# TREE AUTOMATA AND ENRICHED CATEGORY THEORY (*)

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SOMMARIO. - *Si dimostra che un singolo automa ad albero può essere considerato come una categoria basata su un'opportuna bicategoria costruita a partire dagli alberi di input. In questo contesto si estende il teorema di aggiunzione locale fra realizzazione e comportamento.*

SUMMARY. - *It is shown that tree automata can be described as categories enriched on a suitable base bicategory built up with input trees. In this setting the known theorem relating realization and behaviour by a local adjunction still holds true.*

## Introduction

It is known that (nondeterministic) automata can be described as categories enriched on a suitable monoidal biclosed category $\mathcal{V}$, constructed out of the inputs. The $\mathcal{V}$-objects are to be thought of as the truth values for a generalized logic of automata in the sense of Lawvere [11] and they are actually languages, i.e. subsets of a monoid. The basics of the theory and some applications are given in [1], [5] and [6]. An extension to the abstract theory of cofibrations in $\mathcal{V}$-cat (in the sense of Street [14]) is provided in [13], where it is proved that the canonical realization described in [6] holds in that abstract context.

In this paper we wish to generalize the $\mathcal{V}$-categorical setting to encompass the theory of (non deterministic) *tree automata.* To

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do this, we are led to consider a base bicategory built up again with the inputs, which now are trees.

Deferring to another paper [12] the extension to the abstract theory of cofibrations in $B$-cat, we will develop here just the basics, as a first step towards a $B$-categorical treatment of tree-automata. Namely, we prove a behaviour-realization «weak» adjointness, and we characterize deterministic automata, aiming also to stress that even the first properties of automata have a categorical nature.

First we recall the basics of the classical theory of tree automata, following the account of Thatcher and Wright [16], and then we describe automata as $B$-categories (general references on $B$-categories are [2], [4], [15] and [3]).

We conclude proving the results mentioned above.

## 1. - Basic notions on tree automata

*$\Sigma$-automata.* Let $\mathcal{S} = (\Sigma, \sigma)$ be a similarity type, i.e. a set $\Sigma$ of operation symbols and an «arity» map $\sigma : \Sigma \to N$ (non negative integers). We will denote $\Sigma_n = \sigma^{-1}(n)$ the set of symbols of arity $n$ ($\Sigma_0$ is then the set of constant symbols). Following Thatcher and Wright [16], a *deterministic $\Sigma$-automaton* is defined essentially as a $\Sigma$-algebra, i.e. a set $A$ (the *carrier* of the algebra) plus a map $\alpha$ assigning to each symbol $g$ of $\Sigma$ an operation $\alpha_g : A^{\sigma(g)} \to A$. In particular, nullary operations $\alpha_\lambda, \lambda \in \Sigma_0$ are called *constants*. An automaton $\mathcal{A} = (A, \alpha, F)$ has set of *states* $A$, for each input symbol $g \in \Sigma_n$ has the *transition map* $\alpha_g$, and has the set of *final states* $F \subset A$. We call an *initial state* the map $\alpha_\lambda$, relative to each $\lambda \in \Sigma_0$.

The inputs of an automaton are given by the set $T_\Sigma$ of *terms* (or *trees*) determined by the type $\Sigma$, i.e. the least subset of $\Sigma^*$ (the free monoid generated by $\Sigma$) satisfying:

a) $\Sigma_0 \subset T_\Sigma$,

b) if $f \in \Sigma_n$ and $t_1, \ldots, t_n \in T_\Sigma$, then $f(t_1, \ldots, t_n) \in T_\Sigma$.

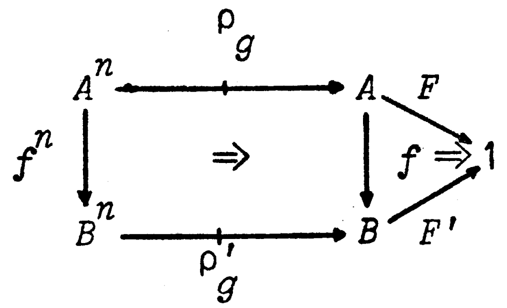
*Remark.* Notice that in a «relational algebra» $(R, \rho)$ the transition $\rho_g : A^{\sigma(g)} \twoheadrightarrow A$ is a relation. So we get the notion of a *non deterministic automaton.* Together with the transition relation $\rho_g$ relative to the input $g$, we have, if $\lambda \in \Sigma_0$ the *set* $\rho_\lambda$ of initial states. Clearly $\alpha$ and $\rho$ can be inductively extended from $\Sigma$ to $T_\Sigma$.

*Reachability and the behaviour functor.* The *reachable states* (definable elements of $A$) are given for non deterministic (deterministic) automata as the $\rho$-images ($\alpha$-images) of the terms.

