ON CERTAIN DECOMPOSITIONS OF CONTINUITY(*)

by JÁN BORSÍK and JOZEF DOBOŠ (in Košice)(**)

SOMMARIO.- Nell'articolo si introduce una nozione indebolita del concetto di continuità. Tale definizione (detta "mild continuity") generalizza vari tipi di continuità considerati precedentemente. Come risultato principale, si riesce a caratterizzare la continuità di una funzione usando forme più deboli di continuità.

SUMMARY.- In the paper the notion of mild continuity is introduced. It generalizes some well-known types of continuity (e.g. cliquishness, quasi-continuity, closedness of the graph). As a main result it is proved that a function is continuous if and only if it is mildly continuous and almost continuous.

In the literature there are many papers dealing with almost continuity in connection with the decomposition of continuity. See for example [4], [13], where it is proved that a function is continuous if and only if it is almost continuous and cliquish, similarly, in [11], [12] continuity is decomposed into almost continuity and quasi-continuity and, in [8], into almost continuity and graph closedness.

In the present paper we give a simultaneous generalization of the decompositions mentioned above. We introduce a class of functions containing cliquish functions, quasi-continuous functions and functions with the closed graph. The corresponding generalized continuity (called mild continuity) combined with almost continuity gives a decomposition of continuity.

In what follows $X$, $Y$ denote topological spaces. For a subset $A$ of a topological space denote $Cl A$ and $Int A$ the closure and the interior of $A$, respectively. The letters $Q$ and $R$ stand for the set of rational and real numbers, respectively.

DEFINITION 1. (See [7].) A function $f : X \rightarrow Y$ is said to be almost continuous (known also as nearly continuous, see [11]) at a point $x \in X$ if

(*) Pervenuto in Redazione il 7 maggio 1989.
(**) Indirizzi degli Autori: J. Borsík: Matematický ústav SAV, Ždanovova 6, 040 01 Košice (Czechoslovakia); J. Doboš: Katedra matematiky VŠT, Švermova 9, 040 01 Košice (Czechoslovakia).
for each neighbourhood \( V \) of \( f(x) \), \( Cl f^{-1}(V) \) is a neighbourhood of \( x \). If \( f \) is almost continuous at every \( x \in X \), then it is called almost continuous.

**Definition 2.** (See [9].) A function \( f : X \to Y \) is said to be quasi-continuous at a point \( x \in X \) if for each neighbourhood \( U \) of \( x \) and each neighbourhood \( V \) of \( f(x) \) there exists a nonempty open set \( G \subseteq U \) such that \( f(G) \subseteq V \). Denote by \( Q_f \) the set of all points at which \( f \) is quasi-continuous. If \( Q_f = X \), then \( f \) is called quasi-continuous.

**Theorem A.** (See [12], also [11].) Let \( Y \) be a regular space. Then \( f : X \to Y \) is continuous if and only if it is almost continuous and quasi-continuous.

The following example shows that the assumption of the regularity of \( Y \) in Theorem A cannot be omitted.

**Example 1.** Let \( X = R \) with the usual topology. Let \( Y = R \) with the topology \( \mathcal{T} = \mathcal{F} \cup \{\emptyset\} \), where \( \mathcal{F} \) is the filter consisting of all sets the complements of which are nowhere dense sets in usual topology of \( R \). Let \( f : X \to Y \) be the identity function. Then \( f \) is almost continuous and quasi-continuous, but it is not continuous.

The following example shows that Theorem A does not hold point-wise.

**Example 2.** Let \( f : R \to R, f(x) = 1 \) for \( x = 1/n \) (\( n = 1, 2, \ldots \)) and \( f(x) = 0 \) otherwise. Then \( f \) is quasi-continuous and almost continuous at the point 0, however it is not continuous at the point 0.

**Theorem 1.** Let \( Y \) be a regular space. Let \( f : X \to Y \) be almost continuous. Then the set \( Q_f \) is closed in \( X \).

**Proof:** Let \( x \in Cl Q_f \). Let \( U \) and \( V \) be open neighbourhoods of \( x \) and \( f(x) \) respectively. Choose a neighbourhood \( W \) of \( f(x) \) such that \( Cl W \subseteq V \). From the almost continuity at \( x \) there is an open neighbourhood \( H \) of \( x \) such that the set \( f^{-1}(W) \) is dense in \( H \). Since \( x \in Cl Q_f \), there is a point \( y \in Q_f \cap H \cap U \). Let \( S \) be a neighbourhood of \( f(y) \). From the quasi-continuity at \( y \) there is a nonempty open set \( T \subseteq H \cap U \) such that \( f(T) \subseteq S \). Since \( f^{-1}(W) \) is dense in \( H \), we have \( f^{-1}(W) \cap T \neq \emptyset \). Then \( \emptyset \neq W \cap f(T) \subseteq W \cap S \). Thus each neighbourhood \( S \) of \( f(y) \) intersects the set \( W \), which yields \( f(y) \in Cl W \subseteq V \). Therefore \( V \) is a neighbourhood of \( f(y) \). From the
quasi-continuity at $y$ there is a nonempty open set $G \subset U$ such that $f(G) \subset V$. Thus $x \in Q_f$.

