APPLYING A FIXED POINT TECHNIQUE TO ASYMPTOTIC BEHAVIOR AND BOUNDARY VALUE PROBLEMS IN ORDINARY DIFFERENTIALS EQUATIONS (*)

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SOMMARIO. - *Si usa un metodo di punti fissi per trovare soluzioni di equazioni differenziali della forma* $Lx = f(t, x)\, x$.

SUMMARY. - *We use a fixed point method to find solutions to differential equations of the form* $Lx = f(t, x)\, x$.

Introduction. - Let $I \subset [0, \infty)$ be an interval and $0 \leq m < n$; let $a_i(t, v_0, \ldots, v_m)$ $(0 \leq i \leq n-1)$ be continuous on $I \times R \times \ldots \times R$; and let $L_u[x](t) = x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, u(t), \ldots, u^{(m)}(t))\, x^{(i)}(t)$.

For a fixed $u \in C^m(I)$,

(1 u) $\quad L_u[x](t) = 0, \; t \in I,$

is a linear differential equation. And

(2) $\quad L_x[x](t) = 0, \; t \in I$

is a nonlinear differential equation. The problem of finding solutions

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of (2) which satisfy a suitable property P can be transformed into the problem of finding fixed points of the mapping $u \to Tu = \{$Solutions of (1 u) satisfying $P\}$.

This technique was used by Schauder to show the existence of solutions of partial differential equations. And it has been used to study systems of the form

(3) $\quad x' = A(t,x)\,x + b(t,x).$

Here $A(t,x)$ is an $n \times n$ matrix-valued function and $b(t,x)$ is a vector-valued function, both defined on $I \times R$. Conti, [3] and [4], studied boundary value problems for (3) (see also Anichini [1] and [2], [5], [12], [15]) and the asymptotic behavior of solutions of (3) was studied by Corduneanu [8] and Kartsatos [13].

We shall use the technique to study two problems.

1) When does the equation

(4) $\quad (r(t)\,x^{(n)})^{(n)} = f(t,x)\,x$

have solutions which are asymptotic to t^m, $0 \leq m < 2n$?

2) When does (2) have a solution which satisfies

$$x(t_i) = b_i,\ 1 \leq i \leq n,\ t_1 < \ldots < t_n\,?$$

1. - In this section we discuss equation (4). To simplify the presentation we shall assume that $n = 2$ and, in sections of the proofs, that $r(t) = 1$. But the results hold for the more general case. We have

(1.1) $\quad (r(t)\,x'')'' = f(t,x)\,x,\ a \leq t < \infty,$

where $f(t,x)$ is continuous and positive on $[a,\infty)$ and $r(t) \in C^2[a,\infty)$ with $\int_0^\infty \frac{du}{r(u)} = \infty$.

Notation.

$$E_k(x(t)) \;=\; \begin{cases} x^{(k)}(t) \text{ for } k = 0,1; \\ (r(t)\,x''(t))^{(k-2)} \text{ for } k = 2,3,4 \end{cases}$$

$$R_k(t,x) \;=\; \begin{cases} \dfrac{(s-t)^k}{k!} \text{ for } k = 0,1; \\ \displaystyle\int_0^s \frac{(u-t)(s-u)^{k-2}}{r(u)}\,du \text{ for } k = 2,3; \end{cases}$$

for $a \leq s,\ t < \infty$.

