GETTING A SOLUTION
BETWEEN SUB- AND SUPERSOLUTIONS
WITHOUT MONOTONE ITERATION (*)

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1. - Introduction and main result.

We consider the following nonlinear boundary value problem:

\begin{equation}
\begin{aligned}
-\Delta u = f(x, u) & \quad \text{in } \Omega, \\
u = g & \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}

where $\Omega$ is a bounded domain of $\mathbb{R}^N$.

For $f$ we only assume

\text{(H 1)} \quad f : \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is continuous.}

We also assume that

\text{(H 2)} \quad g : \partial \Omega \to \mathbb{R} \quad \text{is continuous.}

In this note we are interested in the existence of solutions of

\text{(1)} \quad \text{lying between sub- and supersolutions defined in a rather weak sense. Due to the special form of the left hand side we can define}

\text{Definition 1 - A function $u$ is called a sub (super) solution of (1) if}

\begin{enumerate}
\item[i)] $u \in C(\overline{\Omega}; \mathbb{R})$
\item[ii)] $\int_{\Omega} (u(-\Delta \varphi) - f(x, u) \varphi) \, dx \leq (\geq) 0$ for every $\varphi \in \mathcal{D}^+(\Omega)$
\item[iii)] $u \leq (\geq) g$ on $\partial \Omega$
\end{enumerate}

are satisfied, where $\mathcal{D}^+(\Omega)$ consist of all nonnegative functions in $C_0^\infty(\Omega)$.

\text{Definition 2 - A function $u$ is called a solution of (1) if}

\begin{enumerate}
\item[i)] $u \in C(\overline{\Omega}; \mathbb{R})$
\item[ii)] $\int_{\Omega} (u(-\Delta \varphi) - f(x, u) \varphi) \, dx = 0$ for every $\varphi \in C_0^\infty(\Omega)$
\item[iii)] $u = g$ on $\partial \Omega$
\end{enumerate}

are satisfied.

If $f$ satisfies some additional assumption, like for example $u \to f(\cdot, u) + \omega u$ is increasing for some $\omega \in \mathbb{R}$, and if $\partial \Omega$ satisfies some smoothness condition, then the following is known, see [2] [5] [6, Ch. 10] [3].

If $\underline{u}$ is a subsolution, $\overline{u}$ is a supersolution such that $\underline{u} \leq \overline{u}$, then problem (1) possesses a minimal and a maximal solution in the order interval $[\underline{u}, \overline{u}]$. These solutions are obtained by using the method of monotone iterations.
In [1] another method is used to prove the existence of a solution lying between a sub- and a supersolution for a very general quasilinear elliptic problem. The goal of this note is to show the existence of a solution lying between a sub- and supersolution, assuming only the continuity of \( f \) and for a much larger class of sub- and supersolutions.

We shall use the Schauder fixed point theorem and a version of the strong maximum principle.

Observe that if \( f = 0 \), then problem (1) possesses a solution for every \( g \in C(\partial \Omega) \), if and only if all boundary points are regular, see [4, Th. 2.14]. Therefore we assume

\[(H3) \quad \Omega \text{ is a bounded domain of } \mathbb{R}^N \text{ and every point of } \partial \Omega \text{ is regular.}\]

Then we have

**Theorem - Assume (H1), (H2) and (H3), and let \( u \) respectively \( \bar{u} \) be a sub-respectively a supersolution of problem (1), satisfying \( u \leq \bar{u} \) in \( \bar{\Omega} \).

Then problem (1) possesses at least one solution \( u \) satisfying \( u \leq u \leq \bar{u} \) in \( \bar{\Omega} \).

2. - Proof.

We shall proceed in four steps.

**Step 1 - Reduction to homogeneous boundary condition.**

Let \( h \) denote the unique harmonic function on \( \Omega \), continuous on \( \bar{\Omega} \), satisfying \( h = g \) on \( \partial \Omega \). Set \( v = u - h \). Then \( u \) is a solution of problem (1) if and only if \( v \) is a solution of

\[
\begin{align*}
-\Delta v &= f(x, h(x) + v) \quad \text{in } \Omega, \\
\quad v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Observe that the modified right hand side again satisfies (H1). Since both \( u - h \) and \( \bar{u} - h \) are sub- respectively supersolution for the modified problem and are also ordered, we may assume without loss of generality that \( g = 0 \).

