ASPECTS OF MORSE THEORY (*)

by MICHEL WILLEM (in Louvain) (**)

SOMMARIO. - *Questo lavoro è dedicato alla nozione di gruppi critici, alla disuguaglianza di Morse nel caso di degenerazione, alla relazione tra la teoria di Morse e il grado topologico e all'applicazione della teoria di Morse ai problemi non lineari di valore al contorno.*

SUMMARY. - *This paper is devoted to the notion of critical groups, the Morse inequalities in the degenerate case, the relation between Morse theory and topological degree and the application of Morse theory to nonlinear boundary value problems.*

Introduction.

Let M be a C^2 riemannian manifold, for example a Hilbert space. If φ is a C^2 function on M, consider the filtration induced by the order structure of $\mathbf{R}$:

$$\varphi^a = \{u \in M : \varphi(u) \leqslant a\}.$$

Morse theory shows that, under some assumptions, the relative homology of the pair (φ^b, φ^a), where $a < b$, is related to the set of critical points in $\varphi^{-1}([a, b])$. The method is to study the change of topology near a critical point and to «integrate» globally the local

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(**) Indirizzo dell'Autore: Institut de Mathématiques Pures et Appliquées - Université Catholique de Louvain - B 1348 Louvain la Neuve (Belgio).

results by using algebra. Morse theory implies some estimates on the number of critical points of φ. Those estimates extend the following observation: if $M = \mathbf{R}$, between two local minima there is a local maximum.

1. - Critical groups.

In order to study possibly degenerate critical points we have to introduce critical groups.

Let A be a topological space and let B be a subset of A. For every integer n, we denote by $H_n(A, B)$ the n-th *singular homology group* of the pair (A, B) over a field F (see [14]). The *Betti numbers* to the pair (A, B) are defined by

$$B_n(A, B) = \dim H_n(A, B)$$

and the *Euler-Poincaré characteristic* by

$$\chi_n(A, B) = \sum_{n=0}^{\infty} (-1)^n B_n(A, B).$$

We shall use the following properties:

(i) If $A \supset B \supset B'$ with B' a deformation retract of B then

$$H_n(A, B) \simeq H_n(A, B).$$

(Let us recall that $\simeq$ means «is isomorphic to» and that B' is a deformation retract of B if there is $h \in C([0,1] \times B, B)$ such that $h(1, u) = u$ for every $u \in B'$ and $h(0, u) = u$, $h(1, u) \in B'$ for every $u \in B$). Similarly, if $A \supset A' \supset C$ with A' a deformation retract of A then

$$H_n(A', C) \simeq H_n(A, C).$$

(ii) (Excision). Assume that C is an open subset of A such that the closure of C is contained interior of B. Then

$$H_n(A \backslash C, B \backslash C) \simeq H_n(A, B).$$

(iii) Let S^k be the k-sphere and B^k the k-ball. Then

$$H_n(B^k, S^{k-1}) \simeq \delta_{n,k} F \quad \text{for } n \geqslant 0,\ k \geqslant 1$$

$$H_n(B^\infty, S^\infty) \simeq \{0\} \qquad \text{for } n \geqslant 0.$$

Let us recall that a topological space is regular if every neighborhood of a point contains a closed neighborhood.

Let M be a regular C^2-manifold on a Banach space V and let u be an isolated critical point of $\varphi \in C^2(M, \mathbf{R})$. The *critical groups* (over a field F) of u are defined by

$$C_n(\varphi, u) = H_n(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}), \; n = 0, 1, \ldots$$

where $c = \varphi(u)$ and U is a closed neighborhood of u. By excision, the critical groups are independent of the choice of U. Let x be a chart at u. The *Morse index* of u is the supremum of the dimensions of subspaces of V on which the bounded bilinear form

$$F(v, w) = (\varphi \circ x^{-1})''(x(u))(v, w)$$

is negative definite. (If M is a Banach space, the Morse index of u is thus the supremum of the dimensions of subspaces of V on which $\varphi''(u)$ is negative definite). The *nullity* of u is the supremum of the dimensions of subspaces of V on which the bilinear form F is zero. The critical point u is *nondegenerate* if the linear operator $L: V \to V^*$ defined by

$$\langle Lv, w \rangle = F(v, w), \; \forall v, w \in V,$$

is invertible.

REMARKS - 1. By the chain rule, the above definitions are independent of the choice of the chart x.

