# ON SOME PAIRS OF PARTIAL TRIPLE SYSTEMS (*)

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SOMMARIO. - *Si costruiscono tutte le coppie di Sistemi Parziali di Terne di Steiner (PTS), disgiunte e mutuamente bilanciate (DMB), aventi* $m = 8$ *blocchi e* $M_4 = \emptyset$.

SUMMARY. - *We make all pairs of Partial Steiner Triple Systems (PTS), disjoint and mutually balanced (DMB), with* $m = 8$ *blocks and* $M_4 = \emptyset$.

## 1. - Introduction

A *Partial Triple System (PTS)* is a pair $(P, t)$, where $P$ is a finite set and $t$ is a collection of 3-subsets of $P$, called *blocks* or *triples*, such that every 2-subset of $P$ is contained in *at most* one block of $t$. Two *PTSs* $(P, t_1)$ and $(P, t_2)$ are said to be *disjoint* and *mutually balanced (DMB)* if $t_1 \cap t_2 = \emptyset$ and every 2-subset of $P$ is contained in a block of $t_1$ iff it is contained in a block of $t_2$.

A *Steiner Triple System (STS)* is a *PTS* $(S, t)$ such that every 2-subset of $S$ is contained in *exactly* one block of $t$. The number $|S|$ is the *order* of the triple system and if $|S| = v$ it is well-known that a necessary and sufficient condition for the existence of an *STS* of order $v$ $(STS(v))$ is $v \equiv 1$ or $3$ $(mod\ 6)$. Further, it is

$$t_v = |t| = \frac{v(v-1)}{6}.$$

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Let $(P,t)$ be a *PTS*. Using graph theoretic terminology, we will say that an element $x \in P$ has *degree* $d(x) = h$ if $x$ belongs to exactly $h$ blocks of $t$. Clearly $\sum_{x \in P} d(x) = 3 \cdot |t|$. We will call the *degree-set (DS)* of a *PTS* $(P,t)$ the $n$-uple $DS = [d(x), d(y), \dots]$, where $x, y, \dots$ are the elements of $P$. If there are $r_i$ elements of $P$ having degree $h_i$, for $i = 1, 2, \dots, s$, we will write $DS = [(h_1)_{r_1}, (h_2)_{r_2}, \dots, (h_s)_{r_s}]$, where $r_1 + r_2 + \dots + r_s = |P|$. If $r_i = 1$, for some $i$, then we will write $(h_i)_1 = h_i$. It is immediate that for any two *DMB PTS* $(P, t_1)$ and $(P, t_2)$ it is $|t_1| = |t_2|$.

A partial triple system $(P, \tau)$ is said to be embedded in a triple system $(S, t)$ provided that $P \subseteq S, \tau \subseteq t$.

Given a *PTS* $(P,t)$, we can define a *partial binary operation* $\circ$ on $P$ as follows

(1) $x \circ x = x$ for all $x \in P$, and

(2) if $x \neq y$, $x \circ y$ is defined and $x \circ y = z$ if and only if $\{x, y, z\}$ is a triple of $t$.

It is a routine matter to see that $\circ$ is well defined and that $(P, \circ)$ is *a partial idempotent commutative quasigroup*; i.e., $x \circ x = x$ for all $x \in P$, and whenever $x \circ y$ is defined then so is $y \circ x$, and furthermore $x \circ y = y \circ x$.

If $(P, t_1)$ and $(P, t_2)$ are two *DMB PTSs*, in what follows we will always indicate by $(P, \cdot)$ and $(P, :)$ the two partial idempotent commutative quasigroup associated with them, respectively.

Two triples $(P, t_1, t_2)$ and $(P, t'_1, t'_2)$ are *isomorphic* if $(P, t_1)$ is isomorphic to $(P, t'_1)$ [resp. $(P, t'_2)$] and $(P, t_2)$ is isomorphic to $(P, t'_2)$ [resp. $(P. t'_1)$].