**Step 2 - Modification of \( f \).**

Define \[ \]
\[ f^*(x, u) = \begin{cases} f(x, u) & \text{if } u < \underline{u}(x), \\ f(x, \underline{u}(x)) & \text{if } \underline{u}(x) \leq u \leq \bar{u}(x), \\ f(x, \bar{u}(x)) & \text{if } \bar{u}(x) < u, \end{cases} \quad \text{and } x \in \bar{\Omega}. \]

Then \( f^*: \bar{\Omega} \times \mathbb{R} \to \mathbb{R} \) is continuous and bounded. Note that, if \( u \) is a solution of

\[
\begin{cases} -\Delta u = f^*(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
\]

and \( \underline{u} \leq u \leq \bar{u} \) in \( \bar{\Omega} \), then \( u \) is a solution of (1) with \( g = 0 \). In fact every solution of (3) satisfies \( \underline{u} \leq u \leq \bar{u} \) in \( \bar{\Omega} \). This is done in

**STEP 3 - Use of the maximum principle.**

Let \( u \) be a solution of (3) and set \( \Omega^+ = \{ x \in \Omega ; \bar{u}(x) < u(x) \} \). We want to prove that \( \Omega^+ \) is empty. Assume to the contrary that \( \Omega^+ \) is not empty. First, note that \( \Omega^+ \) is open, since \( u \) and \( \bar{u} \) are continuous. Moreover we have

\[
\int_{\Omega^+} (u - \bar{u}) (-\Delta \varphi) \, dx \leq \int_{\Omega^+} (f^*(x, u(x)) - f(x, \bar{u}(x))) \varphi \, dx = 0
\]

for every \( \varphi \in \mathcal{D}^+(\Omega^+) \).

Then \( u - \bar{u} \in C(\bar{\Omega}^+) \) is subharmonic and nonnegative in \( \Omega^+ \). Such functions achieve its maximum at the boundary, see [4].

Since \( u - \bar{u} = 0 \) on \( \partial\Omega^+ \) it follows that \( u = \bar{u} \) in \( \Omega^+ \). Hence \( \Omega^+ \) is empty, a contradiction. Similarly one proves that \( \underline{u} \leq u \) in \( \bar{\Omega} \).

**STEP 4 - Application of Schauder fixed point theorem.**

It remains to show that problem (3) possesses a solution. Let us recall that problem (1) with \( f \) depending only on \( x \) and \( g = 0 \) has exactly one solution \( u \in C(\bar{\Omega}) \). Let \( K: C(\bar{\Omega}) \to C(\bar{\Omega}) \) denote the solution operator, that is \( u = Kf \). Then it is known that \( K \) is a linear compact operator in \( C(\bar{\Omega}) \) equipped with the usual maximum norm \( \| \cdot \| \) (see also Appendix).

Let \( F: C(\bar{\Omega}) \to C(\bar{\Omega}) \) denote the Niemytski operator associated with \( f^* \), that is

\[
F(u)(x) = f^*(x, u(x)) \quad \text{for } u \in C(\bar{\Omega}), \ x \in \bar{\Omega}.\]

Then \( F \) is continuous and there is \( M > 0 \) such that \( \| F(u) \| \leq M \).
Finally observe that \( u \) is a solution of problem (3) if and only if \( u \) satisfies
\[
u = KF(u).
\]

A straightforward application of the Schauder fixed point theorem guarantees the existence of such solution. This completes the proof of the theorem. ■

**Remark** - If \( u \) is a solution of (1), then it follows from standard regularity theory theorems that \( u \in W^{2,p}_{\text{loc}}(\Omega) \) for all \( p \in [1, \infty) \), although \( \bar{u} \) and \( \tilde{u} \) do not need to possess such regularity.

**3. - Appendix.**

**Proposition** - Let \( \Omega \) satisfy (H3) and \( f \in C(\bar{\Omega}) \), then there exists a unique \( u \in C(\bar{\Omega}) \) satisfying
\[
i \int_{\Omega} (u(-\Delta \varphi) + f \varphi) \, dx = 0 \quad \text{for every} \quad \varphi \in C_0^{\infty}(\Omega),
\]
\[
i u = 0 \quad \text{on} \ \partial \Omega.
\]

Moreover the mapping \( f \mapsto u \) is compact in \( C(\bar{\Omega}) \).

**Proof.** The uniqueness is a direct consequence of the maximum principle for harmonic functions. For the existence we extend \( f \) by 0 outside of \( \bar{\Omega} \) and set
\[
w(x) = \int_{\mathbb{R}^N} \Gamma(x - y) \, f(y) \, dy,
\]
the Newtonian potential of \( f \), see [4, p. 50].

Then \( w \in C^1(\bar{\Omega}) \), see [4, Lemma 4.1], and the mapping \( f \mapsto w \) from \( C(\bar{\Omega}) \) in \( C^1(\bar{\Omega}) \) is continuous, where \( C(\bar{\Omega}) \) and \( C^1(\bar{\Omega}) \) are equipped with the usual norm. Since \( C^1(\bar{\Omega}) \) is compactly imbedded in \( C(\bar{\Omega}) \), the mapping \( f \mapsto w \) from \( C(\bar{\Omega}) \) into \( C(\bar{\Omega}) \) is compact.

Let \( h \in C(\bar{\Omega}) \) be the unique harmonic function satisfying \( h = w \) on \( \partial \Omega \) (here we use (H3)). Then \( u = w - h \) is a solution of i), ii). Since the mapping \( w \mapsto h \) from \( C(\bar{\Omega}) \) into \( C(\bar{\Omega}) \) is continuous we have that the mapping \( f \mapsto u \) from \( C(\bar{\Omega}) \) into \( C(\bar{\Omega}) \) is compact. ■
REFERENCES


