APPLYING A FIXED POINT TECHNIQUE TO ASYMPTOTIC BEHAVIOR AND BOUNDARY VALUE PROBLEMS IN ORDINARY DIFFERENTIALS EQUATIONS (*)

by J. D. SCHUUR (in East Lansing) (**) 

SOMMARIO. - Si usa un metodo di punti fissi per trovare soluzioni di equazioni differenziali della forma $Lx = f(t, x)$.

SUMMARY. - We use a fixed point method to find solutions to differential equations of the form $Lx = f(t, x)$.

**Introduction.** - Let $I \subset [0, \infty)$ be an interval and $0 \leq m < n$; let $a_i(t, v_0, \ldots, v_m) \ (0 \leq i \leq n - 1)$ be continuous on $I \times R \times \ldots \times R$; and let $L_n[x](t) = x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, u(t), \ldots, u^{(m)}(t)) x^{(i)}(t)$.

For a fixed $u \in C^n(I),$

(1 u) $L_n[x](t) = 0, \ t \in I,$

is a linear differential equation. And

(2) $L_n[x](t) = 0, \ t \in I$

is a nonlinear differential equation. The problem of finding solutions

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of (2) which satisfy a suitable property $P$ can be transformed into
the problem of finding fixed points of the mapping $u \rightarrow Tu = \{\text{Solutions of (1) u} \text{ satisfying } P\}$.

This technique was used by Schauder to show the existence of
solutions of partial differential equations. And it has been used to
study systems of the form

\[(3) \quad x' = A(t, x) x + b(t, x).\]

Here $A(t, x)$ is an $n \times n$ matrix-valued function and $b(t, x)$ is a
vector-valued function, both defined on $I \times R$. Conti, [3] and [4],
studied boundary value problems for (3) (see also Anichini [1] and
[2], [5], [12], [15]) and the asymptotic behavior of solutions of
(3) was studied by Corduneanu [8] and Kartsatos [13].

We shall use the technique to study two problems.

1) When does the equation

\[(4) \quad (r(t) x^{(n)})^{(n)} = f(t, x) x\]

have solutions which are asymptotic to $t^m$, $0 \leq m \leq 2n$?

2) When does (2) have a solution which satisfies

\[x(t_i) = b_i, \quad 1 \leq i \leq n, \quad t_1 < \ldots < t_n?\]

1. - In this section we discuss equation (4). To simplify the
presentation we shall assume that $n = 2$ and, in sections of the
proofs, that $r(t) = 1$. But the results hold for the more general case.
We have

\[(1.1) \quad (r(t) x'')'' = f(t, x) x, \quad a \leq t < \infty,\]

where $f(t,x)$ is continuous and positive on $[a, \infty)$ and $r(t) \in C^2[a, \infty)$

with $\int_0^\infty \frac{du}{r(u)} = \infty$.

Notation.

\[E_k(x(t)) = \begin{cases} x^{(k)}(t) & \text{for } k = 0, 1; \\ (r(t) x''(t))^{(k-2)} & \text{for } k = 2, 3, 4 \end{cases}\]

\[R_k(t, x) = \begin{cases} (s - t)^k & \text{for } k = 0, 1; \\ \frac{k!}{s} \int_0^s \frac{(u - t)(s - u)^{k-2}}{r(u)} \, du & \text{for } k = 2, 3; \\ & \text{for } a \leq s, \quad t < \infty. \end{cases}\]

Let $k \in [0, 3]$ be given.
\[ S^i_k(t) = \begin{cases} R_k^{(i)}(t, a) & \text{for } i = 0, 1; \\ r(t) R_k^{(i)}(t, a) & \text{for } i = 2, 3 \end{cases} \begin{array}{c} 0 \leq i \leq k, \ (i) = \frac{d^i}{dt^i} \end{array} \]

\[ S^k_i(t) = \begin{cases} R_i^{[k]}(a, t) & \text{for } k = 0, 1; \\ r(a) R_i^{[k]}(a, t) & \text{for } k = 2 \end{cases} \begin{array}{c} k < i \leq 3, \ [k] = \frac{d^k}{da^k} \end{array} \]