In what follows $(P, t_1)$ and $(P, t_2)$ will be two *DMB PTSs* with $P = \{0, 1, 2, \dots, n\text{-}1\}$ and $|t_1| = |t_2| = m$. Further, for $i = 1, 2$, we will put $\mathbf{L}_i = \bigcup_{b \in t_i} P_2(b)$, where $P_2(b) = \{x : x \in b, |x| = 2\}$. If $x \in P$ then $\mathbf{K}(x, i) = \{b \in t_i : x \in b\}$, and $M_r = \{x \in P : d(x) = r\}$.

In [13] we constructed, to within isomorphism, all *DMB PTSs* with $m \leq 7$ blocks and *DMB PTSs* having $m = 8$ and at least an element of degree 4.

In this paper we complete the construction of *DMB PTS* with $m = 8$, determining all *DMB PTSs* with $m = 8$ blocks and $M_4 = \emptyset$.

These results are useful because, by similar constructions, it is possible to study the parameter $D(2, 3, v, k)$, which is the maximum number of $STS(v)$ such that any two of them have exactly $k$ blocks in common, there $k$ blocks being moreover in each of the $D(2, 3, v, k)$ systems (see J. Doyen [1]).

Further, it is known that an open problem is to determine (if they there exist) a pair of *DMB PQS* (partial quadruple systems) with $m = 17$ blocks. Since, if $(Q, q_1)$ and $(Q, q_2)$ are two *DMB PQSs* with 17 blocks, the maximum degree of an element of $Q$ is 8; it follows that, for $x \in Q$ and $t_i(x)=\{\{a,b,c\}:\{a,b,c,x\} \in q_i\}$ $(i = 1,2)$, $(Q - \{x\}, t_1(x))$ and $(Q - \{x\}, t_2(x))$ are two *DMB PTSs* with $m \leq 8$ blocks.

Therefore the constructions of all *DMB PTSs* with $m \leq 8$ blocks are useful to study the existence of *DMB PQSs* with $m = 17$ blocks.

## 2. - Properties and case $M_4 = M_3 = \varnothing$ for $m = 8$

Let $(P, t_1)$ and $(P, t_2)$ two *DMB PTSs*.

In [13] we proved the following properties:

Prop. 1).

*It is* $|P| \geq 6$, $m \geq 4$ *and* $d(x) \geq 2$ *for every* $x \in P$.

Prop. 2).

*If* $h = max\{d(x) : x \in P\}$, *then* $m \geq 2h$ *and* $n \geq 2h + 1$.

Prop. 3).

*If* $R \subseteq M_2$ *for some* $R \in t_1 \cup t_2$, *then* $m = 4$ *or* $m \geq 7$; *for* $m = 8$ *it is* $DS = [(4)_t, (2)_{n-t}]$, $t = 0,1,2$.

THEOREM 2.1 - *There exists exactly one pair of DMB PTSs with* $m = 8$ *and* $M_4 = M_3 = \varnothing$.

*Proof.* It is $DS = [(2)_{12}]$. From prop. 3) we have:

$t_1 =$

| | |
|---|---|
| 1, 2, 3 | 7, 8, 9 |
| 1, 4, 5 | 7, 0, $A$ |
| 2, 4, 6 | 8, 0, $B$ |
| 3, 5, 6 | 9, $A$, $B$ |

,

$t_2 =$

| | |
|---|---|
| 1, 2, 4 | 7, 8, 0 |
| 1, 3, 5 | 7, 9, $A$ |
| 2, 3, 6 | 8, 9, $B$ |
| 4, 5, 6 | 0, $A$, $B$ |

## 3. - *DMB PTSs* with $m = 8$ blocks and $M_4 = \varnothing, M_3 \neq \varnothing$

LEMMA 1. - *Let* $(P, t_1, t_2)$ *be a pair of DMB PTSs with* $M_4 = \varnothing, M_3 \neq \varnothing$ *and* $m = 8$ *blocks. It is* $|M_3| = 4$ *or* 6 *or* 8. *Further, if there exists a block* $R \in t_i$ $(i = 1$ *or* $2)$ *such that* $R \subseteq M_3$, *then* $|M_3| \geq 6$. *If* $R'$ *is another block belonging to* $t_i$ *and such that* $R' \subseteq M_3, R \cap R' = \varnothing$, *then* $|M_3| = 8$.

