \quad \begin{cases} f(x,t,u) = f'(\infty)u + g(x,t,u) \\ \text{where} \\ \dfrac{g(x,t,u)}{|u|} \to 0 \text{ as } |u| \to +\infty \text{ uniformly in } (x,t) \in [0,\pi] \times R . \end{cases}$$

We set $G(x,t,u) = \int_0^u g(x,t,s)\, ds$. The following theorem holds:

THEOREM 1.6.- Let $\dfrac{T}{2\pi} = \dfrac{b}{a}$, $a,b \in N$, b odd. Assume (W_1),

(W_2) $f(x,t,u)$ is $T/2$-periodic in t, $f(\pi-x,t,u) = f(x,t,u)$;

(W_3) there exist some positive constants a_i, $i = 3, ..., 6$, s and p s.t.

i) $G(x,t,u) \geq a_3 |u|^s - a_4 \quad 1 < s < 2,$
ii) $g(x,t,u) \leq a_5 |u|^p + a_6 \quad 0 < p < 1.$

Then for $k \in N$, k odd, (1.5) possesses a kT-periodic solution u_k. Moreover $\sup_{x,t} |u_k(x,t)| \to \infty$ as $k \to \infty$.

2. Proof of theorems 1.3 and 1.6

Proof of theorem 1.3. By a suitable change of variable, a kT-periodic solution of (1.1) corresponds to a T-periodic solution of

$$\dot{z} = kJH_z(kt,z). \tag{2.1}$$

If we denote by L_k the self-adjoint realization in $L^2(0,T)$ of the operator $z \to -J\dot{z} - kA(t)z$ with T-periodicity conditions, it is easy to see that the functional

$$f_k(z) = -\frac{1}{2} < L_k z,z > + k \int_0^T \hat{H}(kt,z)dt \tag{2.2}$$

is C^1 in $H^{1/2}(R/kZ)$ and its critical points are kT-periodic solutions of (1.1). Moreover standard perturbation results (cf. e.g. [10], pag. 28) permit to deduce that the spectrum $\sigma(L_k)$ of L_k consists (like the spectrum of the operator $z \to -J\dot{z}$) of infinitely many positive and infinitely many negative isolated eigenvalues of finite multiplicity. We denote by $\{\lambda_j\}$ the sequence of the eigenvalues of L_k and by $\{M_j\}$ the sequence of the corresponding eigenspaces. We set

$$H^+ = \overline{\bigoplus_{\lambda_j>0} M_j} \quad , \quad H^- = \overline{\bigoplus_{\lambda_j<0} M_j} \quad , \quad H_o = \ker L_k \ ,$$

where the closure is taken in $H^{1/2}$ and H_0 can be also the space $\{0\}$.

Denote by $\|\bullet\|$ and $|\bullet|_p$ the norms in $H^{1/2}$ and in $L^p(0,T)$; for $R > 0$ let be $B_R = \{u \in H^{1/2} \mid \|u\| < R\}$ and $S_R = \partial B_R$. Moreover denote by $|\bullet|_\infty$ the norm in L^∞.

Then standard calculations (cf. e.g. [13]) show that there exist α, β, $R > 0$ and $\nu \in H^+$ ($\|\nu\| < R$) such that

$$f_k(z) \geq \beta \qquad \text{on } S, \tag{2.3}$$

$$f_k(z) \leq \alpha \qquad \text{on } \partial Q, \tag{2.4}$$

$$\sup_Q f_k(z) < +\infty, \tag{2.5}$$

where $S = \{\tilde{z} + \nu \mid \tilde{z} \in H^- \oplus H^o\}$ and $Q = B_R \cap H^+$.

Moreover f_k satisfies the Palais-Smale condition (even if $H_0 \neq \{0\}$), then by theorem 1.18 in [5] f_k possesses a critical point z_k such that

$$f_k(z_k) \geq \beta.$$

Since z_1 (kt) is also a kT-periodic solution of (1.1), we prove that infinitely many solutions z_k are distinct. Suppose first (cf. [11])

$$z_k(t) = z_1(kt) \qquad \text{for any } k \in N, \tag{2.6}$$

then

$$c_k = f_k(z_k) = k f_1(z_1) = k\, c_1. \tag{2.7}$$

We shall show in fact that, for large k, there exists $\overline{\beta} > 0$ such that

$$c_k \geq \overline{\beta}\, k^{\gamma} \quad \gamma > 1 \tag{2.8}$$

and therefore the solutions z_k are not reparametrizations of z_1 for k large.
Moreover, since

$$c_k = f_k(z_k) = k \int_0^T [\hat{H}(kt, z_k) - \frac{1}{2}(z_k, \hat{H}_z(kt, z_k))]\, dt,$$

if $\{\, |z_k|_\infty \}$ were bounded, we would have

$$c_k \leq Mk, \qquad M > 0,$$

contrary to (2.8). Then $|z_k|_\infty \to +\infty$ as $k \to +\infty$.
It remains to verify (2.8). In the sequel we denote by d_i some positive constants. We shall need the following lemma.