2. The Morse index is a measure of the indefiniteness of $(\varphi \circ x^{-1})''(x(u))$.

3. Bl the implicit function theorem, a nondegenerate critical point is isolated.

We shall prove in a very simple way that, for a nondegenerate critical point, the critical groups depend only on the Morse index.

THEOREM 1 - *Let M be a regular C^2-manifold modelled on a Hilbert space V and let u be an isolated critical point of $\varphi \in C^2(M, \mathbf{R})$.*
a) *If u is a local minimum point*

$$C_n(\varphi, u) = \delta_{n,0} F, \qquad n = 0, 1, \ldots$$

b) *If u is a nondegenerate critical point with Morse index $1 \leqslant k \leqslant \infty$, then*

$$C_n(\varphi, u) = \delta_{n,k} F, \qquad n = 1, 2, \ldots$$

Proof. a) Let B be a closed neighborhood of u so small that $\varphi(v) > c = \varphi(u)$ for $v \in B \setminus \{u\}$. Then, by definition,

$$C_n(\varphi, u) = H_n(\varphi^c \cap B, \varphi^c \cap B \setminus \{u\}) = H_n(\{u\}, \Phi) = \delta_{n,0} F.$$

b) Let x be a chart at u and let $B \subset D(x)$ be a closed neighborhood of u. Then, by definition,

$$C_n(\varphi, u) = H_n(\varphi^c \cap B, \varphi^c \cap B \setminus \{u\}),$$

where $c = \varphi(u)$. It suffices thus to consider the case where M is an open subset of V. We can assume that $u = 0$ and $c = 0$. Let $L: V \to V$ be the self-adjoint operator defined by

$$(Lv, w) = \varphi''(0)(v, w) \qquad \forall v, w \in V.$$

Since L is invertible by assumption, V is the orthogonal sum of Y and Z with L negative definite on Y and positive definite on Z. Let B be a closed ball with center 0 so small that

(1) $B \cap Y \subset \varphi^0$

(2) $B \cap Z \cap \varphi^0 = \{0\}$

(3) $\varphi''(v)(w, w) \geqslant 0 \qquad \forall v \in B, \ \forall w \in Z.$

Let us define

$$\eta : [0,1] \cap B \to B : (t, v) \mapsto (1-t)\, v + t\, Pv$$

where P is the orthogonal projector onto Y. For $v \in \varphi^0 \cap B$, let us write $f(t) = \varphi(\eta(t, v))$. It follows from (3) that, for all $t \in [0,1]$,

$$f''(t) = \varphi''((1-t)\, v + t\, Pv)\, (I - P)\, v) \geqslant 0,$$

so that f is convex on $[0,1]$. But $f(0) = \varphi(v) \leqslant 0$ since $v \in \varphi^0$ and $f(1) = \varphi(Pv) \leqslant 0$ by (1). Thus $\varphi(\eta(t, v)) = f(t) \leqslant 0$ for all $t \in [0,1]$. Moreover if $\eta(t, v) = 0$ for some $t \in [0,1]$ and some $v \in \varphi^0 \cap B$, then $v = 0$. Because $\eta(t, v) = 0$ implies $Pv = 0$ so that, by (2), $v = 0$. Finally $Y \cap B \setminus \{0\}$ is a deformation retract of $\varphi^0 \cap B \setminus \{0\}$ and $Y \cap B$ is a deformation retract of $\varphi^0 \cap B$. Since $\dim Y = k$, we obtain

$$\begin{aligned} H_n(\varphi^0 \cap B, \varphi^0 \cap B \setminus \{0\}) &\simeq H_n(\varphi^0 \cap B, Y \cap B \setminus \{0\}) \\ &\simeq H_n(Y \cap B, Y \cap B \setminus \{0\}) \\ &\simeq H_n(B^k, S^{k-1}) \\ &\simeq \delta_{n,k} F. \qquad \square \end{aligned}$$

2. - Morse inequalities.