*Proof.* Observe that, from Prop. 3), a *DMB PTS* $(P, t_1, t_2)$ with

$m = 8$ and $M_3 \neq \emptyset$ does not contain blocks $R \subseteq M_2$. Further, if $M_3 \neq \emptyset$, since $\sum_{x \in P} d(x)$ is an even number (24), then $|M_3| = 2$ or 4 or 6 or 8. But $|M_3| = 2$ and $M_4 = \emptyset$ imply the existence of a block $R \subseteq M_2$. Therefore $|M_3| = 4$ or 6 or 8.

Let $R = \{1,2,3\} \subseteq M_3$ be a block of a *DMB PTSs* $(P, t_1, t_2)$ with $m = 8$ and $M_4 = \emptyset$. Suppose $R \in t_1$. If $|M_3| = 4$, necessarily $DS = [(3)_4, (2)_6]$.

Let $\{1,2,4\}, \{1,3,5\}, \{2,3,6\} \in t_2$. Since $|\{4,5,6\} \cap M_3| \leq 1$, we can suppose $\{4,5\} \subseteq M_2$. We have

$$\begin{array}{l} \{1,4,x_1\}, \{2,4,x_2\}, \{3,5,x_3\} \\ \{1,5,y_1\}, \{2,6,y_2\}, \{3,6,y_3\} \end{array} \in t_1,$$
$$\{,1\, x_1, y_1\}, \{2, x_2, y_2\}, \{3, x_3, y_3\} \in t_2,$$

further $\{4,5\} \subseteq M_2$ implies that

$$\{4, x_1, x_2\}, \{5, y_1, x_3\} \in t_2.$$

Since $3 \notin \{x_1, x_2\}$ and $2 \notin \{y_1, x_3\}$, it follows $6 \in M_3$ (otherwise, $\{6, y_2, y_3\} \in t_2$ and $m > 8$). But, since for every $b \in t_1 \cup t_2$ $b \not\subseteq M_2$, we have $6 \notin \{x_1, y_1\}$, and this implies $\{4,6\}, \{5,6\} \in \mathbf{L}_2$ with $\{4,6\}$, $\{5,6\} \notin \mathbf{L}_1$.

Therefore, $|M_3| \geq 6$.

Now, suppose that there exists another block $R' \in t_1$, such that $R' \cap R = \emptyset$ and $R' \subseteq M_3$. Further suppose $|M_3| < 8$, hence $|M_3| = 6$, i.e. $DS = [(3)_6, (2)_3]$.

Let $R' = \{4,5,6\}$. It is immediate to see that $\{1,4\}, \{1,5\}, \{2,4\}$, $\{2,6\}, \{3,5\}, \{3,6\}$ are contained in exactly six distinct blocks of $t_1$, respectively.

Let $\{1,4,7\}, \{1,5,8\} \in t_1$. In $t_2$ there are six blocks containing $\{1,2\}, \{1,3\}, \{2,3\}, \{4,5\}, \{4,6\}, \{5,6\}$ and other two blocks $b_1, b_2$. Since $b_i \not\subseteq M_2$, for $i = 1,2$, if $b_1 = \{x, y, z\}$, we can suppose $x = 1$. Further, it is $y \in \{4,5\}$ (otherwise, $\{y, z\} \in \mathbf{L}_2 - \mathbf{L}_1$). Let $y = 4$ (or, likewise, $y = 5$). Considering that $\{4,5,4:5\} \neq b_1$ (i.e. $4:5 \neq 1$), necessarily $\{1,3,5\} \in t_2$, and further $\{1,4,8\}, \{1,2,7\} \in t_2$. It follows $\{2,4,8\} \in t_1$ with $8 \in M_2$, hence $\{2,5,8\} \in t_2$, with $\{2,5\} \in \mathbf{L}_2 - \mathbf{L}_1$.

