ON THE PERIODIC BVP FOR THE FORCED DUFFING EQUATION (*)

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SOMMARIO. - *Si studiano condizioni sufficienti per l'esistenza e la molteplicità esatta di soluzioni di una equazione del tipo di Duffing* $-Ax + a\,\mathrm{sen}(x) = e$, *con*

$$A : \mathrm{dom}\, A \subseteq L^\infty \to L^\infty = L^\infty([0, T], \mathbf{R})$$

lineare, con nucleo le costanti e imagine le funzioni a integrale nullo, con inverso destro compatto, e dove $a > 0$, $e \in L^\infty$. *Come applicazione, si studia il problema ai limiti periodico per l'equazione ordinaria del tipo di Duffing* $d^n x/dt^n + a\,\mathrm{sen}(x) = e(t)$, *che quando* $n = 2$ *è la usuale equazione del pendolo forzato.*

SUMMARY. - *We study the equation of Duffing type* $-Ax + a \sin(x) = e$, *where* $A : \mathrm{dom}\, A \subseteq L^\infty \to L^\infty = L^\infty([0, T], \mathbf{R})$ *is a linear map whose kernel consists of constant mappings, the range is the set of maps with mean value zero, having a compact rigth inverse, and where* $a > 0$, $e \in L^\infty$. *Sufficient conditions for the existence and for the exact multiplicity of the solutions are given. As an application, we consider the periodic BVP for the* $n-th$ *order ODE of Duffing type* $d^n x/dt^n + a \sin(x) = e(t)$, *which is, when* $n = 2$, *the usual forced pendulum equation.*

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§ 1 - **Introduction.**

Let $T > 0$ be given, and let $L^\infty := L^\infty([0, T], \mathbf{R})$, equipped with

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the usual sup-norm. Let $\mathrm{dom}\,A$ be a linear (possibly not closed) manifold of L^∞ and let $A : \mathrm{dom}\,A \subseteq L^\infty \to L^\infty$ be a linear map such that:

(a) $\ker A = \{x \in L^\infty \mid x \text{ is (a.e.) constant}\}$,

(b) $\mathrm{Im} A = \{x \in L^\infty \mid T^{-1} \int_0^T x(s)\, ds = 0\}$.

Let $a > 0$, $e \in L^\infty$ be given. We study the existence and the multiplicity of solutions to the abstract equation of Duffing type

$$-Ax + a \sin(x) = e, \qquad e \in \mathrm{dom}A \tag{1.1}$$

For example, when $Ax = -x''$ the equation (1.1) is the classical forced pendulum equation with BC depending on the domain of A. Namely, (a)-(b) hold for the homogeneous Neumann BC, or for periodic BC. The periodic BVP for the forced pendulum equation has been considered recently by several authors, who have mainly employed variational methods, degree techniques, the method of upper and lower solutions. We recall here only [13] and its references. [13] contains also an excellent historical introduction to this problem, starting from the results by Duffing [7], up to 1982. We remark however that in general the variational methods cannot work when $Ax = -x^{(n)}$ with n odd, and that the other mentioned techniques (except for the method of upper and lower solutions, for which we refer to [11] for a general account, and to [3] for a specific application to an equation of Duffing type) do not give direct constructive existence proofs, which can be useful in the study of nonlinear phoenomena for which the forced pendulum equation is a first-approximation model. For instance, the rolling of a ship amongst waves is of this type (see [15]).

We shall study (1.1) in the following way.

(1) We first pass to the system

$$Q\, a \sin(r + u) = \bar{e} \qquad \text{(bifurcation equation)}, \tag{1.2}$$

$$u = A^+(I - Q)(a \sin(r + u) - e) \qquad \text{(auxiliary equation)}, \tag{1.3}$$

where $Q : L^\infty \to L^\infty$ is the projection $Qx = \bar{x} : T^{-1} \int_0^T x(s)\, ds$, and where A^+ is the right inverse of A, i.e. the inverse of the restriction of A to $\mathrm{dom}A \cap \ker Q$, with values in $\mathrm{Im}A$. Therefore, (1.1) has a solution $x = r + u$ $(r \in \ker Q)$ if and only if the system (1.2)-(1.3) has a solution $(r, u) = (\bar{x}, x - \bar{x})$.