**Theorem 1.1** - In (1.1) assume that \( f(t, x) \) is either increasing or decreasing with respect to \( x \) for \( t \in [a, \infty) \) and let \( k \in [0, 3] \) be given. a) If (1.1) has a solution \( x_k \) satisfying \( \lim_{t \to \infty} x_k(t) / R_k(t, a) = 1 \) (1.2), then

\[ \lim_{t \to \infty} E_i(x_k(t)) / S^i_k(t) = 1 \quad \text{for } 0 \leq i \leq k, \]

\[ \lim_{t \to \infty} E_i(x_k(t)) / S^k_i(t) = 0 \quad \text{for } k < i \leq 3, \quad \text{and} \]

\[ \int_0^\infty R_3(a, s) f(s, cR_k(a, s)) \, ds < \infty \quad \text{for some } c > 0 \quad \text{hold.} \]

b) If (1.4) holds, then (1.1) has a solution \( x_k \) satisfying (1.2).

To prove the theorem by the fixed point technique we need to know the behavior of the linear equation associated with (1.1), viz.

\[ (r(t) x''')'' = p(t) x, \] where \( p \) is continuous and positive on \( [a, \infty) \) and \( r \) satisfies the previous hypotheses. In this theorem we shall let \( r(t) = 1 \) to simplify the discussion.

**Theorem 1.2** - Let (1.5) be given with \( r(t) = 1 \) and let \( k \in [0, 3] \) be given. a) If (1.5) has a solution \( x_k \) satisfying (1.6)

\[ \lim_{t \to \infty} x_k(t) / t^k = 1, \]

then

\[ \lim_{t \to \infty} x_k^{(i)}(t) / t^{k-i} = \begin{cases} 1 & \text{for } 0 \leq i \leq k \\ 0 & \text{for } k < i \leq 3, \end{cases} \]

\[ \int_a^\infty t^i p(t) \, dt < \infty \quad \text{hold.} \]

b) If (1.8) holds, then for each \( k \in [0, 3] \), (1.5) has a solution satisfying (1.6).

**Proof of Theorem 1.2.** We shall outline the proof for the case \( k = 2 \). A complete proof is given in [11].

a) If \( x_2(t) \sim t^2 \), it can then be shown that for all \( t \) sufficiently large

\[ x_2(t), x'_2(t), x''_2(t) > 0 \quad \text{and} \quad x'''_2(t) < 0. \]
We are then justified in differentiating once to obtain \( x'_2(t) \sim t \) and we also see that \( \lim_{t \to \infty} x''_2(t) \) exists.

Using Taylor's theorem we have

\[
(1.10) \quad x_2(t) = \sum_{k=0}^{3} \frac{(-1)^k x^{(k)}_2(b)}{k!} (b - t)^k + \int_{t}^{b} \frac{(s - t)^3}{6} p(s) x_2(s) \, ds
\]

and hence

\[
x''_2(t) - x''_2(b) = - x''_2(b) (b - t) + \int_{t}^{b} (s - t) \, p(s) x_2(s) \, ds.
\]

The limit on the lefthand side exists; each of the terms on the righthand side is positive; hence the limit of each term on the righthand side exists. Using \( x_2(t) \sim t^2 \), (1.8) follows. And by proving that \( \lim_{t \to \infty} x''_2(b) (b - t) = 0 \) we also obtain \( x''_2(b) \sim 1 \).

To prove this part of the theorem for general \( r(t) \) we have \( E_k(x_2(t)) \), \( k = 0, 1, 2, 3 \) in (1.9) and (1.10) becomes

\[
x_2(t) = \sum_{k=0}^{3} \frac{(-1)^k E_k(x_k(b))}{k!} R_k(t, b) + \int_{t}^{b} R_3(t, s) \, p(s) x(s) \, ds.
\]

The \( S_k \)'s are associated with the derivatives of \( R_k \).

b) Assume that (1.8) holds and choose \( c \) such that

\[
\int_{c}^{\infty} \frac{(s - t)^3}{6} p(s) \, ds < 1.
\]

Let \( X \) be the space of functions which are continuous on \([c, \infty)\) and for which \( \| x \| = \sup \{ |x(t)| / t^2 : 0 \leq t < \infty \} \) exists. Then

\[
Tx(t) = t^2 + \int_{c}^{t} \frac{(s - t)^3}{6} p(s) x(s) \, ds
\]

is a contraction mapping on \( X \) and its fixed point will be \( x_2 \). This completes the proof.