Let us consider the following framework:

(A) M is a complete riemannian manifold, σ is the flow defined by

$$\dot{\sigma}(t, u) = -\nabla \varphi(\sigma(t, u))$$
$$\sigma(0, u) = u \in M$$

where $\varphi \in C^2(M, \mathbf{R})$, X is the closure of an open subset of M such that $\sigma(t, u) \in X$ for $u \in X$ and $0 < t < \omega + (u)$, $a < b$ are real numbers. The critical points in $\varphi^{-1}([a, b]) \cap X$ are isolated and the boundary of $\varphi^{-1}([a, b]) \cap X$ is free of critical points. Finally the Palais-Smale condition over $\varphi^{-1}([a, b]) \cap X$ is satisfied: every sequence $(u_i) \subset \varphi^{-1}([a, b]) \cap X$ such that $|\nabla \varphi(u_i)| \to 0$ contains a convergent subsequence.

It is now easy to verify that $\varphi^{-1}([a,b]) \cap X$ contains only a finite number of critical points $u_1, \ldots, u_j$. The *Morse numbers* are defined by

$$M_n^{a,b}(X) = \sum_{i=1}^{j} \dim C_n(\varphi, u_i) \qquad n = 0, 1, \ldots$$

Let us also define

$$\varphi_X^a = \varphi^a \cap X$$

and

$$B_n^{a,b}(X) = B_n(\varphi_X^b, \varphi_X^a).$$

THEOREM 2 - *If the Morse numbers $M_n^{a,b}(X)$ are all finite and zero for n sufficiently large, then*

$$\sum_{n=0}^{m} (-1)^{m-n} B_n^{a,b}(X) \leqslant \sum_{m}^{n=0} (-1)^{m-n} M_n^{a,b}(X) \tag{4}$$

$$B_n^{a,b}(X) \leqslant M_n^{a,b}(X) \tag{5}$$

$$\chi(\varphi_X^b, \varphi_X^a) = \sum_{n=0}^{\infty} (-1)^n M_n^{a,b}(X). \tag{6}$$

REMARKS - 1. For the rather delicate proof of (4) and (6) see [10]. Morse theory was first extended in infinite dimension in [12] and [13].

2. To obtain (5) it suffices to add successives lines of (4).

3. When $X = M$ and all the critical point are nondegenerate the

classical interpretation of the Morse inequalities follows from Theorem 1.

4. The Morse inequalities (4) can be used to compute some homology groups ([15]) and the Morse equality (6) is a basic ingredient in the proof of the Poincaré-Hopf theorem ([11]). In a recent paper ([4]) K. C. Chang obtains in a rather direct way the main results of [1], [2], [6] and [7] from Theorem 2. For other applications see [8], [10] and [15].

Let us recall that $\chi(M) = \chi(M, \Phi)$.

COROLLARY 1 - *Let M be a complete riemannian manifold and assume that $\varphi \in C^2(M, \mathbf{R})$ satisfies the Palais-Smale condition over M. If φ has a global minimum and $\chi(M)$ nondegenerate critical points with finite Morse index, then φ has at least $\chi(M) + 2$ critical points.*

Proof. If φ has only $\chi(M) + 1$ critical points, let us apply Theorem 2 with $X = M$ and

$$a < \min_M \varphi \, , b > \sup \{ \varphi(u) : \nabla \varphi(u) = 0 \} .$$

Then φ^b is a deformation retract of M (see [10]) and

$$H_n(\varphi^b, \varphi^a) = H_n(M, \Phi).$$

Equality (6) becomes

$$M_0 - M_1 + M_2 - \ldots = \chi(M) \tag{7}$$

where $M_n = M_n^{a,b}(M)$. By Theorem 1, M_0 is the number of (local or global minimum points and $M_n, n \geqslant 1$, is the number of critical points with Morse index n. Thus

$$M_0 + M_1 + M_2 + \ldots = \chi(M) + 1 . \tag{8}$$

It follows from (7) and (8) that

$$2M_0 + 2M_2 + \ldots = 2\chi(M) + 1 ,$$

a contradiction. □

Example 1 : $\chi(M) = 0$. Let us consider the periodic boundary value problem:

$$\begin{aligned} &\ddot{u}(t) + \sin u(t) = f(t) \\ &u(0) - u(T) = \dot{u}(0) - \dot{u}(T), \end{aligned} \tag{9}$$

where $f \in L^1([0,T]; \mathbf{R})$ and $\int_0^T f(t)\, dt = 0$. The corresponding Lagrangian action is given by

$$\varphi(u) = \int_0^T \left[\frac{\dot{u}(t)^2}{2} + \cos u(t) + f(t)\, u(t) \right] dt .$$