The following example shows that the assumption of the regularity of $Y$ in Theorem 1 cannot be omitted.

**Example 3.** Let $X = R$ with the usual topology. Let $Y = \{a, b\}, \mathcal{T} = \{\emptyset, \{b\}, Y\}$. Let $f : X \to Y, f(x) = a$ for $x \in Q, f(x) = b$ otherwise. Then $f$ is almost continuous, $Q_f$ is dense in $X$, however $f$ is not quasi-continuous.

From Theorem A and Theorem 1 we obtain the following

**Theorem B.** (See [6].) Let $Y$ be a regular space. Then $f : X \to Y$ is continuous if and only if it is almost continuous and the set $Q_f$ is dense in $X$.

**Definition 3.** We say that a function $f : X \to Y$ is pointwise discontinuous if the set of all continuity points of $f$ is dense in $X$.

**Corollary 1.** (See [1].) Let $Y$ be a regular space. Then $f : X \to Y$ is continuous if and only if it is almost continuous and pointwise discontinuous.

The following example shows that a quasi-continuous real function need not be pointwise discontinuous.

**Example 4.** Let $Q = \{q_1, q_2, q_3, \ldots\}$. Let $f : Q \to R, f(x) = \sum_{n,q_n \leq x} 2^{-n}$ for each $x \in Q$. Then $f$ is a quasi-continuous function without points of continuity.

**Definition 4.** (See [8].) We say that a function $f : X \to Y$ has the closed graph if the set $\{(x,y) \in X \times Y; y = f(x)\}$ is a closed subset of the product $X \times Y$.

**Corollary 2.** Let $X$ be a Baire space and $Y$ be a $\sigma$-compact $T_3$ space. Then $f : X \to Y$ is continuous if and only if it is almost continuous and has the closed graph.

**Proof.** Suppose that $f : X \to Y$ has the closed graph. Then by [3] the function $f$ is pointwise continuous.
DEFINITION 5. Let \( f : X \to Y \). Put \( S_f = \{ x \in X ; \text{there is a base } \mathcal{A} \text{ of} \} \) neighbourhoods of \( f(x) \) such that for every \( A \in \mathcal{A} \) and for every neighbourhood \( U \) of \( x \) the set \( f^{-1}(A) - \text{Int } f^{-1}(A) \) is not dense in \( U \). We say that \( f \) is mildly continuous, if the set \( S_f \) is dense in \( X \).

LEMMA 1. Let \( f : X \to Y \). Then \( Q_f \subseteq S_f \).

Proof. Let \( x \in Q_f \). Let \( U \) and \( V \) be neighbourhoods of \( x \) and \( f(x) \) respectively. We shall show that \( f^{-1}(U) - \text{Int } f^{-1}(V) \) is not dense in \( U \). Since \( x \in Q_f \), there is a nonempty open set \( G \subseteq U \) such that \( f(G) \subseteq V \). Thus \( G \subseteq \text{Int } f^{-1}(V) \), which yields \( f^{-1}(V) - \text{Int } f^{-1}(V) \cap G = \emptyset \).

LEMMA 2. Let \( f : X \to Y \) be almost continuous. Then \( S_f \subseteq Q_f \).

Proof. Let \( x \in S_f \). Let \( U \) and \( V \) be neighbourhoods of \( x \) and \( f(x) \) respectively. Let \( A \) be a neighbourhood of \( f(x) \) such that \( A \subseteq V \) and for each neighbourhood \( T \) of \( x \) the set \( f^{-1}(A) - \text{Int } f^{-1}(A) \) is not dense in \( T \). From the almost continuity at \( x \) there is a neighbourhood \( W \) of \( x \) such that \( f^{-1}(A) \) is dense in \( W \). Now there exists a nonempty open set \( H \subseteq U \cap W \) such that \( f^{-1}(A) - \text{Int } f^{-1}(A) \cap H = \emptyset \). Since \( f^{-1}(A) \) is dense in \( W \), we obtain \( f^{-1}(A) \cap H \neq \emptyset \). Hence \( \text{Int } f^{-1}(A) \cap H \neq \emptyset \). Put \( G = \text{Int } f^{-1}(A) \cap H \). Then \( G \) is a nonempty open set such that \( G \subseteq U \) and \( f(G) \subseteq V \). Thus \( x \in Q_f \).

