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On the Solvability Conditions for a Linearized Cahn-Hilliard Equation

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Abstract. *We derive solvability conditions in $H^4(\mathbb{R}^3)$ for a fourth order partial differential equation which is the linearized Cahn-Hilliard problem using the results obtained for a Schrödinger type operator without Fredholm property in our preceding work [17].*

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1. Introduction

Consider a binary mixture and denote by c its composition, that is the fraction of one of its two components. Then the evolution of the composition is described by the Cahn-Hilliard equation (see, e.g., [1, 11]):

$$\frac{\partial c}{\partial t} = M\Delta\left(\frac{d\phi}{dc} - K\Delta c\right), \tag{1}$$

where M and K are some constants and ϕ is the free energy density. From the Flory-Huggins solution theory it follows that

$$\frac{d\phi}{dc} = k_1 + k_2 c + k_3 T(\ln c - \ln(1-c)),$$

k_i, $i = 1, 2, 3$, are some thermodynamical constants and T is the temperature (see, e.g., [8]). We note that the constants k_1, k_2 and K characterize interaction of components in the binary medium. They can be positive or negative. If the components are identical, that is the medium is not in fact binary, they are equal to zero. In this case, equation (1) is reduced to the diffusion equation.

Denote the right-hand side of the last equality by $F(T, c)$. If the variation of the composition is small, then we can linearize it around some constant $c = c_0$:

$$F(T, c) \approx k_1 + k_2 c + k_3 T(\alpha + \beta(c - c_0)).$$

where $\alpha = \ln(c_0/(1-c_0))$ and $\beta = 1/(c_0(1-c_0))$. Substituting the expression for $F(T,c)$ into (1), we obtain

$$\frac{\partial u}{\partial t} = M\Delta\left(k_1 + k_2 c_0 + \alpha k_3 T + (k_2 + k_3\beta T)u - K\Delta u\right), \tag{2}$$

where $u = c - c_0$.

The existence, stability and some properties of solutions of the Cahn-Hilliard equation have been studied extensively in recent years (see, e.g., [3, 6, 11]). In this work we investigate the existence of stationary solutions of equation (2), which we write as

$$\Delta(\Delta u + V(x)u + au) = f(x), \tag{3}$$

where

$$V(x) = -\frac{k_3\beta T_0(x)}{K}, \quad f(x) = \frac{\alpha k_3}{K}\Delta T_0(x) + g(x), \quad a = -\frac{k_2 + k_3\beta T_\infty}{K}.$$

Here we use the representation $T(x) = T_\infty + T_0(x)$, where T_∞ denotes the value of the temperature at infinity and $T_0(x)$ decays as $|x| \to \infty$; $g(x)$ is a source term.

Thus, from the physical point of view, we study the existence of stationary composition distributions depending on the temperature distribution, which enters both in the coefficient of the equation and in the right-hand side. If the temperature distribution is constant, that is $T_0(x) \equiv 0$, then we obtain a homogeneous equation with constant coefficients. It can have either only trivial solution, in which case the composition distribution is also constant, $c \equiv c_0$, or, if the spectrum contains the origin, a nonzero eigenfunction. This case corresponds to the phase separation.

In this work we study the case of a nonuniform temperature distribution, such that $T_0(x)$ does not vanish identically. We will formulate the conditions of the existence of the solution. If a solution does not exist, then this can signify that the assumption about small variation of the composition is not applicable. Instability of the homogeneous in space solution results in phase separation with strong composition gradients.

From the mathematical point of view, we consider a linear elliptic equation of the fourth order in $\mathbb{R}^3$. There are two principally different cases. If the essential spectrum of the corresponding elliptic operator does not contain the origin, then the operator satisfies the Fredholm property, its image is closed and equation (3) is solvable if and only if $f(x)$ is orthogonal to all solutions of the homogeneous adjoint equation. The essential spectrum can be determined through limiting operators [15]. If the coefficients of the operator have limits at infinity, the essential spectrum can be easily found by means of Fourier transform (see below). If it contains the origin, then the operator does not satisfy the Fredholm property and the Fredholm alternative is not applicable.

In spite of apparent simplicity of equation (3), its solvability conditions in the non-Fredholm case are not known. In the case of the second order equation, solvability conditions were recently obtained in our previous works [17]–[20]. In this work we will apply these results to study the fourth order equation.

Let us assume that the potential $V(x)$ is a smooth function vanishing at infinity. The precise conditions on it will be specified below. The function $f(x)$ belongs to the appropriate weighted Hölder space, which will imply its square integrability, and a is a nonnegative constant. We will study this equation in $\mathbb{R}^3$.

