GETTING A SOLUTION BETWEEN SUB - AND SUPERSOLUTIONS WITHOUT MONOTONE ITERATION (*)

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SOMMARIO. - *Se esiste una sotto-sopra soluzione per un problema semilineare ellittico allora si può provare l'esistenza di una soluzione usando il metodo della iterazione monotona. Per applicare questo metodo è necessario assumere una regolarità del secondo membro più forte della continuità.*
In questa nota si prova l'esistenza di una soluzione nella sola ipotesi di continuità del secondo membro usando il teorema di Schauder e una versione del principio di massimo forte assumendo l'esistenza di una sotto (sopra) soluzione debole.

SUMMARY. - *If there exist a sub- and a supersolution for a semilinear elliptic problem, then one can show the existence of a solution by a monotone iteration scheme. In order to do this one needs more than continuity of the right hand side.*
In this note the Schauder fixed point theorem and a version of the strong maximum principle is used to get existence of a solution with only continuity of the right hand side under the existence of a weak sub- and supersolution.

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1. - Introduction and main result.

We consider the following nonlinear boundary value problem:

$$(1) \qquad \begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbf{R}^N$.

For f we only assume

(H 1) $\quad f : \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ is continuous.

We also assume that

(H 2) $\quad g : \partial\Omega \to \mathbf{R}$ is continuous.

In this note we are interested in the existence of solutions of (1) lying between sub- and supersolutions defined in a rather weak sense. Due to the special form of the left hand side we can define

DEFINITION 1 - A function u is called a sub (super) solution of (1) if

i) $u \in C(\bar{\Omega}; \mathbf{R})$

ii) $\int_\Omega (u(-\Delta\varphi) - f(x,u)\varphi)\, dx \leqslant (\geqslant)\, 0$ for every $\varphi \in \mathfrak{D}^+(\Omega)$

iii) $u \leqslant (\geqslant)\, g$ on $\partial\Omega$
are satisfied, where $\mathfrak{D}^+(\Omega)$ consist of all nonnegative functions in $C_0^\infty(\Omega)$.

DEFINITION 2 - A function u is called a solution of (1) if

i) $u \in C(\bar{\Omega}; \mathbf{R})$

ii) $\int_\Omega (u(-\Delta\varphi) - f(x,u)\varphi)\, dx = 0$ for every $\varphi \in C_0^\infty(\Omega)$

iii) $u = g$ on $\partial\Omega$

are satisfied.

If f satisfies some additional assumption, like for example $u \to f(\cdot, u) + \omega u$ is increasing for some $\omega \in \mathbf{R}$, and if $\partial\Omega$ satisfies some smoothness condition, then the following is known, see [2] [5] [6, Ch. 10] [3].

If $\underline{u}$ is a subsolution, $\bar{u}$ is a supersolution such that $\underline{u} \leqslant \bar{u}$, then problem (1) possesses a minimal and a maximal solution in the order interval $[\underline{u}, \bar{u}]$. These solutions are obtained by using the method of monotone iterations.

In [1] another method is used to prove the existence of a solution lying between a sub- and a supersolution for a very general quasilinear elliptic problem. The goal of this note is to show the existence of a solution lying between a sub- and supersolution, assuming only the continuity of f and for a much larger class of sub- and supersolutions.

We shall use the Schauder fixed point theorem and a version of the strong maximum principle.

Observe that if $f = 0$, then problem (1) possesses a solution for every $g \in C(\partial\Omega)$, if and only if all boundary points are regular, see [4, Th. 2.14]. Therefore we assume
(H 3) Ω is a bounded domain of $\mathbf{R}^N$ and every point of $\partial\Omega$ is regular.

Then we have

THEOREM - *Assume* (H 1), (H 2) *and* (H 3), *and let* $\underline{u}$ *respectively* $\bar{u}$ *be a sub-respectively a supersolution of problem* (1), *satisfying* $\underline{u} \leqslant \bar{u}$ *in* $\bar{\Omega}$.

Then problem (1) *possesses at least one solution* u *satisfying* $\underline{u} \leqslant u \leqslant \bar{u}$ *in* $\bar{\Omega}$.

2. - Proof.

We shall proceed in four steps.

STEP 1 - Reduction to homogeneous boundary condition.

Let h denote the unique harmonic function on Ω, continuous on $\bar{\Omega}$, satisfying $h = g$ on $\partial\Omega$. Set $v = u - h$. Then u is a solution of problem (1) if and only if v is a solution of

$$(2) \qquad \begin{cases} -\Delta v = f(x, h(x) + v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Observe that the modified right hand side again satisfies (H 1). Since both $\underline{u} - h$ and $\bar{u} - h$ are sub- respectively supersolution for the modified problem and are also ordered, we may assume without loss of generality that $g \equiv 0$.

STEP 2 - Modification of f.