We are now able to establish the main theorem, which follows from Lemmas 1 and 2 and Theorem B.

THEOREM 2. Let \( Y \) be a regular space. Then \( f : X \to Y \) is continuous if and only if it is almost continuous and mildly continuous.

COROLLARY 3. (See [8].) Let \( Y \) be a locally compact Hausdorff space. Then \( f : X \to Y \) is continuous if and only if it is almost continuous and has the closed graph.

Proof. Suppose that \( f : X \to Y \) has the closed graph. Let \( x \in X \). Let \( \mathcal{A} \) be a base of compact neighbourhoods of the point \( f(x) \). Since \( f \) has the closed graph, by [5] the set \( f^{-1}(A) \) is closed in \( X \) for each \( A \in \mathcal{A} \). Thus \( f^{-1}(A) - \text{Int } f^{-1}(A) \) is nowhere dense in \( X \). This shows that \( x \in S_f \).

DEFINITION 6. (See [9].) Let \( Y \) be a metric space with a metric \( d \). A function \( f : X \to Y \) is said to be cliquish at a point \( x \in X \) if for each \( \varepsilon > 0 \) and each neighbourhood \( U \) of the point \( x \) there exists a nonempty open set
ON CERTAIN DECOMPOSITIONS OF CONTINUITY

$G \subset U$ such that $d(f(y), f(z)) < \varepsilon$ for each $y, z \in G$. Denote by $A_f$ the set of all points at which $f : X \to Y$ is cliquish. If $A_f = X$, then $f$ is called cliquish.

The following example shows that the cliquishness is not a topological notion.

**Example 5.** Let $X = Q \cap (0, 1)$ with the usual topology. Put $Y = \{1/n; n = 1, 2, 3, \ldots\}$. Let $d_1$ be the usual metric on $Y$ and $d_2$ the discrete metric on $Y$ (i.e. $d_2(x, y) = 1$ for $x \neq y$). Then $d_1$ and $d_2$ are topologically equivalent. Let $f : X \to Y$ be the Riemann function (i.e. $f(x) = 1/q$ for $x = p/q$, where $p, q$ are relatively prime integers, $q > 0$). Then $f : X \to (Y, d_1)$ is cliquish, while $f : X \to (Y, d_2)$ is not cliquish.

**Lemma 3.** Let $Y$ be a metric space with a metric $d$. Let $f : X \to Y$. Then $A_f \subset S_f$.

**Proof.** Let $x \in A_f - S_f$. Let $\varepsilon_0 > 0$ be a such that for each neighbourhood $V$ of $f(x)$, $V \subset S(f(x), \varepsilon_0)$, there is a neighbourhood $T$ of $x$ such that the set $f^{-1}(V) - \text{Int} f^{-1}(V)$ is dense in $T$. Let $\varepsilon > 0$. We may assume $\varepsilon < \varepsilon_0$. Then there is a neighbourhood $W$ of $x$ such that the set $H = f^{-1}(S(f(x), \varepsilon/2)) - \text{Int} f^{-1}(S(f(x), \varepsilon/2))$ is dense in $W$. Let $U$ be a neighbourhood of $x$. Since $x \in A_f$, there is a nonempty open set $G \subset U \cap W$ such that $d(f(y), f(z)) < \varepsilon/2$ for each $y, z \in G$. Since $H$ is dense in $W$, we have $G \cap H \neq \emptyset$. Choose $y \in G \cap H$. Let $z \in G$. Then $d(f(x), f(z)) = d(f(x), f(y)) + d(f(y), f(z)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $f(z) \in S(f(x), \varepsilon)$. Thus $f(G) \subset S(f(x), \varepsilon)$. Hence $x \in Q_f$, which contradicts to Lemma 1. Thus $A_f - S_f = \emptyset$.

**Corollary 4.** (See [13], also [4]a.) Let $Y$ be a metric space. Then $f : X \to Y$ is continuous if and only if it is almost continuous and cliquish.

**Definition 7.** (See [2]a.) A function $f : X \to Y$ is said to be simply continuous if for every open set $V$ of $Y$, $f^{-1}(V)$ is the union of an open set of $X$ and a nowhere dense set of $X$.