**Proof of Theorem 1.1.** Again assuming \( r(t) = 1 \) and \( k = 2 \).

a) If \( x \) is a solution of (1.1) which satisfies (1.2), then \( x_2 \) is a solution of the linear equation \( x'' = f(t, x_2(t)) \) \( x \) and, by applying Theorem 1.2, (1.3) and (1.4) hold.

b) Assume that (1.4) holds and that \( f(t, x) \) \( x \) is increasing with respect to \( x \). We shall apply the following fixed point theorem of Pan and Glicksberg (see [17]).

1.11 - Thm.) If \( K \) is a closed, convex, nonempty subset of a Frechet space \( X \) and if \( T \) satisfies: i) for each \( u \in K \), \( Tu \) is a nonempty, compact, convex subset of \( X \); ii) \( T \) is a closed mapping; and iii) \( TK \) is contained in a compact subset of \( K \); then there is a \( u \in K \) such that \( u \in Tu \).
Let $X$ be the space of functions which are continuous on $[a, \infty)$. For $x_n, x \in X, \|x_n - x\| \to 0$ means that $\sup_{t \in I} |x_n(t) - x(t)| \to 0$ uniformly on each compact subset $I \subset [a, \infty)$. Then $(X, \| \cdot \|)$ is a Fréchet space.

Let $K = \{x \in X : (c - 1) t^2 \leq x(t) \leq ct^2 \text{ on } [a, \infty)\}$. Then $K$ is closed, convex, and nonempty. For $u \in K$, let $Tu = \{x \in K : x \text{ is a solution of } (1u) x'' = f(t, u(t))\}$. By Theorem 1.2, $Tu$ is nonempty. Since $Tu$ is a set of solutions of a linear equation it is convex.

$Tu$ is compact: Let $I_m = [a, m], m > a$ an integer, let $u \in K$, and let $\{x_n\}$ be a sequence in $Tu$. On $I_m$, $u$ and $x_n$, and hence $x''_n$ and $x'_n$, are uniformly bounded. Hence by Ascoli's theorem some subsequence of $\{x_n\}$ converges uniformly on $I_m$, say to $z_m$. And $z_m$ is seen to be a solution of $(1u)$ with $(c - 1) t^2 \leq z_m(t) \leq ct^2$ on $I_m$. Using a sequence of $I_m$'s and a diagonalization argument on the $z_m$'s, it will follow that $T(u)$ is compact in $X$.

$Tu$ is a closed mapping (i.e., $u \in K$ with $u_t \to u_0 \in K$ and $x_t \in T(u_t)$ with $x_t \to x_0$ implies $x_0 \in T(u_0)$) and $TK$ is compact: The arguments are similar to those of the preceding paragraph.

Thus there is a $u \in K$ with $u \in Tu$, this is our $x_2$, and Theorem 1.1 is proved.

REMARKS - Theorem 1.1, the analogous theorem with $f$ negative, and related results on the behavior of solutions may be found in Edelson, Perri, and Schuur, [9], [10], [16] and in the references listed there.

The equation $(p(t) f(x) x')' = q(t) g(x)$ also includes some interesting examples and problems - see [14].

2. In this section we let

$$L_u [x](t) = x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, u(t), u'(t)) x^{(i)}(t) = 0$$

for $0 \leq t \leq 1$, where the $a_i(t, u, v)$ are continuous on $[0, 1] \times R \times R$ and we have the conditions

\begin{equation}
(2.1) \quad x^{(i-1)}(t_j) = b_j^{i-1}. \quad 0 = t_1 < \ldots < t_m = 1
\end{equation}

where $1 \leq m \leq n, 1 \leq i \leq I(j), \sum_{j=1}^{m} I(j) = n$ and the $b_j^i$'s are constant.

We shall also write (2.1) as

\begin{equation}
(2.1a) \quad Mx = b \quad \text{where } M : C^{(n)}[0, 1] \to R^n \text{ is continuous and linear}
\end{equation}

and $b = \col (b_1^{0}, \ldots, b_m^{I(m)-1})$. 