Let $k \in [0,3]$ be given.

$$S^i_k(t) = \begin{cases} R^{(i)}_k(t,a) & \text{for } i = 0,1; \\ r(t)\, R^{(i)}_k(t,a) & \text{for } i = 2,3 \end{cases} \qquad 0 \le i \le k, \quad (i) = d^i/dt^i$$

$$S^k_i(t) = \begin{cases} R^{[k]}_i(a,t) & \text{for } k = 0,1; \\ r(a)\, R^{[k]}_i(a,t) & \text{for } k = 2 \end{cases} \qquad k < i \le 3, \quad [k] = d^k/da^k.$$

THEOREM 1.1 - *In* (1.1) *assume that* $f(t,x)\,x$ *is either increasing or decreasing with respect to* x *for* $t \in [a,\infty)$ *and let* $k \in [0,3]$ *be given.* a) *If* (1.1) *has a solution* x_k *satisfying* $\lim_{t\to\infty} x_k(t)/R_k(t,a) = 1$ (1.2), *then*

$$(1.3) \qquad \begin{cases} \lim_{t\to\infty} E_i(x_k(t))/S^i_k(t) = 1 \text{ for } 0 \le i \le k, \\ \lim_{t\to\infty} E_i(x_k(t))/S^k_i(t) = 0 \text{ for } k < i \le 3, \text{ and} \end{cases}$$

$$(1.4) \qquad \int_0^\infty R_3(a,s)\, f(s, cR_k(a,s))\, ds < \infty \text{ for some } c > 0 \text{ hold.}$$ b) *If* (1.4) *holds, then* (1.1) *has a solution* x_k *satisfying* (1.2).

To prove the theorem by the fixed point technique we need to know the behavior of the linear equation associated with (1.1), viz.

(1.5) $(r(t)\,x'')'' = p(t)\,x$, where p is continuous and positive on $[a,\infty)$ and r satisfies the previous hypotheses. In this theorem we shall let $r(t) = 1$ to simplify the discussion.

THEOREM 1.2 - *Let* (1.5) *be given with* $r(t) = 1$ *and let* $k \in [0,3]$ *be given.* a) *If* (1.5) *has a solution* x_k *satisfying* (1.6)

$$\lim_{t\to\infty} x_k(t)/t^k = 1,$$

then

$$(1.7) \qquad \lim_{t\to\infty} x^{(i)}_k(t)/t^{k-i} = \begin{cases} 1 \text{ for } 0 \le i \le k \\ 0 \text{ for } k < i \le 3, \text{ and} \end{cases}$$

$$(1.8) \qquad \int_a^\infty t^3 p(t)\, dt < \infty \text{ hold.}$$

b) *If* (1.8) *holds, then for each* $k \in [0,3]$, (1.5) *has a solution satisfying* (1.6).

Proof of Theorem 1.2. We shall outline the proof for the case $k = 2$. A complete proof is given in [11].

a) If $x_2(t) \sim t^2$, it can then be shown that for all t sufficiently large

$$(1.9) \qquad x_2(t),\ x_2'(t),\ x_2''(t) > 0 \text{ and } x_2'''(t) < 0.$$

We are then justified in differentiating once to obtain $x_2'(t) \sim t$ and we also see that $\lim_{t\to\infty} x_2''(t)$ exists.

Using Taylor's theorem we have

$$(1.10) \quad x_2(t) = \sum_{k=0}^{3} (-1)^k x_2^{(k)}(b) \frac{(b-t)^k}{k!} + \int_t^b \frac{(s-t)^3}{6} p(s)\, x_2(s)\, ds$$

and hence

$$x_2''(t) - x_2''(b) = -x_2'''(b)(b-t) + \int_t^b (s-t)\, p(s)\, x_2(s)\, ds .$$

The limit on the lefthand side exists; each of the terms on the righthand side is positive; hence the limit of each term on the righthand side exists. Using $x_2(t) \sim t^2$, (1.8) follows. And by proving that $\lim_{t\to\infty} x_2'''(b)(b-t) = 0$ we also obtain $x_2''(b) \sim 1$.