The operator

$$Lu = \Delta(\Delta u + V(x)u + au)$$

considered as acting from $H^4(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$ (or in the corresponding Hölder spaces) does not satisfy the Fredholm property. Indeed, since $V(x)$ vanishes at infinity, then the essential spectrum of this operator is the set of all complex λ for which the equation

$$\Delta(\Delta u + au) = \lambda u$$

has a nonzero bounded solution. Applying the Fourier transform, we obtain

$$\lambda = -\xi^2(a - \xi^2), \quad \xi \in \mathbb{R}^3.$$

Hence the essential spectrum contains the origin. Consequently, the operator does not satisfy the Fredholm property, and solvability conditions of equation (3) are not known. We will obtain solvability conditions for this equation using the method developed in our previous papers [17]–[20]. This method is based on spectral decomposition of self-adjoint operators.

Obviously, the problem above can be conveniently rewritten in the equivalent form of the system of two second order equations

$$\begin{cases} -\Delta v = f(x), \\ -\Delta u - V(x)u - au = v(x) \end{cases} \tag{4}$$

in which the first one has an explicit solution due to the fast rate of decay of its right side stated in Assumption 3, namely

$$v_0(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy \tag{5}$$

with properties established in Lemma A1 of the Appendix. Note that both equations of the system above involve second order differential operators without Fredholm property. Their essential spectra are $\sigma_{ess}(-\Delta) = [0, \infty)$ and $\sigma_{ess}(-\Delta - V(x) - a) = [-a, \infty)$ for $V(x) \to 0$ at infinity (see, e.g., [9]), such that neither of the operators has a finite dimensional isolated kernel. Solvability conditions for operators of that kind have been studied extensively in recent

works for a single Schrödinger type operator (see [17]), sums of second order differential operators (see [18]), the Laplacian operator with the drift term (see [19]). Non Fredholm operators arise as well while studying the existence and stability of stationary and travelling wave solutions of certain reaction-diffusion equations (see, e.g., [5, 7, 16]). For the second equation in system (4) we introduce the corresponding homogeneous problem

$$-\Delta w - V(x)w - aw = 0. \tag{6}$$

We make the following technical assumptions on the scalar potential and the right side of equation (3). Note that the first one contains conditions on $V(x)$ analogous to those stated in Assumption 1.1 of [17] (see also [18, 19]) with the slight difference that the precise rate of decay is assumed not a.e. as before but pointwise since in the present work the potential function is considered to be smooth.

Assumption 1. *The potential function $V(x) : \mathbb{R}^3 \to \mathbb{R}$ satisfies the estimate*

$$|V(x)| \leq \frac{C}{1+|x|^{3.5+\delta}}$$

with some $\delta > 0$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)}\|V\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} < 1 \quad \textit{and} \quad \sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi.$$

Here and further down C stands for a finite positive constant and c_{HLS} given on p.98 of [12] is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\Big| \leq c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)}, \quad f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3).$$

Here and below the norm of a function $f_1 \in L^p(\mathbb{R}^3),\ 1 \leq p \leq \infty$ is denoted as $\|f_1\|_{L^p(\mathbb{R}^3)}$.

Assumption 2. $\Delta V \in L^2(\mathbb{R}^3)$ *and* $\nabla V \in L^\infty(\mathbb{R}^3)$.

We will use the notation

$$(f_1(x), f_2(x))_{L^2(\mathbb{R}^3)} := \int_{\mathbb{R}^3} f_1(x)\bar{f}_2(x)dx,$$

with a slight abuse of notations when these functions are not square integrable, like for instance some of those used in the Assumption 3 below. Let us introduce the auxiliary quantity

$$\rho(x) := (1+|x|^2)^{\frac{1}{2}},\ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \tag{7}$$

and the space $C_a^\mu(\mathbb{R}^3)$, where a is a real number and $0 < \mu < 1$ consisting of all functions u for which

$$u\rho^a \in C^\mu(\mathbb{R}^3).$$

Here $C^\mu(\mathbb{R}^3)$ stands for the Hölder space such that the norm on $C_a^\mu(\mathbb{R}^3)$ is defined as

$$\|u\|_{C_a^\mu(\mathbb{R}^3)} := \sup_{x\in\mathbb{R}^3}|\rho^a(x)u(x)| + \sup_{x,y\in\mathbb{R}^3}\frac{|\rho^a(x)u(x) - \rho^a(y)u(y)|}{|x-y|^\mu}.$$

Then the space of all functions for which

$$\partial^\alpha u \in C^\mu_{a+|\alpha|}(\mathbb{R}^3),\ |\alpha| \le l,$$

where l is a nonnegative integer is being denoted as $C_a^{\mu+l}(\mathbb{R}^3)$. Let $P(s)$ be the set of polynomials of three variables of the order less or equal to s for $s \ge 0$ and $P(s)$ is empty when $s < 0$. We make the following assumption on the right side of the linearized Cahn-Hilliard problem.