It is easy to verify that $\varphi(u + 2\pi) = \varphi(u)$. Thus it is natural to define φ on

$$M = S^1 \times \tilde{H}^1_T$$

where

$$\tilde{H}^1_T = \{ u : [0,T] \to \mathbf{R} ; \; u \text{ is absolutely continuous, } \int_0^T \dot{u}^2 < \infty , \; \int_0^T u = 0 , \; u(0) = u(T) \} .$$

Now φ is bounded from below and satisfies the Palais-Smale condition over M, so that φ has a global minimum [10]. Since $\chi(M) = \chi(S^1) = 0$, Corollary 2 implies the existence of at least 2 critical points. Thus problem (9) has at least two solutions in M. (This result was first obtained in [9] from the moutain pass theorem).

Example 2 : $\chi(M) = 1$. Let M be a Hilbert space. If $\varphi \in C^2(M, \mathbf{R})$

satisfies the Palais-Smale condition, has a global minimum and a nondegenerate critical point with finite Morse index, then φ hat at least three critical points. Indeed in this case $\chi(M) = 1$. (This result is due to Castro and Lazer in finite dimension [3] and to K.C. Chang in the general case [4]).

3. - Perturbation.

In some cases, it is possible to «resolve» degenerate critical points into finitely many nondegenerate critical points.

A linear continuous operator L between two Banach spaces is a *Fredholm operator* if

a) $\dim \ker L < \infty$,

b) $R(L)$ is closed,

c) $\operatorname{codim} R(L) < \infty$.

Let M and N be a pair of C^1-manifolds. A map $f \in C^1(M, N)$ is a *Fredholm map* if, for every $u \in M$, for every chart x at u and for every chart y at $f(u)$, the operator

$$(y \circ f \circ x^{-1})' x(u)$$

is a Fredholm operator.

The following result is due to Marino and Prodi:

THEOREM 3 - *Let V be a Hilbert space, U an open subset of V and $\varphi \in C^2(U, \mathbf{R})$. If $\nabla\varphi$ is a Fredholm map and if u_0 is the only critical point of φ, then for every $\bar{\varepsilon} > 0$ there exists $\varepsilon \in]0, \bar{\varepsilon}]$ and $\psi \in C^2(U, \mathbf{R})$ such that*

(i) *The critical points of ψ, if any, are in finite number and non-degenerate.*

(ii) $|u - u_0| \geqslant \varepsilon \Rightarrow \psi(u) = \varphi(u)$.

(iii) *For every $u \in U$,*
$$|\psi(u) - \varphi(u)| + |\psi'(u) - \varphi'(u)|_{V^*} + |\psi''(u) - \varphi''(u)|_{\mathcal{L}(V, V^*)} \leqslant \varepsilon .$$

(iv) Every sequence $(u_j) \subset B[u_0, \varepsilon]$ such that $\nabla\psi(u_j) \to 0$ contains a convergent subsequence.

Theorem 3 follows from the Sard-Smale lemma. We shall give some applications to degree theory. For other applications see [8] and [10].

4. - Degree theory.

In this section, besides (A) we assume that M is a Hilbert space and that $K = \mathrm{id} - \nabla\varphi$ maps bounded sets into relatively compact sets. Thus K is completely continuous and $\nabla\varphi$ is a Fredholm map (see [10]). If the critical points of φ in $\varphi^{-1}([a,b]) \cap X$ are contained in the interior of a closed ball B, the Leray-Schauder degree

$$d(\nabla\varphi, \varphi^{-1}([a,b]) \cap X \cap B)$$

is well defined.

THEOREM 4 - *Under the above assumptions*

$$d(\nabla\varphi, \varphi^{-1}([a,b]) \cap X \cap B) = \chi(\varphi_X^b, \varphi_X^a).$$

Proof. By (A), the set $\varphi^{-1}([a,b]) \cap X$ contains a finite number of critical points $u_1, \ldots, u_j$. Let $\bar{\varepsilon} > 0$ be such that the balls $B[u_i, \bar{\varepsilon}]$ are disjoint and contained in the interior of $X \cap \varphi^{-1}([a,b]) \cap B$. By Theorem 3, for $i = 1, \ldots, j$ there is a perturbatiion φ_i of φ satisfying (i) to (iv). Let us define $\tilde{\varphi}$ by

$$\begin{aligned} \tilde{\varphi}(u) &= \varphi_i(u) \text{ if } u \in B[u_i, \varepsilon], \quad i = 1, \ldots, j \\ &= \varphi(u) \text{ elsewhere in } M. \end{aligned}$$