To prove this part of the theorem for general $r(t)$ we have $E_k(x_2(t))$, $k = 0, 1, 2, 3$ in (1.9) and (1.10) becomes

$$x_2(t) = \sum_{k=0}^{3} (-1)^k E_k(x_k(b))\, R_k(t,b) + \int_t^b R_3(t,s)\, p(s)\, x(s)\, ds .$$

The S_k^i's are associated with the derivatives of R_k.

b) Assume that (1.8) holds and choose c such that

$$\int_c^\infty \frac{(s-t)^3}{6} p(s)\, ds < 1 .$$

Let X be the space of functions which are continuous on $[c, \infty)$ and for which $\|x\| = \sup\{|x(t)| / t^2 : 0 \leq t < \infty\}$ exists. Then

$Tx(t) = t^2 + \int_c^t \frac{(s-t)^3}{6} p(s)\, x(s)\, ds$ is a contraction mapping on X

and its fixed point will be x_2. This completes the proof.

Proof of Theorem 1.1. Again assuming $r(t) = 1$ and $k = 2$.

a) If x_2 is a solutiion of (1.1) which satisfies (1.2), then x_2 is a solution of the linear equation $x'' = f(t, x_2(t))\, x$ and, by applying Theorem 1.2, (1.3) and (1.4) hold.

b) Assume that (1.4) holds and that $f(t,x)\, x$ is increasing with respect to x. We shall apply the following fixed point theorem of Fan and Glicksberg (see [17]).

1.11 - Thm.) If K is a closed, convex, nonempty subset of a Frechet space X and if T satisfies: i) for each $u \in K$, Tu is a nonempty, compact, convex subset of X; ii) T is a closed mapping; and iii) TK is contained in a compact subset of K; then there is a $u \in K$ such that $u \in Tu$.

Let X be the space of functions which are continuous on $[a, \infty)$. For $x_n, x \in X$, $\|x_n - x\| \to 0$ means that $\sup_{t \in I} |x_n(t) - x(t)| \to 0$ uniformly on each compact subset $I \subset [a, \infty)$. Then $(X, \|\cdot\|)$ is a Fréchet space.

Let $K = \{x \in X : (c-1)\, t^2 \leq x(t) \leq ct^2 \text{ on } [a, \infty)\}$. Then K is closed, convex, and nonempty. For $u \in K$, let $Tu = \{x \in K : x$ is a solution of (1u) $x'' = f(t, u(t))\}$. By Theorem 1.2, Tu is nonempty. Since Tu is a set of solutions of a linear equation it is convex.

Tu is compact: Let $I_m = [a, m]$, $m > a$ an integer, let $u \in K$, and let $\{x_n\}$ be a sequence in Tu. On I_m, u and x_n, and hence x''_n and x'_n, are uniformly bounded. Hence by Ascoli's theorem some subsequence of $\{x_n\}$ converges uniformly on I_m, say to z_m. And z_m is seen to be a solution of (1u) with $(c-1)\, t^2 \leq z_m(t) \leq ct^2$ on I_m. Using a sequence of I_m's and a diagonalization argument on the z_m's, it will follow that $T(u)$ is compact in X.

Tu is a closed mapping (i.e., $u_i \in K$ with $u_i \to u_0 \in K$ and $x_i \in T(u_i)$ with $x_i \to x_0$ implies $x_0 \in T(u_0)$) and TK is compact: The arguments are similar to those of the preceding paragraph.

Thus there is a $u \in K$ with $u \in Tu$, this is our x_2, and Theorem 1.1 is proved.

REMARKS - Theorem 1.1, the analogous theorem with f negative, and related results on the behavior of solutions may be found in Edelson, Perri, and Schuur, [9], [10], [16] and in the references listed there.

The equation $(p(t)\, f(x)\, x')' = q(t)\, g(x)$ also includes some interesting examples and problems - see [14].

2. - In this section we let

$$L_u[x](t) = x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t, u(t), u'(t))\, x^{(i)}(t) = 0$$

for $0 \leq t \leq 1$, where the $a_i(t, u, v)$ are continuous on $[0,1] \times R \times R$ and we have the conditions

(2.1) $\quad x^{(i-1)}(t_j) = b_j^{i-1}, \quad 0 = t_1 < \ldots < t_m = 1$

where $1 \leq m \leq n$, $1 \leq i \leq I(j)$, $\sum_{j=1}^{m} I(j) = n$ and the b_j^i's are constant.