**Corollary 5.** Let $Y$ be a regular space. Then $f : X \to Y$ is continuous if and only if it is almost continuous and simply continuous.

**Definition 8.** (See [10]a.) A function $f : X \to Y$ is said to be barely continuous if $f|_M$ has a point of the continuity for each nonempty closed set $M$ in $X$.

**Lemma 4.** Let $Y$ be a regular space. If $f : X \to Y$ is barely continuous, then it is mildly continuous.
Proof. Suppose that \( f : X \to Y \) is barely continuous. Let \( x \in X \). If \( x \in Q_f \), by Lemma 1 we have \( x \in S_f \). Suppose that \( x \notin Q_f \). Then there is an open neighbourhood \( W \) of \( f(x) \) and an open neighbourhood \( U \) of \( x \) such that for each nonempty open set \( G \subseteq U \) there is \( y \in G - f^{-1}(W) \). Let \( V \) be a neighbourhood of \( f(x) \) such that \( \text{Cl} \ V \subseteq W \). The set \( \mathcal{A} = \{ T \subseteq Y; \ T \text{ is a neighbourhood of } f(x) \text{ and } T \subseteq V \} \) is a base of neighbourhoods of \( f(x) \). Let \( S \) be a neighbourhood of \( x \) and \( T \in \mathcal{A} \). We shall show that \( f^{-1}(T) - \text{Int} f^{-1}(T) \) is not dense in \( S \). Clearly \( P = U \cap \text{Int} S \) is an open neighbourhood of \( x \). Let \( z \) be a continuity point of \( f|_{\text{Cl} P} \). We shall show that \( f(z) \notin W \). If \( f(z) \in W \), then \( W \) is a neighbourhood of \( f(z) \) and hence there is an open neighbourhood \( A \) of \( z \) in \( X \) such that \( f(A \cap \text{Cl} P) \subset W \). However, \( A \cap P \) is a nonempty open subset of \( U \) and hence there is \( y \in A \cap P \) with \( f(y) \notin W \), a contradiction. Thus \( f(z) \notin W \). This implies that \( Y - \text{Cl} V \) is a neighbourhood of \( f(z) \). Hence there is an open neighbourhood \( B \) of \( z \) in \( X \) such that \( f(B \cap \text{Cl} P) \subseteq Y - \text{Cl} V \). Then \( B \cap P \) is a nonempty open subset of \( S \) and \( f(B \cap P) \subseteq Y - \text{Cl} V \). Therefore \( (B \cap P) \cap f^{-1}(V) = \emptyset \), i.e. \( f^{-1}(V) \) is not dense in \( S \). Hence \( f^{-1}(T) \) is not dense in \( S \). Therefore \( f^{-1}(T) - \text{Int} f^{-1}(T) \) is not dense in \( S \).

The assumption of the regularity of \( Y \) is essential, as the following example shows.

**Example 6.** Let \( X = Y = \{ a, b, c \} \), \( \mathcal{S} = \{ \emptyset, \{ a, c \}, X \} \), \( \mathcal{T} = \{ \emptyset, \{ a \}, \{ c \}, \{ a, c \}, Y \} \). Let \( f : (X, \mathcal{S}) \to (Y, \mathcal{T}) \) be the identity function. Then \( f \) is barely continuous, but it is not mildly continuous.

**Corollary 6.** Let \( Y \) be a regular space. Then \( f : X \to Y \) is continuous if and only if it is almost continuous and barely continuous.

The following example shows that the assumption of the mild continuity in Theorem 2 cannot be replaced by the assumption “first Baire class” or by the assumption that the set of all discontinuity points is of the first category.

**Example 7.** Let \( A \) be a dense subset of \( Q \) such that \( Q - A \) is dense in \( Q \). Let \( f : Q \to R \), \( f(x) = 0 \) for each \( x \in A \) and \( f(x) = 1 \) otherwise. Then \( f \) is almost continuous of the first Baire class, but \( S_f = \emptyset \).

It is well-known that the set of all points of the discontinuity of a quasi-continuous (cliquish, pointwise discontinuous, simply continuous, barely continuous, with the closed graph) real function of a real variable is of the
first category. We shall show that there is a mildly continuous real function of a real variable without points of the continuity.

**Example 8.** Let \( f : \mathbb{R} \to \mathbb{R} \), \( f(x) = q \) for \( x = p/q \), where \( p, q \) are relatively prime integers, \( q > 0 \), \( f(x) = 0 \) otherwise. Then \( f \) is mildly continuous, nevertheless \( f \) is discontinuous at each point.
REFERENCES


[6] L. Holá, *Some conditions that imply continuity of almost continuous multifunc-