Therefore, necessarily it is $|M_3| \geq 8$, i.e. $|M_3| = 8$ and $P = M_3$. △

THEOREM 3.1 - *There exists exactly one pair of DMB PTSs with* $m = 8$ *blocks and* $|M_3| = 8$.

*Proof.* If $(P, t_1, t_2)$ is a pair of *DMB PTSs* with $|M_3| = 8$, then $P = M_3$ and $DS = [(3)_8]$.

If $R = \{1,2,3\} \in t_1$, necessarily there exist a block $R' \in t_1$ such that $R \cap R' = \emptyset$. Let $R' = \{4,5,6\}$. It follows that

$t_1 =$

| | |
|---|---|
| 1, 2, 3 | 2, 4, 8 |
| 4, 5, 6 | 2, 6, 7 |
| 1, 4, 7 | 3, 5, 7 |
| 1, 5, 8 | 3, 6, 8 |

Since $\{1,2\},\{1,3\},\{2,3\},\{4,5\},\{4,6\},\{5,6\}$ are contained in six distinct blocks of $t_2$, necessarily $\{4,5,6\} \cap \{1:2, 1:3, 2:3\} \neq \emptyset$ and $\{1,2,3\} \cap \{4:5, 4:6, 5:6\} \neq \emptyset$. Further

$$1:2 \in \{4,7,8\},\ 1:3 \in \{5,7,8\},\ 2:3 \in \{6,7,8\},$$

with $\{7,8\} \subseteq \{1:2, 1:3, 2:3\}$ and $\{7,8\} \subseteq \{4:5, 4:6, 5:6\}$. Suppose $\{1,2,4\} \in t_2$ (or, likewise, $\{1,3,5\} \in t_2$ or $\{2,3,6\} \in t_2$). It follows $\{3,5,6\} \in t_2$, hence

$t_2 =$

| | |
|---|---|
| 1, 2, 4 | 2, 3, 7 |
| 3, 5, 6 | 2, 6, 8 |
| 1, 3, 8 | 4, 5, 8 |
| 1, 5, 7 | 4, 6, 7 |

We obtain so two *DMB PTSs* with $DS = [(3)_8]$. $\triangle$

THEOREM 3.2 - *There exists exactly one pair of DMB PTSs with* $m = 8$, $|M_3| = 4$ *and* $M_4 = \emptyset$.

*Proof.* Let $M_3 = \{1,2,3,4\}$. Necessarily it is $DS = [(3)_4, (2)_6]$. Since $|M_3| = 4$, from Lemma 1, it follows that $|b \cap M_i| \leq 2$ for every $b \in t_1 \cup t_2$ and $i = 2,3$. It is easy to verify that there are in $t_i$ $(i = 1,2)$ four blocks $b_{ij}$ $(j = 1,2,3,4)$ containing exactly one element $x_j$ of $M_3$ and other four blocks $b_{ij}$ $(j = 5,6,7,8)$ containing two elements of $M_3$ and one element of $M_2$. It is easy to see that (for $i = 1$ or 2) $w = |\bigcup_{j=1}^{4} (M_3 \cap b_{ij})| \geq 3$. Consider $w = 3$, with $x_1 = x_2$, and let $b_{11} = \{x_1, a, b\}$, $b_{12} = \{x_1, c, d\}$. We can suppose $\{x_1, b, c\}, \{x_3, a, b\}, \{x_4, c, d\} \in t_2$. Since $c \in M_2$, it follows $\{c, b, x_4\} \in t_1$, with $\{b, x_4\} \in \mathbf{L}_2$, $x_3 \neq x_4$, hence $d(b) > 2$. Therefore, it is $w = 4$, with $x_i \neq x_j$ for every $i, j, i \neq j$.

Let $x_i = i$, and let $\{1,2\},\{1,3\},\{2,4\},\{3,4\} \in \mathbf{L}_1 \cap \mathbf{L}_2$.