Assumption 3. *Let $f(x) \in C^\mu_{6+\varepsilon}(\mathbb{R}^3)$ for some $0 < \varepsilon < 1$ and the orthogonality relation*

$$(f(x), p(x))_{L^2(\mathbb{R}^3)} = 0 \tag{8}$$

holds for any polynomial $p(x) \in P(3)$ satisfying the equation $\Delta p(x) = 0$.

Remark. *A good example of such polynomials of the third order is*

$$\frac{a}{2}x_1^3 + \frac{b}{2}x_1x_2^2 + \frac{c}{2}x_1x_3^2,$$

where a, b and c are constants, such that $3a + b + c = 0$. The set of admissible $p(x)$ includes also constants, linear functions of three variables and many more examples.

By means of Lemma 2.3 of [17], under our Assumption 1 above on the potential function, the operator $-\Delta - V(x) - a$ is self-adjoint and unitarily equivalent to $-\Delta - a$ on $L^2(\mathbb{R}^3)$ via the wave operators (see [10, 14])

$$\Omega^\pm := s - \lim_{t\to\mp\infty} e^{it(-\Delta - V)}e^{it\Delta}$$

with the limit understood in the strong L^2 sense (see, e.g., [13] p.34, [4] p.90). Therefore, $-\Delta - V(x) - a$ on $L^2(\mathbb{R}^3)$ has only the essential spectrum $\sigma_{ess}(-\Delta - V(x) - a) = [-a,\ \infty)$. Via the spectral theorem, its functions of the continuous spectrum satisfying

$$[-\Delta - V(x)]\varphi_k(x) = k^2\varphi_k(x), \quad k \in \mathbb{R}^3, \tag{9}$$

in the integral formulation the Lippmann-Schwinger equation for the perturbed plane waves (see, e.g., [13] p.98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} + \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{10}$$

and the orthogonality relations

$$(\varphi_k(x), \varphi_q(x))_{L^2(\mathbb{R}^3)} = \delta(k-q), \quad k, q \in \mathbb{R}^3 \tag{11}$$

form the complete system in $L^2(\mathbb{R}^3)$. We introduce the following auxiliary functional space (see also [19, 20])

$$\tilde{W}^{2,\infty}(\mathbb{R}^3) := \{w(x) : \mathbb{R}^3 \to \mathbb{C} \mid w, \nabla w, \Delta w \in L^\infty(\mathbb{R}^3)\}. \tag{12}$$

As distinct from the standard Sobolev space we require here not the boundedness of all second partial derivatives of the function but of its Laplacian. Our main result is as follows.

THEOREM 4. *Let Assumptions 1, 2 and 3 hold, $a \geq 0$ and $v_0(x)$ is given by (5). Then problem (3) admits a unique solution $u_a \in H^4(\mathbb{R}^3)$ if and only if*

$$(v_0(x), w(x))_{L^2(\mathbb{R}^3)} = 0 \tag{13}$$

for any $w(x) \in \tilde{W}^{2,\infty}(\mathbb{R}^3)$ satisfying equation (6).

REMARK. *Note that $\varphi_k(x) \in \tilde{W}^{2,\infty}(\mathbb{R}^3)$, $k \in \mathbb{R}^3$, which was proven in Lemma A3 of [19]. By means of (9) these perturbed plane waves satisfy the homogeneous problem (6) when the wave vector k belongs to the sphere in three dimensions centered at the origin of radius $\sqrt{a}$.*

2. Proof of the Main Result

Armed with the technical lemma of the Appendix we proceed to prove the main result.