By construction, the critical points of $\tilde{\varphi}$ in $\varphi^{-1}([a,b]) \cap X$ are in finite number and nondegenerate. Let v be a critical point of $\tilde{\varphi}$ with Morse index k. Since k is the number of eigenvalues of $K'(v)$ strictly greater than 1, the Leray-Schauder index is given by

$$i(\nabla\varphi, v) = (-1)^k. \tag{10}$$

Using boundary dependance, additivity of degree and (10), we obtain, in obvious notations.

$$\begin{aligned} & d(\nabla\varphi, \varphi^{-1}([a,b]) \cap X \cap B) \\ &= d(\nabla\tilde{\varphi}, \tilde{\varphi}^{-1}([a,b]) \cap X \cap B) \\ &= \tilde{M}_0^{a,b}(X) - \tilde{M}_1^{a,b}(X) + \ldots \end{aligned} \tag{11}$$

The Morse equality (6) implies that

$$\begin{aligned} \tilde{M}_0^{a,b}(X) - \tilde{M}_1^{a,b}(X) + \ldots &= \chi(\tilde{\varphi}_X^b, \tilde{\varphi}_X^a) \\ &= \chi(\varphi_X^b, \varphi_X^a). \end{aligned} \tag{12}$$

The result follows then from (11) and (12). □

THEOREM 5 - *If w is a critical point of φ in $\varphi^{-1}([a,b]) \cap X$, then*

$$i(\nabla\varphi, w) = \sum_{n=0}^{\infty} (-1)^n \dim C_n(\varphi, w).$$

Proof. Let $u_1 = w,\ u_2, \ldots, u_j$ be the critical points of φ in $\varphi^{-1}([a,b]) \cap X$. Let $\varepsilon > 0$ and $\varphi_2, \ldots, \varphi_j$ be as in the proof of Theorem 4. The function $\hat{\varphi}$ is defined by

$$\begin{aligned}\hat{\varphi}(u) &= \varphi_i(u) \ \text{ if } u \in B[u_i, \varepsilon], \quad i = 2, \ldots, j \\ &= \varphi(u) \qquad \text{elsewhere in } M.\end{aligned}$$

By construction, $\hat{\varphi}$ has a finite number of critical points in $\hat{\varphi}^{-1}([a,b]) \cap X : v_1 = w,\ v_2, \ldots, v_m$ and $v_2, \ldots, v_m$ are nondegenerate. Theorem 4 and the Morse equality (6) imply that

$$\begin{aligned}& \mathrm{i}(\nabla\hat{\varphi}, w) + \sum_{j=2}^{m} \mathrm{i}(\nabla\hat{\varphi}, v_j) \\ &= \deg(\nabla\hat{\varphi}, \hat{\varphi}^{-1})([a,b]) \cap X \cap B) \\ &= \chi(\hat{\varphi}_X^b, \hat{\varphi}_X^a) \\ &= \hat{M}_0^{a,b}(X) - \hat{M}_1^{a,b}(X) + \ldots\end{aligned}$$

By (10) we have

$$\hat{M}_0^{a,b}(X) - \hat{M}_1^{a,b}(X) + \ldots = \sum_{n=0}^{\infty} (-1)^n \dim C_n(\hat{\varphi}, w) + \sum_{j=2}^{m} \mathrm{i}(\nabla\hat{\varphi}, v_j).$$

Thus, after simplification,

$$\mathrm{i}(\nabla\hat{\varphi}, w) = \sum_{n=0}^{\infty} (-1)^n \dim C_n(\hat{\varphi}, w).$$

But, by construction, $\mathrm{i}(\nabla\hat{\varphi}, w) = \mathrm{i}(\nabla\varphi, w)$ and $C_n(\hat{\varphi}, w) = C_n(\varphi, w)$, so that the proof is complete. □

REMARKS - 1. Theorem 5 is due to Rothe [13], but the Morse theoretical proof is new.

2. By Theorem 5 the Leray-Schauder index depends only on the critical groups.

3. Theorems 4 and 5 give nice relations between Leray-Schauder degree and Morse theory, but in the (variational) applications it is in general simpler to use directly Morse theory.

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