The *behaviour* of an automaton $\mathcal{A} = (A, \rho, F)$ is the set of terms *recognized* by $\mathcal{A}$, i.e.

$$\beta\mathcal{A} = \{t \in T_\Sigma \text{ such that } \rho(t) \cap F \neq \Phi\}.$$

The notion of a morphism between automata of the same type $\mathfrak{A} = (A, \beta, F) \to \mathfrak{B} = (B, \rho', F')$ is obtained by onsidering a map $f: A \to B$ which respects transition relations and final states, i.e. such that for each $g \in \Sigma_n$ it holds:




where the 2-cells are inclusions and the left hand square has to be commutative in the deterministic case.

Notice that this notion of morphism allows the definition of a *behaviour functor* from the category of Σ-automata to the poset of the subsets of $T_{\Sigma}$.

*T-algebras.* With this said, we are now able to give an «invariant presentation» in terms of functorial semantics (Lawvere [10]), recalling that the «dynamics» (i.e. forgetting terminal states) of an automaton is just an algebra.

Following Eilenberg and Wright [8], let us recall that an algebraic (one sorted) theory $T$ is a category whose objects are finite sets $[n] = \{1, \dots, n\}$, $n = 0, 1, \dots$, and which admits the category of finite sets as a subcategory.

Notice that $[0] = \Phi$ is the initial object and that $[m]$ is the $m$-fold coproduct of $[1]$, which means that an arrow $a : [m] \to [n]$ is equivalent to an $m$-ple of «injections» $a_i : [1] \to [n]$.

A *T-algebra* is then a product preserving functor $A : T^{op} \to \mathcal{S}$ and a morphism of $T$-algebras is just a natural transformation.

The carrier of a $T$-algebra is $A[1]$ and to each arrow $f : [1] \to [n]$ corresponds an $n$-ary operation of the algebra. Again arrows $g : [n] \to [m]$ correspond to $n$-tuples of $m$-ary operations $A^m \to A$.

The *initial algebra* (i.e. the initial object of the category of $T$-algebras) has carrier $\hom_T([1], [0])$, while the operations are as follows: if $f : [1] \to [n]$ is an arrow of $T$, it becomes the operation that assigns to each $n$-uple $a_i : [1] \to [0]$ the arrow $f.\langle a_1, \dots, a_n \rangle : [1] \to [0]$, where $\langle a_1, \dots, a_n \rangle : [n] \to [0]$ is uniquely determined by the universal property of the coproduct. We call the elements of $\hom_T([1], [0])$ still *terms* or *trees*.

Things beeing so, it looks quite obvious to call a *deterministic automaton* the pair $(A,F)$ where $A: T^{op} \to \mathcal{S}$ is a $T$-algebra and $F \subset A[1]$ is the set of *final states*.

*Relational T-algebras.* As for relational (or non deterministic algebras) let us now diverge from [8] and use the quasi products of the bicategory Rel $\mathcal{S}$ of sets and relations (for lax notions of limit, see for instance [7] or [9]). Hence a relational $T$-algebra can be defined as a lax functor $A: T^{op} \to \mathrm{Rel}\,\mathcal{S}$, sending coproducts of $T$ into quasi-products of Rel $\mathcal{S}$.

A non deterministic automaton will be still a pair $(A,F)$, where $A$ is a relational $T$-algebra and $F \subset A[1]$ is the set of final states.

Notice that in order to get the category of non deterministic $T$-automata, one has to take quasi-natural transformations ([7], [9]) as morphisms.

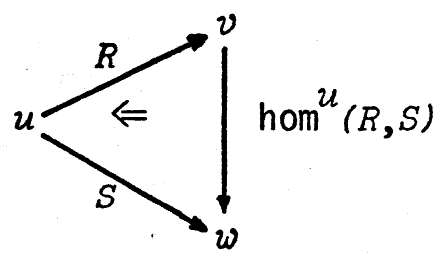
*Remark.* Let us finally observe that the notion of non deterministic $T$-automaton aims to describe the computation of «many valued» operations: in this context $T$ does not represent at all a «theory» of «relations».

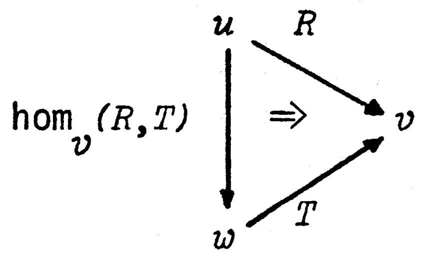
## 2. - *T*-algebras as enriched categories

*The base bicategory.* Let us now introduce a bicategory $\mathcal{B}(T)$ constructed out of the theory $T$, over which automata will be enriched. In the case of sequential automata, $\mathcal{B}(T)$ specializes to the biclosed category $\mathfrak{X}$ described in [1], [5] and [6].