Suppose $\{1,2,0\},\{1,3,9\} \in t_1$. Observe that for every pair $\{x,y\} \subseteq b_{1j} \cap M_2$, for some $j = 1,2,3,4$, there exists a $j' \in \{1,2,3,4\}$ such that $\{x,y\} \subseteq b_{2,j'}$. Therefore we can suppose that $\{1,3,0\}$, $\{1,2,8\} \in t_2$. Hence $\{3,4,9\} \in t_2$, $\{2,4,8\} \in t_1$, and (since $2,3 \in M_3$) $\{2,4,7\} \in t_2$, $\{3,4,7\} \in t_1$. It follows

$$\begin{matrix}\{1,8,5\}, \{3,0,6\} \\ \{2,7,6\}, \{4,9,5\}\end{matrix} \in t_1 \quad \text{and} \quad \begin{matrix}\{1,9,5\}, \{3,7,6\} \\ \{2,0,6\}, \{4,8,5\}\end{matrix} \in t_2.$$

We have so two *DMB PTSs* with $DS = [(3)_4, (2)_6]$. $\triangle$

THEOREM 3.3 - *There exists exactly two pairs of DMB PTSs with* $m = 8, |M_3| = 6, M_4 = \emptyset$. *They have* $DS = [(3)_6, (2)_3]$.

*Proof.* It is $DS = [(3)_6, (2)_3]$.

Observe that there exist in $t_i\,(i = 1$ or $2)$ at least two blocks $R, R' \subseteq M_3$. From Lemma 1, it is $R \cap R' \neq \emptyset$.

Suppose

$$\begin{matrix} R = \{1,2,3\} \\ R' = \{1,4,5\} \in t_1 \\ \{1,6,7\} \end{matrix} \quad , \quad \begin{matrix} \{1,2,4\} \\ \{1,3,7\} \in t_2 \\ \{1,5,6\} \end{matrix}$$

Further, it is $\{6,7\} \cap M_2 \neq \emptyset$. Let $7 \in M_2$ (or, likewise, $6 \in M_2$). If $\{6,7,x\} \in t_2$, we have $\{3,7,x\} \in t_1$, with $x = 4$ or 8.

First, suppose $x = 4$. It follows $\{3,4,5\} \in t_2$, and $\{2,3,8\} \in t_2$, necessarily. Hence

$t_1 =$

| | |
|---|---|
| 1, 2, 3 | 3, 5, 8 |
| 1, 4, 5 | 2, 4, 6 |
| 1, 6, 7 | 5, 6, 9 |
| 3, 7, 4 | 2, 8, 9 |

, $t_2 =$

| | |
|---|---|
| 1, 2, 4 | 3, 4, 5 |
| 1, 3, 7 | 2, 3, 8 |
| 1, 5, 6 | 5, 8, 9 |
| 6, 7, 4 | 2, 6, 9 |

We obtain so two *DMB PTSs* with $DS = [(3)_6, (2)_3]$.

Now, suppose $x = 8$. It is $2:3 \neq 8$ (otherwise $3 \notin M_3$) and $4:5 \in \{8,9\}$ (observe that $\{3,4,5\} \in t_2$ implies $\{3,4\}, \{3,5\} \in \mathbf{L}_1$, hence $3 \notin M_3$). Prove that $4:5 \neq 8$.

If $\{4,5,8\} \in t_2$, then $8 \in M_3$, hence $6 \in M_2$. It follows $\{5,6,8\} \in t_1$, with $5 \notin M_3$.

Necessarily, $\{4,5,9\} \in t_2$. If $\{2,4,y\} \in t_1$, it is $y \in \{6,8,9\}$. Since for $y = 9$ it is $4 \notin M_3$ and for $y = 6$ we have $\{5,6,8\} \in t_1$ (necessarily), hence $\{2,4,6\} \in t_2$, with $\{2,4,6\} \in t_1 \cap t_2$, it follows $y = 8$. It is $8 \in M_3$, $6 \in M_2$. We have

$t_1 =$

| | |
|---|---|
| 1, 2, 3 | 2, 4, 8 |
| 1, 4, 5 | 5, 6, 8 |
| 1, 6, 7 | 2, 5, 9 |
| 3, 7, 8 | 3, 4, 9 |