We shall also write (2.1) as

(2.1a) $\quad Mx = b$ where $M : C^{(n)}[0,1] \to R^n$ is continuous and linear and $b = \text{col}\,(b_1^0, \ldots, b_m^{I(m)-1})$.

We seek a solution of

(2.2) $L_x[x](t) = 0$

which satisfies (2.1). We also use the linear equation

(2.3) $L_u[x](t) = 0, \; u \in C^1[0,1]$ given.

THEOREM 2.1 - *Assume that the roots of the algebraic equation*

$$\lambda^n + a_{n-1}(t,v,w)\lambda^{n-1} + \ldots + a_0(t,v,w) = 0$$

are real and satisfy

(2.4) $$\mu_0 \leq \lambda_1(t,v,w) \leq \mu_1 \leq \lambda_2(t,v,w) \leq \ldots \leq \mu_{n-1} \leq \lambda_{n-1}(t,v,w) \leq \mu_n .$$

For $0 \leq t \leq 1, \; -\infty < v, w < \infty$ *where* $\mu_0 < \mu_1 < \ldots < \mu_n$. *Then* (2.2), (2.1) *has a solution.*

REMARKS - The bounds on the a_i's are not given explicitly, but they are contained in (2.4). From (2.4) it follows that there exist constants A_i such that

(2.5) $|a_i(t,v,w)| \leq A_i, \; A_i = A_i(\mu_0, \ldots, \mu_n),$

For $0 \leq i \leq n-1, \; 0 \leq t \leq 1, \; -\infty < v, w < \infty$.

First we need some results for a linear equation. Consider

(2.6) $x^{(n)}(t) + \sum_{i=0}^{n-1} \alpha_i(t) x^{(i)}(t) = 0$ and

(2.7) $\lambda^n + \alpha_{n-1}(t)\lambda^{n-1} + \ldots + \alpha_0(t) = 0$

where the α_i's are continuous and $|\alpha_i(t)| \leq A_i$ (A_i given in (2.5)) for $0 \leq t \leq 1, \; 0 \leq i \leq n-1$.

Assume that the roots of the algebraic equation (2.7) are real and satisfy

(2.8) $\mu_0 \leq \lambda_1(t) \leq \mu_1 \leq \ldots \leq \mu_{n-1} \leq \lambda_n(t) \leq \mu_n, \; 0 \leq t \leq 1,$

where the μ_i's are given in (2.4). Then the following hold.

i) There exists a set of n solutions $x_1, \ldots, x_n$ of (2.6) satisfying

$$x_1(0) = 1, \; x_i(t) > 0, \; \mu_{i-1} \leq x'_i(t)/x_i(t) \leq \mu_i,$$

for $0 \leq t \leq 1, \; 1 \leq i \leq n$. (Hence $|x_i(t)| \leq e^{\mu_n}, \; 0 \leq t \leq 1$).

Proof. See [7], Chapter 3.

ii) If there exists a constant B_1 such that a solution x of (2.6) satisfies $|x(t)| \leq B_1, \; 0 \leq t \leq 1$, then there is a constant

$$B = B(B_1, A_0, \dots, A_{n-1})$$

such that $|x^{(j)}(t)| \le B$, $0 \le t \le 1$, $0 \le j \le n$.

Proof. See [6], Chapter 5.

iii) Let $x_1, \dots, x_n$ be the fundamental set of solutions of (2.6) given in (i) and let X be the matrix with $x_i^{(k-1)}$, $1 \le k \le n$, as its i-th column. From (ii) we have a constant $B = B(\mu_0, \dots, \mu_n)$ such that $|x_i^{(k-1)}(t)| \le B$, $0 \le t \le 1$, $1 \le i, k \le n$. Also there is a constant $\beta = \beta(\mu_0, \dots, \mu_n)$ such that $|\det X(t)| \ge \beta > 0$ for $0 \le t \le 1$.