(2) We assume $|\bar{e}| < a$, and we prove the existence of two real continuous maps $r_j (j = 0, 1)$, defined in a same ball of $\ker Q$ and not differing by $2\pi k$, $k \in \mathbf{Z}$, such that $r = r_j(u)$ solves (1.2).

(3) Putting $r = r_j(u)$ in (1.3), we solve the resulting equations by the Schauder fixed point Theorem (the complete continuity of $A^+(I-Q)$ is assumed). For small values of the parameters involved, these two fixed point equations can be solved by the Contraction Mapping Principle, too. In this way we get also the exact multiplicity of the solutions of (1.1), and their approximability by the method of successive approximations in L^∞. To study the multiplicity, we can also use Kellogg's Uniqueness Theorem (instead of the Contraction Mapping Principle). See §§ 2, 3, 4.

We have that (1)-(2)-(3) is, essentially, a classic alternative method for weakly nonlinear equations (see, for instance, [16]), except for the particular feature that here the bifurcation equation is not uniquely solvable, but has two different solutions.

In §§ 5, 6 we apply these results to the particular case of the periodic BVP

$$x^{(n)} + a\sin(x) = e\,,\quad x^{(k)}(T) = x^{(k)}(0) \quad (k = 0, 1, \dots, n-1).$$

§ 2 - The bifurcation equation.

For any $f \in L^\infty$, we denote by $M_\infty(f)$ (resp. $m_\infty(f)$) the infimum of all $c \in \mathbf{R}$ such that $f(\cdot) \leqq c$ (resp. $f(\cdot) \geqq c$) a.e. on $[0, T]$. Therefore, the usual norm on L^∞ is $f \to |f|_\infty := M_\infty(|f|)$. Let B the set of all functions $u \in L^\infty$ such that

$$0 = Qu = \left(T^{-1}\int_0^T u(s)\,ds\right), \tag{2.1}$$

$$M_\infty(u) - m_\infty(u) < \pi \tag{2.2}$$

and let $Z: B \to \mathbf{C}$ the continuous map defined by

$$Z(u) := T^{-1}\int_0^T \exp(iu(s))\,ds\,.$$

For any $u \in B$, let D_u be the closed convex hull of the set

$$\{\exp(ix) \mid x \in [m_\infty(u), M_\infty(u)]\}\,.$$

We can assume (after a possible modification of $u \in B$ on a set of zero measure) that $u(s) \in [m_\infty(u), M_\infty(u)]$ for every $s \in [0, T]$.

Therefore we have

$$Z(u) \in D_u$$

for every $u \in B$. On the other hand, the inequality (2.2) implies that $0 \notin D_u$. Thus we have

$$Z(u) \neq 0$$

for every $u \in B$. From (2.1) we deduce $0 \in [m_\infty(u), M_\infty(u)]$, and so $1 \in D_u$. Since D_u is convex, we get

$$|Z(u)|^{-1} Z(u) \neq -1 .$$

Let now $\psi : \mathbf{S}^1 \setminus \{-1\} \to]-\pi, \pi[$ be the inverse map of $x \in]-\pi, \pi[\to \exp(ix) \in \mathbf{S}^1 \setminus \{-1\}$, and let

$$E : B \to]0, 1], \qquad E(u) := |Z(u)|,$$

$$\theta : B \to]-\pi, \pi[, \qquad \theta(u) := \psi(|Z(u)|^{-1} Z(u)).$$

The addition formula for the sine function gives easily that the equality

$$T^{-1} \int_0^T \sin(r + u(s))\, ds = E(u) \sin(r + \theta(u)) \tag{2.3}$$

holds for every $u \in B$ and every $r \in \mathbf{R}$.