The construction $\mathcal{B}(T)$ basically consists of taking the subsets of arrows with the same domain and codomain. Namely, if $T$ is a small category $\mathcal{B}(T)$ has the same objects as $T$, the 1-cells from $u$ to $v$ in $\mathcal{B}(T)$ are the subsets of $\hom_T(u,v)$ and 2-cells are inclusions. Composition of 1-cells and identities are the obvious one.

The bicategory $\mathcal{B}(T)$ is locally complete and cocomplete. Moreover it is a biclosed bicategory, with $\hom^u(R,S) = \{h: v \to w$ in $T$, such that $R.h \subset S\}$ and $\hom_u(R,T) = \{k: w \to u$ in $T$, such that $k.R \subset T\}$ as in the diagram below:



There is a natural embedding $T \hookrightarrow \mathfrak{B}(T)$ sending an arrow $f: u \to v$ into itself seen as an «atom» of $\mathfrak{B}(T)(u, v)$.

By $\mathfrak{B}(T)$-cat we mean, as usual, the bicategory of $\mathfrak{B}(T)$-enriched categories (see [2] and [4], for definitions).

*Reachability.* If $X$ is a $\mathfrak{B}(T)$-category with just one object $x_0$ over $[0]$, we will call *reachable* an $X$-object a over $[n]$ if it is the cotensor (see [2] and [15]) of $x_0$ along an atom.

A skeletal $\mathfrak{B}(T)$-category $X$ will be called *reachable* if all the objects are reachable.

Notice that a reachable $\mathfrak{B}(T)$-category $X$ satisfies the following condition:

$$X_{[n]} = X^{n}_{[1]}$$

where $X_{[k]}$ is the set of $X$-objects over $[k]$.

THEOREM 1 - *The category of non deterministic reachable T-algebras is isomorphic to the category of reachable $\mathfrak{B}(T)$-categories.*

*Proof.* Let $A: T^{op} \to \mathrm{Rel}\,\mathcal{S}$ be a $T$-algebra with carrier $A = A[1]$. We construct a $\mathfrak{B}(T)$-category $X$ taking as objects over $[n]$ the elements of $A[n] = A^n$, i.e. an $X$-object over $[n]$ is identified with an $n$-tuple $(a_1, \ldots, a_n), a_i \in A$. So $X_{[1]} = A[1]$ and for each $k$, we have $X_{[k]} = A[k] = A^k$. For each pair $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_m)$, the hom $X(a, b)$ is given by:

$$X(a, b) = \{f: [n] \to [m] \text{ such that } (b, a) \in A(f)\}.$$

On the other hand, let $X$ be a reachable $\mathfrak{B}(T)$-category. We can recover a $T$-algebra $A$ by taking as carrier $A = X_{[1]}$. If $f: [n] \to$ $\to [m]$ is an arrow of $T$, then $A(f)$ is defined as a relation $A^m \to A^n$ in the following way:

$$A(f) = \{(a, b) \text{ such that } f \in X(b, a)\}.$$

The bijection on objects being so established, a straightforward calculation shows the correspondence between quasi-natural transformations and $\mathfrak{B}(T)$-functors.

q.e.d.

Let us characterize the reachable categories which correspond, in the bijection above, to deterministic $T$-algebras.

THEOREM 2 - *The reachable $\mathfrak{B}(T)$-category $X$ corresponds to a deterministic reachable T-algebra $A$ if and only if:*

i) *the underlying category of $X$ is discrete;*

ii) *$X$ is cotensored along the atoms.*

*Proof.* Let $A : T^{op} \to \mathcal{S}$ be a deterministic $T$-algebra. To check that the corresponding category $X$ has a discrete one underlying it is sufficient to consider a pair $(a,b)$ of objects over $[n]$. If it holds $1_{[n]} \leqslant X(a,b)$ we have $A(1_{[n]})(a) = b$ and, since $A$ is a functor, $a=b$, i.e. the underlying of $X$ is discrete. Let now $x$ be an object over $[n]$ and $h : [m] \to [n]$ be an arrow of $T$. The cotensor $x \pitchfork h$ is then just the image $A(h)(x)$, where $A(h) : A[n] \to A[m]$. It is easy to check the universal property of the cotensor. The other way round, if a reachable category $X$, which satisfies i) and ii), is considered as a $T$-algebra $A : T^{op} \to \mathrm{Rel}\,\mathcal{S}$ it turns out to be deterministic. Indeed, identities are preserved by discreteness and for each $h : [m] \to [n]$ we are able to define a set map $A(h) : A[n] \to A[m]$ by setting $A(h)(x) = x \pitchfork h$ (which is unique by discreteness).