We seek a solution of

\[(2.2) \quad L_x [x] (t) = 0 \]

which satisfies (2.1). We also use the linear equation

\[(2.3) \quad L_u [x] (t) = 0, \quad u \in C^1 [0,1] \text{ given} . \]

**Theorem 2.1** - Assume that the roots of the algebraic equation

\[\lambda^n + a_{n-1} (t, \nu, w) \lambda^{n-1} + \ldots + a_0 (t, \nu, w) = 0\]

are real and satisfy

\[(2.4) \quad \mu_0 \leq \lambda_1 (t, \nu, w) \leq \lambda_2 (t, \nu, w) \leq \ldots \leq \lambda_{n-1} \leq \lambda_n.\]

For \(0 \leq t \leq 1, -\infty < \nu, w < \infty\) where \(\mu_0 < \mu_1 < \ldots < \mu_n\). Then (2.2), (2.1) has a solution.

**Remarks** - The bounds on the \(a_i\)'s are not given explicitly, but they are contained in (2.4). From (2.4) it follows that there exist constants \(A_i\) such that

\[(2.5) \quad |a_i (t, \nu, w)| \leq A_i, \quad A_i = A_i (\mu_0, \ldots, \mu_n),\]

For \(0 \leq i \leq n - 1, \quad 0 \leq t \leq 1, -\infty < \nu, w < \infty .\)

First we need some results for a linear equation. Consider

\[(2.6) \quad x^{(n)} (t) + \sum_{i=0}^{n-1} a_i (t) x^{(i)} (t) = 0\]

\[(2.7) \quad \lambda^n + a_{n-1} (t) \lambda^{n-1} + \ldots + a_0 (t) = 0\]

where the \(a_i\)'s are continuous and \(|a_i (t)| \leq A_i (A_i \text{ given in (2.5)})\) for \(0 \leq t \leq 1, \quad 0 \leq i \leq n - 1 .\)

Assume that the roots of the algebraic equation (2.7) are real and satisfy

\[(2.8) \quad \mu_0 \leq \lambda_1 (t) \leq \mu_1 \leq \ldots \leq \mu_{n-1} \leq \lambda_n (t) \leq \mu_n, \quad 0 \leq t \leq 1 ,\]

where the \(\mu_i\)'s are given in (2.4). Then the following hold.

i) There exists a set of \(n\) solutions \(x_1, \ldots, x_n\) of (2.6) satisfying

\[x_1 (0) = 1, \quad x_i (t) > 0, \quad \mu_{i-1} \leq x_i (t) / x_i (t) \leq \mu_i,\]

for \(0 \leq t \leq 1, \quad 1 \leq i \leq n .\) (Hence \(|x_i (t)| \leq e^{\mu_i}, \quad 0 \leq t \leq 1\).

**Proof.** See [7], Chapter 3.

ii) If there exists a constant \(B_1\) such that a solution \(x\) of (2.6) satisfies \(|x (t)| \leq B_1, \quad 0 \leq t \leq 1,\) then there is a constant
\[ B = B(B_1, A_0, \ldots, A_{n-1}) \]
such that \( |x^{(j)}(t)| \leq B, \ 0 \leq t \leq 1, \ 0 \leq j \leq n. \)

Proof. See [6], Chapter 5.

(iii) Let \( x_1, \ldots, x_n \) be the fundamental set of solutions of (2.6) given in (i) and let \( X \) be the matrix with \( x^{(k-1)}_i, \ 1 \leq k \leq n, \) as its \( i \)-th column. From (ii) we have a constant \( B = B(\mu_0, \ldots, \mu_n) \) such that \( |x^{(k-1)}(t)| \leq B, \ 0 \leq t \leq 1, \ 1 \leq i, k \leq n. \) Also there is a constant \( \beta = \beta(\mu_0, \ldots, \mu_n) \) such that \( \det X(t) \geq \beta > 0 \) for \( 0 \leq t \leq 1. \)

Proof. See [7], Chapter 3.