We have the following

LEMMA 1 - *Let $a > 0$, $\bar{e} \in \mathbf{R}$, $\delta \in]0, \pi/2[$ be given, and suppose*

$$a^{-1} |\bar{e}| < \sin(\delta). \tag{2.4}$$

Define $B_\delta := \{u \in L^\infty \mid Qu = 0,\ |u|_\infty \leqq (\pi/2) - \delta\}$. Then for every $u \in B_\delta$ there are two solutions $r = r_j(u)$ $(j = 0, 1)$ of the bifurcation equation

$$Q\, a \sin(r + u) = \bar{e}, \tag{2.5}$$

namely

$$r = r_0(u) := \arcsin(E(u)^{-1} a^{-1} \bar{e}) - \theta(u), \tag{2.6}$$

$$r = r_1(u) := -\arcsin(E(u)^{-1} a^{-1} \bar{e}) - \theta(u) + \pi . \tag{2.7}$$

Moreover, for any $u \in B_\delta$, $r_1(u) - r_0(u) \notin 2\pi\mathbf{Z}$, and every $r \in \mathbf{R}$ solving (2.5) has the form $r = r_j(u) + 2\pi k$; $j = 0, 1$; $k \in \mathbf{Z}$.

Proof. Let $u \in B_\delta$. Then $\sin(\delta) \leqq E(u) \leqq 1$. From (2.4) we have $|E(u)^{-1} a^{-1} \bar{e}| < 1$. If r solves (2.5), using (2.3), we deduce $r = r_j(u) + 2\pi k$ for suitable $j = 0, 1$ and $k \in \mathbf{Z}$. Moreover, (2.6) and (2.7) solve trivially (2.5). For any $u \in B_\delta$, the difference $r_1(u) - r_0(u) = \pi - 2 \arcsin(E(u)^{-1} a^{-1} \bar{e})$ lies in $]0, 2\pi[$.

REMARK 1 - Condition (2.2) ensure that the functional

$$Z : u \in L^\infty \cap \ker Q \to T^{-1} \int_0^T \exp(iu(s))\, ds \in \mathbf{C} \tag{2.8}$$

does not vanish on the set B. It is sharp: obviously Z vanishes in a square-wave with mean value zero, T-periodic, with semiaplitude equal $\pi/2$.

REMARK 2 - The functional (2.8) appears also in [20].

§ 3 - The auxiliary equation.

Let us consider the auxiliary equation

$$u = A^+(I - Q)\ (a\sin(r + u) - e). \tag{3.1}$$

We shall denote for short $L_0^\infty := \ker Q \subseteq L^\infty$. Let

$$\tilde{e} := e - (M_\infty(e) + m_\infty(e))/2.$$

Since $(I - Q)\,e = (I - Q)\,\tilde{e}$, we have that (3.1) is equivalent to

$$u = A^+(I - Q)\ (a\sin(r + u) - \tilde{e}) \tag{3.2}$$

(this change gives that

$$|\tilde{e}|_\infty = (M_\infty(e) - m_\infty(e))/2 = \min\{|e - k|_\infty \mid k \in \ker(I - Q)\}).$$

THEOREM 1 - *Suppose that:*

(i) *the map* $A^+(I - Q): L^\infty \to L^\infty$ *is completely continuous; let* $\alpha := \|A^+(I - Q)\|_\infty$ *be its norm.*

(ii) *there is a number* $\delta \in]0, \pi/2[$ *such that*

$$a^{-1}|e| < \sin(\delta), \tag{3.3}$$

$$\alpha(a + |e|_\infty) \leqq (\pi/2) - \delta. \tag{3.4}$$

Then the problem (1.1), *i.e.*

$$x \in \operatorname{dom} A, \quad -Ax + a\sin x = e,$$

has (at least) two solutions $x_j := u_j + r_j(u_j)$ $(j = 0, 1)$, *where* $u_j \in L_0^\infty$, $|u_j|_\infty \leqq (\pi/2) - \delta$, *and* r_j *are defined by* (2.6)-(2.7). *Moreover,* $x_1 - x_0$ *is not* $2\pi k$ *for* $k \in \mathbf{Z}$.