, $t_2 =$

| | |
|---|---|
| 1, 2, 4 | 4, 5, 9 |
| 1, 3, 7 | 3, 4, 8 |
| 1, 5, 6 | 2, 3, 9 |
| 6, 7, 8 | 2, 5, 8 |

which are two *DMB PTSs* with $DS = [(3)_6, (2)_3]$. $\triangle$

## 4. - Appendix

All *DMB PTSs* having $m \leq 8$ blocks (obtained in [13] and in the previous sections) are the following:

$m = 4$

$n = 6$ $DS = [(2)_6]$

| | |
|---|---|
| 1, 2, 3 | 1, 2, 4 |
| 1, 4, 5 | 1, 3, 5 |
| 2, 4, 6 | 2, 3, 6 |
| 3, 5, 6 | 4, 5, 6 |

$m = 6$

$n = 7$ $DS = [(3)_4, (2)_3]$

| | |
|---|---|
| 1, 4, 5 | 1, 4, 6 |
| 1, 6, 7 | 1, 5, 7 |
| 2, 4, 6 | 2, 4, 7 |
| 2, 5, 7 | 2, 5, 6 |
| 3, 4, 7 | 3, 4, 5 |
| 3, 5, 6 | 3, 6, 7 |

$n = 8$ $DS = [(3)_2, (2)_6]$

| | |
|---|---|
| 1, 3, 4 | 1, 3, 5 |
| 1, 5, 6 | 1, 4, 7 |
| 1, 7, 8 | 1, 6, 8 |
| 2, 3, 5 | 2, 3, 4 |
| 2, 4, 7 | 2, 5, 6 |
| 2, 6, 8 | 2, 7, 8 |

$m = 7$

$n = 7$ $DS = [(3)_7]$

| | |
|---|---|
| 1, 2, 3 | 1, 2, 4 |
| 1, 4, 7 | 1, 3, 5 |
| 1, 5, 6 | 2, 3, 6 |
| 2, 4, 5 | 1, 6, 7 |
| 2, 6, 7 | 2, 5, 7 |
| 3, 5, 7 | 3, 4, 7 |
| 3, 4, 6 | 4, 5, 6 |

$n = 9$ $DS = [(3)_3, (2)_6]$

| | |
|---|---|
| 1, 2, 3 | 1, 2, 4 |
| 1, 4, 5 | 1, 3, 5 |
| 2, 4, 6 | 2, 3, 6 |
| 3, 5, 6 | 4, 5, 7 |
| 4, 7, 8 | 4, 6, 8 |
| 5, 7, 9 | 5, 6, 9 |
| 6, 8, 9 | 7, 8, 9 |

$m = 8$

$n = 8$ $DS = [(3)_8]$

| | |
|---|---|
| 1, 2, 3 | 1, 2, 4 |
| 4, 5, 6 | 3, 5, 6 |
| 1, 4, 7 | 1, 3, 8 |
| 1, 5, 8 | 1, 5, 7 |
| 2, 4, 8 | 2, 3, 7 |
| 2, 6, 7 | 2, 6, 8 |
| 3, 5, 7 | 4, 5, 8 |
| 3, 6, 8 | 4, 6, 7 |

$m = 8$

$n = 9$

$DS = [(4)_2, (3)_2, (2)_5]$

| | |
|---|---|
| 1, 2, 3 | 1, 2, 4 |
| 1, 4, 5 | 1, 7, 9 |
| 1, 6, 7 | 1, 3, 6 |
| 1, 8, 9 | 1, 5, 8 |
| 2, 7, 9 | 2, 6, 7 |
| 2, 5, 8 | 2, 8, 9 |
| 2, 4, 6 | 2, 3, 5 |
| 3, 5, 6 | 4, 5, 6 |

$DS = [4, (3)_4, (2)_4]$

| | |
|---|---|
| 1, 2, 3 | 1, 2, 4 |
| 1, 4, 5 | 1, 3, 5 |
| 1, 6, 7 | 1, 6, 8 |
| 1, 8, 9 | 1, 7, 9 |
| 2, 4, 6 | 2, 3, 6 |
| 3, 5, 9 | 4, 5, 9 |
| 3, 6, 8 | 4, 6, 7 |
| 4, 7, 9 | 3, 8, 9 |