LEMMA 2.9.- Suppose that H satisfies (H_1), (H_2). Then, for k sufficiently large, there exists $\lambda(k) \in \sigma(L_k)$ such that

$$d_1 \leq \lambda(k) \leq d_2.$$

Proof. Observe that μ is an eigenvalue of the operator $z \to -J\dot{z} - A(t)z$ in $L^2(0,kT)$ iff $k\mu$ is an eigenvalue of the operator $z \to -J\dot{z} - k\,A(t)z$ in $L^2(0,T)$. Then the proof follows by lemma 2.1 of [2].

In order to verify (2.8), it suffices to prove that

$$f_k(z) \geq \overline{\beta}\, k^{\gamma} \quad \text{on } S,$$

where $S = \{\tilde{z} + \delta\Phi_k \mid \tilde{z} \in H^- \oplus H^\circ\}$, Φ_k being a normalized eigenvector of L_k corresponding to $\lambda(k)$, and δ a suitable constant.

Let be $z = \tilde{z} + \delta\Phi_k$, and δ is free for the moment. Then by (H_3) i) and (2.2)

$$f_k(z) \geq -\frac{1}{2}\lambda(k)\,\delta^2 + k\int_0^T \hat{H}(kt,z)dt \geq -\frac{1}{2}d_2\delta^2 + a_1 k|z|_s^s - a_2 k. \tag{2.10}$$

Arguing as in [11], it is easy to prove that

$$|z|_s \geq d_3\,\delta. \tag{2.11}$$

By (2.10) and (2.11) it follows

$$f_k(z) \geq -\frac{1}{2}d_2\,\delta^2 + d_4\,\delta^s k - a_2 k \geq d_5 k(\delta^s - 1) - d_6\,\delta^2. \tag{2.12}$$

If we choose $\delta = k^\eta$, with $1 + s\eta > 2\eta > 1$, there exists $\bar{\beta} > 0$ s.t. for k large, we have

$$f_k(z) \geq \bar{\beta}\,k^\gamma \qquad \gamma = 1 + s\eta > 1.$$

Proof of theorem 1.6. By a suitable change of variable, a kT-periodic solution of (1.5) corresponds to a T-periodic solution of

$$\begin{cases} u_{tt} - k^2 u_{xx} = k^2 f(x,kt,u) & t \in \mathbf{R},\ x \in [0,\pi] \\ u(0,t) = u(\pi,t) = 0 & t \in \mathbf{R}. \end{cases} \tag{2.13}$$

We set $\Omega = [0,\pi] \times [0,T]$. Let us denote by L_k (resp. A_k) the selfadjoint realization in $L^2(\Omega)$ of the operator $u_{tt} - k^2 u_{xx} - k^2 f'(\infty)u$ (resp. $u_{tt} - k^2 u_{xx}$) with boundary and periodicity conditions.

It is easy to see that the functional

$$g_k(u) = -\frac{1}{2} < L_k u,u > + k^2 \iint_\Omega G(x,kt,u)\,dx\,dt \tag{2.14}$$

is C^1 on a suitable Hilbert space E (cf. e.g. [1] for its definition) and its critical points are kT-periodic solutions of (1.5). But in this case we cannot apply theorem 1.18 of [5] to the functional g_k, 0 being an eigenvalue of A_k

of infinite multiplicity. Therefore, as in [8] (cf. also [1], [7]), we restrict g_k to the subspace

$$\bar{E} = \{u \in E \mid u(x,t) = u(x,t + T/2), u(\pi - x,t) = u(x,t)\}.$$

It is known that $\bar{E}$ is a closed subspace of E and verifies the following properties

$$\begin{cases} \text{i)} & \bar{E} \cap \ker A_k = \{0\}, \\ \text{ii)} & \bar{E} \text{ is invariant under } A_k \text{ and } f. \end{cases} \tag{2.15}$$

Condition (2.15) ii) assures that the critical points of $g_{k|\bar{E}}$ are critical points of g_k. In the sequel we still denote by g_k (resp. L_k) the restriction $g_{k|\bar{E}}$ (resp. $L_{k|\bar{E}}$).

Arguing as in the hamiltonian case, it is possible to prove that for any $k \in N$, k odd, g_k has a critical point u_k. In order to prove that infinitely many solutions u_k are distinct, suppose first

$$u_k(x,t) = u_1(x,kt) \qquad \text{for any } k \in N, k \text{ odd},$$

then (cf. [12]):

$$g_k(u_k) = k^2 g_1(u_1). \tag{2.16}$$

Since $f'(\infty) \in R$, it is easy to prove (cf. e. g. [6]) the following lemma

LEMMA 2.17.- For $k \in N$, k odd, there exists $\lambda(k) \in \sigma(L_k)$ s.t.

$$\bar{d}_1 k \leq \lambda(k) \leq \bar{d}_2 k, \quad \text{with} \quad \bar{d}_1, \bar{d}_2 > 0.$$

Then by lemma 2.17 and by arguments similar to those used to prove (2.8), it is possible to show that, for k large, k odd, it results

$$g_k(u_k) \geq \bar{d}\, k^\gamma \quad \text{with} \quad \bar{d} > 0 \text{ and } \gamma > 2. \tag{2.17}$$

The conclusion of theorem 1.6 follows as in the hamiltonian case.

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