Proof. Condition (3.3) coincides with (2.4) and enables us to solve the bifurcation equation (1.2). Condition (3.4) implies that the compact mappings

$$N_j : B_\delta \subseteq L_0^\infty \to L_0^\infty,$$

$$N_j(u) := A^+(I - Q)\ (a\sin(r_j(u) + u) - \tilde{e})$$

map B_δ into itself. Therefore (Schauder) we have for each N_j a fixed point $u_j = N_j(u_j)$ in B_δ. Thus $(r, u) = (r_j(u_j), u_j)$ are two solutions to (3.2), (2.5), and x_0, x_1 are solutions to (1.1). Assuming $x_1 - x_0 = (r_1(u_1) - r_0(u_0)) + (u_1 - u_0) = 2\pi k$ for some $k \in \mathbf{Z}$, we deduce $u_1 = u_0$, and so $r_1(u_0) - r_0(u_0) = 2\pi k$, a contradiction with Lemma 1.

§ 4 - Multiplicity results.

PROPOSITION 1 - *Suppose that the hypotheses of Theorem 1 hold. Assume that each* $N_j (j = 0, 1)$ *has at most one fixed point* $u_j \in B_\delta$. *If* x *is a solution to* (1.1), *then* $x = x_j + 2\pi k$ *for some* $j = 0, 1$ *and* $k \in \mathbf{Z}$. *Thus* (1.1) *has exactly two solutions not differing for* $2\pi k$, $k \in \mathbf{Z}$.

Proof. We have from Lemma 1 that any solution x to (1.1) has mean value

$$Qx = r_j((I - Q)\, x) + 2\pi k,$$

with $j = 0, 1$ and $k \in \mathbf{Z}$. Furthermore, $u = (I - Q)\, x$ must satisfy

$$\begin{aligned} u &= A^+(I - Q)\,(a \sin(r_j(u) + 2\pi k + u) - e) \\ &= A^+(I - Q)\,(a \sin(r_j(u) + u) - \tilde{e}). \end{aligned} \tag{4.1}$$

Thus $|u|_\infty \leqq \alpha(a + |e|_\infty) \leqq (\pi/2) - \delta$ from (3.4), i.e. $u \in B_\delta$. Then (4.1) implies $u = N_j(u)$, and so $u = u_j$, and

$$x = Qx + (I - Q)\, x = r_j(u_j) + u_j + 2\pi k = x_j + 2\pi k.$$

To fulfill the uniqueness assumption in Proposition 1 we can apply, for instance, the Contraction Mapping Principle. In fact, N_j satisfies a global Lipschitz condition on the complete metric space B_δ. A Lipschitz constant for N_j on B_δ is

$$\begin{aligned} L(N_j) :&= \alpha a (1 + (\pi - 2\delta)/\sin(2\delta) \\ &\quad + |\bar{e}|\,(a^2 \sin(\delta)^2 - |\bar{e}|^2)^{-1/2} \sin(\delta)^{-1}). \end{aligned}$$

The computation of $L(N_j)$ is quite tedious (see [8]), and can be omitted. As a numerical example, if $a = 1$ and $\bar{e} = 0.4$ we can fix $\delta = \pi/6$, and show that the maps N_j are contracting on B_δ provided $|\tilde{e}| < 5.3723$ and $\alpha < 0.1643$.