$DS = [(3)_6, (2)_3]$

| | | | |
|---|---|---|---|
| 1, 2, 4 | 1, 2, 3 | 1, 2, 3 | 1, 2, 4 |
| 1, 3, 7 | 1, 4, 5 | 1, 4, 5 | 1, 3, 7 |
| 1, 5, 6 | 1, 6, 7 | 1, 6, 7 | 1, 5, 6 |
| 4, 6, 7 | 3, 4, 7 | 3, 7, 8 | 6, 7, 8 |
| 3, 4, 5 | 3, 5, 8 | 2, 4, 8 | 4, 5, 9 |
| 2, 3, 8 | 2, 4, 6 | 5, 6, 8 | 3, 4, 8 |
| 5, 8, 9 | 5, 6, 9 | 2, 5, 9 | 2, 3, 9 |
| 2, 6, 9 | 2, 8, 9 | 3, 4, 9 | 2, 5, 8 |

$n = 10$

$DS = [(4)_2, (2)_8]$

| | | | | | |
|---|---|---|---|---|---|
| 1, 2, 3 | 1, 2, 4 | 1, 2, 3 | 1, 2, 4 | 1, 2, 3 | 1, 2, 4 |
| 1, 4, 5 | 1, 7, 9 | 1, 4, 5 | 1, 7, 9 | 1, 4, 5 | 1, 7, 9 |
| 1, 6, 7 | 1, 3, 5 | 1, 6, 7 | 1, 3, 6 | 1, 6, 7 | 1, 3, 5 |
| 1, 8, 9 | 1, 6, 8 | 1, 8, 9 | 1, 5, 8 | 1, 8, 9 | 1, 6, 8 |
| 0, 2, 4 | 0, 2, 3 | 0, 2, 4 | 0, 2, 3 | 2, 7, 9 | 2, 6, 7 |
| 0, 7, 9 | 0, 4, 5 | 0, 7, 9 | 0, 4, 5 | 2, 6, 8 | 2, 8, 9 |
| 0, 3, 5 | 0, 6, 7 | 0, 3, 6 | 0, 6, 7 | 2, 4, 0 | 2, 3, 0 |
| 0, 6, 8 | 0, 8, 9 | 0, 5, 8 | 0, 8, 9 | 3, 5, 0 | 4, 5, 0 |

$DS = [(3)_4, (2)_6]$

| | |
|---|---|
| 1, 2, 0 | 1, 3, 0 |
| 1, 3, 9 | 1, 2, 8 |
| 2, 4, 8 | 2, 4, 7 |
| 3, 4, 7 | 3, 4, 9 |
| 1, 5, 8 | 1, 5, 9 |
| 2, 6, 7 | 2, 6, 0 |
| 3, 6, 0 | 3, 6, 7 |
| 4, 5, 9 | 4, 5, 8 |

$n = 11$ $DS = [4, (2)_{10}]$

| | |
|---|---|
| 1, 2, 3 | 1, 2, 4 |
| 1, 4, 5 | 1, 3, 5 |
| 1, 6, 7 | 1, 6, 8 |
| 1, 8, 9 | 1, 7, 9 |
| 2, 4, 0 | 2, 3, 0 |
| 3, 5, 0 | 4, 5, 0 |
| 6, 8, *A* | 6, 7, *A* |
| 7, 9, *A* | 8, 9, *A* |

$n = 12$ $DS = [(2)_{12}]$

| | |
|---|---|
| 1, 2, 3 | 1, 2, 4 |
| 1, 4, 5 | 1, 3, 5 |
| 2, 4, 6 | 2, 3, 6 |
| 3, 5, 6 | 4, 5, 6 |
| 7, 8, 9 | 7, 8, 0 |
| 7, 0, *A* | 7, 9, *A* |
| 8, 0, *B* | 8, 9, *B* |
| 9, *A*, *B* | 0, *A*, *B* |

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