Define

$$f^*(x,u)=\begin{cases} f(x,\underline{u}(x)) & \text{if } u<\underline{u}(x), \\ f(x,u) & \text{if } \underline{u}\,(x)\leqslant u\leqslant \bar{u}(x), \\ f(x,\bar{u}(x)) & \text{if } \bar{u}(x)<u, \qquad \text{and } x\in\bar{\Omega}. \end{cases}$$

Then $f^*:\bar{\Omega}\times\mathbf{R}\to\mathbf{R}$ is continuous and bounded. Note that, if u is a solution of

$$\text{(3)} \qquad \begin{cases} -\Delta u=f^*(x,u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

and $\underline{u}\leqslant u\leqslant \bar{u}$ in $\bar{\Omega}$, then u is a solution of (1) with $g=0$. In fact every solution of (3) satisfies $\underline{u}\leqslant u\leqslant \bar{u}$ in $\bar{\Omega}$. This is done in

STEP 3 - Use of the maximum principle.

Let u be a solution of (3) and set $\Omega^+=\{x\in\Omega\,;\bar{u}(x)<u(x)\}$. We want to prove that Ω^+ is empty. Assume to the contrary that Ω^+ is not empty. First, note that Ω^+ is open, since u and $\bar{u}$ are continuous. Moreover we have

$$\int_{\Omega^+}(u-\bar{u})\,(-\Delta\varphi)\,dx\leqslant\int_{\Omega^+}(f^*(x,u(x))-f(x,\bar{u}(x)))\,\varphi dx=0$$
$$\text{for every } \varphi\in\mathfrak{D}^+(\Omega^+).$$

Then $u-\bar{u}\in C(\overline{\Omega^+})$ is subharmonic and nonnegative in Ω^+. Such functions achieve its maximum at the boundary, see [4].

Since $u-\bar{u}=0$ on $\partial\Omega^+$ it follows that $u=\bar{u}$ in Ω^+. Hence Ω^+ is empty, a contradiction. Similarly one proves that $\underline{u}\leqslant u$ in $\bar{\Omega}$.

STEP 4 - Application of Schauder fixed point theorem.

It remains to show that problem (3) possesses a solution. Let us recall that problem (1) with f depending only on x and $g=0$ has exactly one solution $u\in C(\bar{\Omega})$. Let $K:C(\bar{\Omega})\to C(\bar{\Omega})$ denote the solution operator, that is $u=Kf$. Then it is known that K is a linear compact operator in $C(\bar{\Omega})$ equipped with the usual maximum norm $\|\cdot\|$ (see also Appendix).

Let $F:C(\bar{\Omega})\to C(\bar{\Omega})$ denote the Niemytski operator associated with f^*, that is

$$F(u)\,(x)=f^*(x,u(x)) \qquad \text{for } u\in C(\bar{\Omega}),\ x\in\bar{\Omega}.$$

Then F is continuous and there is $M>0$ such that $\|F(u)\|\leqslant M$.

Finally observe that u is a solution of problem (3) if and only if u satisfies

$$u = KF(u).$$

A straightforward application of the Schauder fixed point theorem guarantees the existence of such solution. This completes the proof of the theorem. ■

REMARK - If u is a solution of (1), then it follows from standard regularity theory theorems that $u \in W^{2,p}_{\text{loc}}(\Omega)$ for all $p \in [1, \infty)$, although $\underline{u}$ and $\bar{u}$ do not need to possess such regularity.

3. - Appendix.

PROPOSITION - *Let Ω satisfy* (H 3) *and $f \in C(\bar{\Omega})$, then there exists a unique $u \in C(\bar{\Omega})$ satisfying*

i) $\int_\Omega (u(-\Delta\varphi) + f\varphi)\, dx = 0$ for every $\varphi \in C_0^\infty(\Omega)$,

ii) $u = 0$ on $\partial\Omega$.

Moreover the mapping $f \to u$ is compact in $C(\bar{\Omega})$.

Proof. The uniqueness is a direct consequence of the maximum principle for harmonic functions. For the existence we extend f by 0 outside of $\bar{\Omega}$ and set

$$w(x) = \int_{\mathbf{R}^N} \Gamma(x-y)\, f(y)\, dy,$$

the Newtonian potential of f, see [4, p. 50].

Then $w \in C^1(\bar{\Omega})$, see [4, Lemma 4.1], and the mapping $f \to w$ from $C(\bar{\Omega})$ in $C^1(\bar{\Omega})$ is continuous, where $C(\bar{\Omega})$ and $C^1(\bar{\Omega})$ are equipped with the usual norm. Since $C^1(\bar{\Omega})$ is compactly imbedded in $C(\bar{\Omega})$, the mapping $f \to w$ from $C(\bar{\Omega})$ into $C(\bar{\Omega})$ is compact.

Let $h \in C(\bar{\Omega})$ be the unique harmonic function satisfying $h = w$ on $\partial\Omega$ (here we use (H 3)). Then $u = w - h$ is a solution of i), ii). Since the mapping $w \to h$ from $C(\bar{\Omega})$ into $C(\bar{\Omega})$ is continuous we have that the mapping $f \to u$ from $C(\bar{\Omega})$ into $C(\bar{\Omega})$ is compact. ■

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