Since N_j are smooth in the interior $\mathring{B}_\delta$ of B_δ, we can also use Kellogg's result [10]. We remark first that there is no loss of generality if we assume in (3.4) the strict inequality sign. In fact, if the assumption (ii) of Theorem 1 holds with some $\delta \in]0, \pi/2[$, then it holds with $\delta^* \in]0, \pi/2[$, $\delta^* < \delta$. In other words we can assume that the fixed points of N_j do not lie on the boundary ∂B_δ. Therefore we have, as a direct application of Kellogg's Theorem, the following result.

PROPOSITION 2 - *Suppose all the hypotheses of Theorem 1. Assume that for any* $u \in B_\delta$ *the linear problems* $(j = 0, 1)$

$(k_j) \quad v \in L_0^\infty, \quad v = N'_j(u)\, v$

have only the trivial solution $v = 0$. *Then the conclusion of Proposition 1 holds.*

The Fréchet differential $N'_j(u)$ can be easily computed. For the sake of simplicity we assume that $\bar{e} = 0$. In this case (k_j) is equivalent to

$(k_j)_1 \quad v \in L_0^\infty,$

$$v = A^+(I - Q)\,[a \cos(x_j) \cdot (v + (-1)^j E(u_j)^{-1} T^{-1} \int_0^T \cos(x_j(s) \cdot v(s)\, ds)\,].$$

An obvious condition which implies $\ker(N'_j(u) - I) = \{0\}$ is $\|N'_j(u)\| < 1$. Evaluating $\|N'_j(u)\|$ we get that

$$\alpha a\,(1 + \sin(\delta)^{-1}) < 1 \tag{4.2}$$

implies the uniqueness of the fixed point of N_j. An estimate for $\|N'_j(u)\|$ when $\bar{e} \neq 0$ can be found in [14]. Of course, other conditions ensuring that (k_j) have only the trivial solution can be derived from particular expressions of the operator A (e.g. when A is a differential operator).

§ 5 - Estimate of α in some special cases.

We shall estimate the value of the norm $\alpha = \| A^+(I - Q\|_\infty$ in the special case of the periodic BVP

$$x^{(n)} + a \sin(x) = e \quad ({}^{(n)} = d^n/dt^n\,;\; n \geqq 1) \text{ a.e. on } [0, T],$$
$$x^{(k)}(T) - x^{(k)}(0) = 0\,,\; k = 0, 1, \ldots, n-1\,. \tag{5.1}$$

We define in this case

$\operatorname{dom} A := \{x : [0, T] \to \mathbf{R} \mid x, x', \ldots, x^{(n-1)}$ exist and are absolutely continuous; (5.1) holds;

$$x^{(n)} \in L^\infty\}, \tag{5.2}$$

$$Ax := -x^{(n)}. \tag{5.3}$$

It is well known that the sequence $L^\infty \xrightarrow{Q} \operatorname{dom} A \xrightarrow{A} L^\infty \xrightarrow{Q} L^\infty$ is exact, and that $A^+(I - Q)$ is completely continuous. Moreover, if $P : L^\infty \to L^\infty$ is the primitivation operator

$$(Px)\,(s) := \int_0^s x(t)\, dt\,,$$

then $u \in \operatorname{Im} A$ and $-((I - Q)\,P)^n u = v$ if and only if

$$v \in \mathrm{dom} A \cap \ker Q \text{ and } -v^{(n)} = u.$$

Therefore, $A^+ = -((I-Q)P)^n|_{\mathrm{Im} A}$

LEMMA 2 - *If* $u \in L^\infty$, $T^{-1}\int_0^T u(s)\,ds = 0$, *then, for each* $n = 1,2,\ldots$

$$\sup |((I-Q)P)^n u(t)| \leqq T^n |B_{2n}/(2n)!|^{1/2} (T^{-1}\int_0^T |u(s)|^2 ds)^{1/2},$$

$$t \in [0,T] \tag{5.4}$$

where B_{2n} *is the sequence of non-zero Bernoulli numbers.*

Therefore we obtain the following estimate.