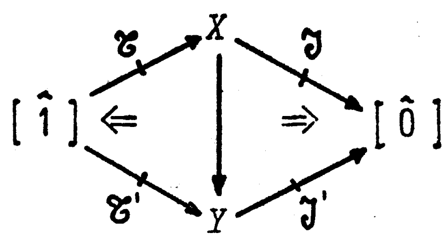
q. e. d.

## 3. - The realization

*T-automata.* Having thus characterized the $T$-algebras in the enriched context, we are now able to introduce non deterministic $T$-automata, taking initial and final states into account.

*Definition.* A (non deterministic) *T-automaton* is a reachable $\mathcal{B}(T)$-category with a final module $\mathfrak{T} : [\hat{1}] \longrightarrow\!\!\!\!\!\!\mapsto X$ and an initial module $\mathfrak{I} : X \longrightarrow\!\!\!\!\!\!\mapsto [\hat{0}]$ (where $[\hat{n}]$ denotes the trivial category with one object over $[n]$).

A morphism of $T$-automata $(X, \mathfrak{I}, \mathfrak{T}) \to (Y, \mathfrak{I}', \mathfrak{T}')$ is a $\mathcal{B}(T)$-functor $X \to Y$ satisfying the inclusions of the diagram below:




Given a tree automaton, i.e. a $T$-algebra $A$ with a set of final states $F$, initial and final modules in the definition above are given as follows: if $b = (b_1, \ldots, b_n)$ is an $X$-object over $[n]$, then $\mathfrak{I}(b) = X(b, x_0)$, and $\mathfrak{T}(b) = \{g \in \hom_T([1], [n])$ such that $\exists a \in F \subset X_1$ and $(b,a) \in A(g)\}$.

The module $\mathfrak{I}$ thus provides $n$-tuples of trees (or terms) whereas $\mathfrak{T}$ picks out operations which are «successful» if performed on those trees. Their composition $\mathcal{T} \,.\, \mathfrak{J}$ is a set of terms of $\mathrm{hom}_T([1],[0])$ as we will see in the following:

*Remark.* Notice that the above definition exhibits automata as «gamuts» or cofibrations in the sense of Street [14] and that the morphisms between automata are defined to be just morphisms of gamuts.

*Remark.* Observe that it is possible to extend Theorem 1 to (respectively) the category of non deterministic automata and the category of «reachable» gamuts from $[\hat{1}]$ to $[\hat{0}]$, which we will call *T-Aut.*

, *The behaviour.* At this stage we are able to define the behaviour of a non deterministic automaton as the composite $\mathcal{T} . \mathfrak{J}$, which turns out obviously to be a functor $\beta$ from the category of $T$-automata to $\mathfrak{B}(T)$ ([1], [0]). As we observed earlier, the behaviour is the set of terms computed by the operations of the automaton which are recognizable, i.e. which belong to the set of final states $F$.

We can carry the analogy with sequential automata further on: it is possible to define (as in [6] and [13]) a «realization» functor which has a quasi-universal property with respect to the behaviour. We have in fact:

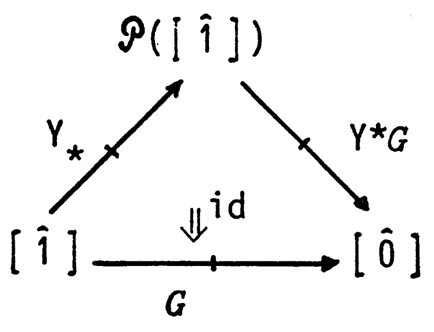
THEOREM 3 - *The functor* $\beta : T\text{-Aut} \rightarrow \mathfrak{B}(T)$ ([1], [0]) *has a local right adjoint* $\rho$.

This means essentially that $\rho$ is a functor from $\mathfrak{B}(T)$ ([1], [0]) to $T$-Aut which gives rise to an adjunction $R \dashv B$:

$$T\text{-Aut}\,(X, \rho G) \underset{R}{\overset{B}{\rightleftarrows}} \mathfrak{B}(T)\,([1],[0])\,(\beta X, G)$$

for each pair of objects $X$ in $T$-Aut and $G$ in $\mathfrak{B}(T)$ ([1], [0]) with suitable naturality conditions (see for instance [6] or [9]). Notice anyway that the right hand member above is just a set.

As for the proof it is sufficient to refer to [12] and to define $\rho$ in the following way: if $G$ is in $\mathfrak{B}(T)$ ([1], [0]) then $\rho(G)$ is the $T$-automaton




where $\mathcal{S}([\hat{1}])$ is the category whose objects over $[n]$ are modules

$[\hat{1}] \longrightarrow\!\!\!\!\!+\; [\hat{n}]$ and the hom is given by the right Kan extension (see [3] and [15]).

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