(iv) For each \( b \in \mathbb{R}^n \) the problem (2.6), (2.1) has a unique solution, call it \( x_b, \) and there exists a constant \( C = C(M, b, \mu_0, \ldots, \mu_n) \) such that \( |x^{(k)}(t)| \leq C, \ 0 \leq k \leq n-1, \ 0 \leq t \leq 1. \)

Proof. Let \( X \) be the matrix given in (iii). Then (2.1a) can be written as

\[ (2.9) \quad MXc = b, \ c \text{ an } n \times 1 \text{ vector} \]

we have a mapping from \( \mathbb{R}^n \) into \( \mathbb{R}^n; \) we shall denote it by \( (MX). \)

Condition (2.8) implies that equation (2.6) is disconjugate, i.e. each nontrivial solution of (2.6) has less than \( n \) zeros. (See [7], Chapter 3). Using (2.1) we see that \( (MX) c = 0 \) has only the trivial solution and hence \( (MX) c = b \) has a unique solution for each \( b. \)

Let \( M_j = (m_{rs}) \) be the \( I(j) \times n \) matrix with \( m_{rs} = 1 \) if \( r = s, \)

\( m_{rs} = 0 \) if \( r \neq s \) and let \( b_j = \text{col} (b_j^0, \ldots, b_j^{(l-1)}). \) Then (2.9) can be written as

\[ M_j X(t_j) c = b_j, \ 1 \leq j \leq m, \] or

\[ \text{diag}(M_1, \ldots, M_m) \text{col}(X(t_1), \ldots, X(t_m)) c = \text{col}(b_1, \ldots, b_m). \]

We know that the \( n \times n \) matrix on the left is nonsingular, call it \( (MX). \) And since each \( M_j \) is a constant matrix; \( |x^{(k)}(t_j)| \leq B; \) and \( |\det X(t_j)| \geq \beta, \) we can show that \( (MX)^{-1} \) is bounded. Hence there is a \( C_0 = C_0(M, B, \beta) \) such that \( ||c|| \leq C_0 ||b|| \) and a \( C = C(b, M, B, \beta) \) such that \( ||\text{col}(x^{(1)}(t), \ldots, x^{(n-1)}(t))|| = ||X(t) c|| \leq C, \ 0 \leq t \leq 1, \) (\( || \cdot || \) is absolute value; \( || \cdot || \) is Euclidean norm) and (iv) is proved.

Proof of Theorem 2.1. This time we can use the Schauder-Tychonov Theorem.

Let \( X \) be the Banach space of \( C^1 \)-functions on \( [0,1] \) with

\[ |x| = |x|_0 + |x'|_0 \text{ for } x \in X. \] (\( |x|_0 = \text{sup} \{|x(t)|: 0 \leq t \leq 1\} \).
Let \( K = \{ x \in X : |x|_1 \leq 2C \} \). Then \( K \) is closed and convex.

For \( u \in K \) let \( Tu \) be the solution of (2.3), (2.1). Then \( Tu \) is well-defined and \( Tu \in K \).

\( T \) is continuous: Let \( u_1, u_2, \ldots \rightarrow u_0 \) in \( K \) and let \( Tu_n = x_n, n \geq 0 \).

From (iv), \( |x^{(k)}_n|_0 \leq C, 0 \leq k \leq n \). By Ascoli’s theorem we have a subsequence \( \{x_m\} \) such that \( x^{(k)}_m \rightarrow z^k, 0 \leq k \leq n \), as \( m \rightarrow \infty \). (We take limits in the differential equation for the case \( k = n \)). Then by taking the limit in

\[
x^{(n)}_m + a_{n-1}(t, u_m(t), u'_m(t)) x^{(n-1)}_m + \ldots = 0, \quad Mx_m = b
\]

and knowing that the solution to (2.3), (2.1) is unique we see that \( z^k = x^{(k)}_0, 0 \leq k \leq n \). The uniqueness also tells us that the full sequence \( \{x_n\} \) converges to \( x_0 \).

\( TK \) is closed and compact: The proof is similar to that of the preceding paragraph.

Hence the mapping \( T : K \rightarrow K \) has a fixed and the theorem is proved.

REFERENCES


[10] EDELSON, A. L. and J. D. SCHUUR, Nonoscillatory solutions of \( (r_2 x^{(n)}(t))^n \pm f(t, x) x = 0 \), Pacific J. Math. 10 (1983), 313-325.
\[ (r(t)x^{(n)})^{(n)} = f(t,x)x, \]
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