PROPOSITION 3 - $\|A^+(I-Q)\|_\infty \leqq T^n |B_{2n}/(2n)!|^{1/2}$.

The sequence B_n can be computed using the recurrence relation [1, Th. 12.17]

$$B_0 = 1, \quad B_n = \Sigma_{k=0,n} \binom{n}{k} B_k,$$

and so, for a fixed T, the upper bound of α given in Proposition 3 can be explicitely computed. For example, if $1 \leqq n \leqq 6$ we obtain:

n	$\lvert B_{2n}/(2n)!\rvert^{1/2}$
1	$1/\sqrt{12}$
2	$1/\sqrt{720}$
3	$1/\sqrt{30\ 240}$
4	$1/\sqrt{1\ 209\ 600}$
5	$1/\sqrt{47\ 900\ 160}$
6	$\sqrt{691}/\sqrt{1\ 307\ 674\ 368\ 000}$

The proof of Lemma 2 is a straightforward application of Fourier series, Parseval identity, and of the relation

$$\zeta(2n) = (-1)^{n+1} 2^{2n-1} \pi^{2n} B_{2n}/(2n)! \quad (n = 1,2,\ldots)$$

between the Riemann function $\zeta(s) = \Sigma_{k=1,\infty} k^{-s}$ and the Bernoulli sequence [1, Th. 12.17]. If $n = 1$, this proof can be found in [5], or [12]. See [8] for the general case. The same technique works for the estimate of $\|A^+(I-Q)\|_\infty$ for other expressions of A; for example, if

$$\mathrm{dom} A := \{x : [0,T] \to \mathbf{R} \mid x, x' \text{ are absolut. cont.},$$

$$x(0) = x(T),\ x'(0) = x'(T),\ x'' \in L^\infty\},$$

$$Ax := -x'' + cx' \quad (c \in \mathbf{R}),$$

one can show (see [2]) that

$$\| A^+(I-Q)\|_\infty \leqq 2^{1/2} w^{-2} (\Sigma_{k=1,\infty} \frac{1}{k^2} \cdot \frac{1}{k^2+(c/w)^2})^{1/2}, \quad w = 2\pi/T.$$

§ 6 - **Applications.**

Let A be the operator defined in (5.2)-(5.3). The preceding results, jointly with the estimate of § 5, give

THEOREM 2 - *Let us consider the n-th order periodic BVP*

$$x^{(n)} + a\sin(x) = e \text{ a.e. on } [0,T], \tag{6.1}$$

$$x^{(k)}(T) - x^{(k)}(0), \quad k = 0,1,\dots,n-1, \tag{6.2}$$

with $a > 0$, $e \in L^\infty$. *Assume that*

$$a^{-1}|\bar{e}| < 1 \qquad (\bar{e} = T^{-1}\int_0^T e(s)\,ds) \tag{6.3}$$

$$a + |\tilde{e}|_\infty < (\pi/2)\,T^{-n}|B_{2n}/(2n)!|^{-1/2} (|\tilde{e}|_\infty = (M_\infty(e) - m_\infty(e))/2) \tag{6.4}$$

$$\arcsin(a^{-1}|\bar{e}|) < (\pi/2) - T^n|B_{2n}/(2n)!|^{1/2}(a + |\tilde{e}|_\infty) \tag{6.5}$$

Then (6.1)-(6.2) *has at least two solutions not differing for* $2\pi k$, $k \in \mathbf{Z}$.

Proof. Apply Theorem 1. To verify condition (ii), choose

$$\delta := \arcsin(a^{-1}|\bar{e}|) + \mu, \tag{6.6}$$

$$\mu := (\frac{1}{2})\{\frac{\pi}{2} - T^n|B_{2n}/(2n)!|^{1/2}(a + |\tilde{e}|_\infty) - \arcsin(a^{-1}|\bar{e}|)\}. \tag{6.7}$$

Then it can easily checked that

$$0 < \delta < \pi/2,$$

$$a^{-1}|\bar{e}| < \sin(\delta),$$

$$\alpha(a + |\tilde{e}|_\infty) < (\pi/2) - \delta,$$

i.e. condition (ii) is satisfied.