Proof. See [7], Chapter 3.

iv) For each $b \in R^n$ the problem (2.6), (2.1) has a unique solution, call it x_b, and there exists a constant $C = C(M, b, \mu_0, \dots, \mu_n)$ such that $|x_b^{(k)}(t)| \le C$, $0 \le k \le n-1$, $0 \le t \le 1$.

Proof. Let X be the matrix given in (iii). Then (2.1a) can be written as

(2.9) $\quad MXc = b$, c an $n \times 1$ vector

we have a mapping from R^n into R^n; we shall denote it by (MX).

Condition (2.8) implies that equation (2.6) is disconjugate, i.e. each nontrivial solution of (2.6) has less than n zeros. (See [7], Chapter 3). Using (2.1) we see that $(MX)\,c = 0$ has only the trivial solution and hence $(MX)\,c = b$ has a unique solution for each b.

Let $M_j = (m_{rs})$ be the $I(j) \times n$ matrix with $m_{rs} = 1$ if $r = s$, $m_{rs} = 0$ if $r \ne s$ and let $b_j = \operatorname{col}(b_j^0, \dots, b_j^{I(j)-1})$. Then (2.9) can be written as

$$M_j X(t_j)\, c = b_j,\ 1 \le j \le m, \text{ or}$$

$$\operatorname{diag}(M_1, \dots, M_m) \operatorname{col}(X(t_1), \dots, X(t_m))\, c = \operatorname{col}(b_1, \dots, b_m).$$

We know that the $n \times n$ matrix on the left is nonsingular, call it (MX). And since each M_j is a constant matrix; $|x_i^{(k)}(t_j)| \le B$; and $|\det X(t_j)| \ge \beta$, we can show that $(MX)^{-1}$ is bounded. Hence there is a $C_0 = C_0(M, B, \beta)$ such that $\|c\| \le C_0 \|b\|$ and a $C = C(b, M, B, \beta)$ such that

$$\|\operatorname{col}(x_b^{(1)}(t), \dots, x_b^{(n-1)}(t))\| = \|X(t)\,c\| \le C,\ 0 \le t \le 1,$$

($|\cdot|$ is absolute value; $\|\cdot\|$ is Euclidean norm) and (iv) is proved.

Proof of Theorem 2.1. This time we can use the Schauder-Tychonov Theorem.

Let X be the Banach space of C^1-functions on [0,1] with $|x|_1 = |x|_0 + |x'|_0$ for $x \in X$. ($|x|_0 = \sup\{|x(t)| : 0 \le t \le 1\}$).

Let $K = \{x \in X : |x|_1 \leq 2C\}$. Then K is closed and convex.

For $u \in K$ let Tu be the solution of (2.3), (2.1). Then Tu is well-defined and $Tu \in K$.

T is continuous: Let $u_1, u_2, \ldots \to u_0$ in K and let $Tu_n = x_n, n \geq 0$. From (iv), $|x_n^{(k)}|_0 \leq C$, $0 \leq k \leq n$. By Ascoli's theorem we have a subsequence $\{x_m\}$ such that $x_m^{(k)} \to z^k$, $0 \leq k \leq n$, as $m \to \infty$. (We take limits in the differential equation for the case $k = n$). Then by taking the limit in

$$x_m^{(n)} + a_{n-1}(t, u_m(t), u'_m(t))\, x_m^{(n-1)} + \ldots = 0\,, \; Mx_m = b$$

and knowing that the solution to (2.3), (2.1) is unique we see that $z^k = x_0^{(k)}$, $0 \leq k \leq n$. The uniqueness also tells us that the full sequence $\{x_n\}$ converges to x_0.

TK is closed and compact: The proof is similar to that of the preceding paragraph.

Hence the mapping $T : K \to K$ has a fixed and the theorem is proved.

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