Proof of Theorem 4. The linearized Cahn-Hillard equation (3) is equivalent to system (4) in which the first equation admits a solution $v_0(x)$ given by (5). The function $v_0(x) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and $|x|v_0(x) \in L^1(\mathbb{R}^3)$ by means of Lemma A1 and Assumption 3. Then according to Theorem 3 of [20] the second equation in system (4) with $v_0(x)$ in its right side admits a unique solution in $H^2(\mathbb{R}^3)$ if and only if the orthogonality relation (13) holds. This solution of problem (3) $u_a(x) \in H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ via the Sobolev embedding theorem, $a \geq 0$ satisfies the equation

$$-\Delta u_a - V(x)u_a - au_a = v_0(x).$$

We use the formula

$$\Delta(Vu_a) = V\Delta u_a + 2\nabla V.\nabla u_a + u_a\Delta V \tag{14}$$

with the "dot" denoting the standard scalar product of two vectors in three dimensions. The first term in the right side of (14) is square integrable since $V(x)$ is bounded and $\Delta u_a(x) \in L^2(\mathbb{R}^3)$. Similarly $u_a\Delta V \in L^2(\mathbb{R}^3)$ since $u_a(x)$ is bounded and ΔV is square integrable by means of Assumption 2. For the second term in the right side of (14) we have $\nabla u_a(x) \in L^2(\mathbb{R}^3)$ and ∇V is bounded via Assumption 2, which yields $\nabla V.\nabla u_a \in L^2(\mathbb{R}^3)$ and therefore, $\Delta(Vu_a) \in L^2(\mathbb{R}^3)$. The right side of problem (3) belongs to $L^2(\mathbb{R}^3)$ due to Assumption 3. Indeed, since $\sup_{x\in\mathbb{R}^3}|\rho^{6+\varepsilon}f| \le C$, we arrive at the estimate

$$|f(x)| \le \frac{C}{(\rho(x))^{6+\varepsilon}},\ x \in \mathbb{R}^3 \tag{15}$$

with $\rho(x)$ defined explicitly in (7). Hence from equation (3) we deduce that $\Delta^2 u_a \in L^2(\mathbb{R}^3)$. Any partial third derivative of u_a is also square integrable due to the trivial estimate in terms of the $L^2(\mathbb{R}^3)$ norms of u_a and $\Delta^2 u_a$, which are finite. This implies that $u_a \in H^4(\mathbb{R}^3)$.

To investigate the issue of uniqueness we suppose $u_1, u_2 \in H^4(\mathbb{R}^3)$ are two solutions of problem (3). Then their difference $u(x) = u_1(x) - u_2(x) \in H^4(\mathbb{R}^3)$ satisfies equation (3) with vanishing right side. Clearly $u, \Delta u \in L^2(\mathbb{R}^3)$ and $Vu \in L^2(\mathbb{R}^3)$. Therefore, $v(x) = -\Delta u - V(x)u - au \in L^2(\mathbb{R}^3)$ and solves the equation $\Delta v = 0$. Since the Laplace operator does not have any nontrivial square integrable zero modes, $v(x) = 0$ a.e. in $\mathbb{R}^3$. Hence, we arrive at the homogeneous problem $(-\Delta - V(x) - a)u = 0, \quad u(x) \in L^2(\mathbb{R}^3)$. The operator in brackets is unitarily equivalent to $-\Delta - a$ on $L^2(\mathbb{R}^3)$ as discussed above and therefore $u(x) = 0$ a.e. in $\mathbb{R}^3$. □

3. Appendix

Lemma A1. *Let Assumption 3 hold. Then* $v_0(x) \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ *and* $xv_0(x) \in L^1(\mathbb{R}^3)$.

Proof. According to the result of [2], for the solution of the Poisson equation (5) under the condition $f(x) \in C^{\mu}_{6+\varepsilon}(\mathbb{R}^3)$ and orthogonality relation (8) given in Assumption 3 we have $v_0(x) \in C^{\mu+2}_{4+\varepsilon}(\mathbb{R}^3)$. Hence $\sup_{x\in\mathbb{R}^3}|\rho^{4+\varepsilon}v_0| \le C$, such that

$$|v_0(x)| \le \frac{C}{(\rho(x))^{4+\varepsilon}},\ x \in \mathbb{R}^3.$$

The statement of the lemma easily follows from definition (7). □

Remark. *Note that the boundedness of $v_0(x)$ can be easily shown via the argument of Lemma 2.1 of [17], which relies on Young's inequality. The square integrability of $v_0(x)$ can be proven by applying the Fourier transform to it, using the facts that $f(x) \in L^2(\mathbb{R}^3)$, $|x|f(x) \in L^1(\mathbb{R}^3)$ and its Fourier image vanishes at the origin since it is orthogonal to a constant by means of Assumption 3.*

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