As corollaries of Theorem 2, some exact multiplicity results for the periodic BVP (6.1)-(6.2) can be obtained from § 4, using the estimate of α in Proposition 3 and the value of δ defined in (6.6), (6.7) to find restrictions on the coefficient a and on the forcing term e which ensure that each N_j has exactly one fixed point.

§ 7 - **Remarks.**

(1) If $n = 2$, $T = 2$, the inequality (6.5) becomes

$$\arcsin(a^{-1}|\bar{e}|) < (\pi/2) - (1/\sqrt{5})(\pi/6)(2\pi a + 2\pi|\tilde{e}|_\infty),$$

which is essentially the assumption of [13, Th. 1] (see also [17], [21], [22]). We recall that the problem of characterizing the range of the operator $x \to x'' + a\sin(x)$ is not completely solved: see [20] for several important results in this direction.

(2) The results of the preceding sections remain true if the argument is deviated. Namely, let $\sigma : [0, T] \to [0, T]$ be any map sucht that, for every $f \in L^\infty$,

$$f \circ \sigma \in L^\infty,$$
$$M_\infty(f \circ \sigma) = M_\infty(f), m_\infty(f \circ \sigma) = m_\infty(f),$$
$$Q(f \circ \sigma) = Qf.$$

For example, for a fixed $d \in [0, T]$, the map

$$\sigma(t) = t - d \quad \mathrm{mod}\, T$$

satisfies these assumptions. Then our result are valid for the problem

$$x \in \mathrm{dom}\, A, \quad -Ax + a\sin(x \circ \sigma) = e.$$

Therefore, we have results of existence, multiplicity and approximation for the T-periodic solutions of the so-called «sunflower equation» (see [9], [4], [3])

$$x''(t) + (c/d)\, x'(t) + (b/d)\sin(x(t - d)) = e(t),$$

where b, c, d are given positive numbers, and $e \in L^\infty(\mathbf{R}, \mathbf{R})$ is T-periodic.

(3) The technique we use depends in a crucial way upon the representation formula (2.3), which is based on the addition property of the sine function. Therefore it seems hard to extend this technique verbatim to different types of nonlinearities, even if periodic ones. The periodic BVP for a Duffing equation of the form

$$-x'' + cx' + f(t, x) = e(t),$$

with an 'oscillating' nonlinearity f (i.e., roughly speaking, when f has no linear asymptotes as $x \to \pm\infty$) is studied in [6]. In [6], under suitable assumptions on the upper and lower limits of $x^{-1}f(t, x)$ as $x \to \pm\infty$, we get an existence result, via the Leray-Schauder degree, for every $e \in L^2([0, T], \mathbf{R})$. In the present paper, we have confined ourselves to forcing terms e in the space $L^\infty([0, T], \mathbf{R})$, because pointwise estimates are needed to ensure that the functional Z does not have zeroes on the set B (which is the basis for (2.3)).

(4) Results concerning BVP's for systems of coupled forced Duffing equations

$$-x''_i + c_i x'_i + a_i \sin(x_i) = e_i(t, x_1, \dots, x_m), \quad (i = 1, \dots, m),$$

will appear in a subsequent paper [19] (see also [18]).

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APPENDIX

This paper was presented at the «Colloquium on Topological Methods in BVP's for ODE's», held at the International School for Advanced Studies, Trieste, may 1984. Subsequent and related papers are the following.

[18] CARISTI, G., *On Periodic Solutions of Systems of Coupled Pendulum-like Equations,* to appear.

[19] DRABEK, P. & INVERNIZZI, S., *Periodic Solutions for Systems of Forced Coupled Pendulum-like Equations,* J. Differential Equations, to appear.

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