

# Vector Bundles on Elliptic Curves and Factors of Automorphy

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*Communicated by Ugo Bruzzo*

ABSTRACT. *We translate Atiyah's results on classification of vector bundles on elliptic curves to the language of factors of automorphy.*

Keywords: Elliptic Curves, Factors of Automorphy, Theta Functions, Vector Bundles.  
MS Classification 2010: 14H60, 14H52, 14D20, 14K25

## 1. Introduction

### 1.1. Motivation

The problem of classification of vector bundles over an elliptic curve was considered and completely solved by Atiyah in [1].

For a group  $\Gamma$  acting on a complex manifold  $Y$ , an  $r$ -dimensional factor of automorphy is a holomorphic function  $f : \Gamma \times Y \rightarrow \mathrm{GL}_r(\mathbb{C})$  satisfying  $f(\lambda\mu, y) = f(\lambda, \mu y)f(\mu, y)$ . Two factors of automorphy  $f$  and  $f'$  are equivalent if there exists a holomorphic function  $h : Y \rightarrow \mathrm{GL}_r(\mathbb{C})$  such that  $h(\lambda y)f(\lambda, y) = f'(\lambda, y)h(y)$ .

Given a complex manifold  $X$  and the universal covering  $Y \xrightarrow{p} X$ , let  $\Gamma$  be the fundamental group of  $X$  acting naturally on  $Y$  by deck transformations. Then there is a one-to-one correspondence between equivalence classes of  $r$ -dimensional factors of automorphy and isomorphism classes of vector bundles on  $X$  with trivial pull-back along  $p$ . In particular, if  $Y$  does not possess any non-trivial vector bundles, one obtains a one-to-one correspondence between equivalence classes of  $r$ -dimensional factors of automorphy and isomorphism classes of vector bundles on  $X$ . In particular this is the case for complex tori.

Since it is known that one-dimensional complex tori correspond to elliptic curves and since the classification of holomorphic vector bundles on a projective variety over  $\mathbb{C}$  is equivalent to the classification of algebraic vector bundles (cf. [13]), it is possible to formulate Atiyah's results in the language of factors of automorphy. So for example in the case of vector bundles of rank 1 and 2

such a formulation using factors of automorphy was given in [4], Theorems 4.4 and 4.5.

This paper is a shortened version of the diploma thesis [7] and aims to prove some results used without any proofs by different authors, in particular in [12] and [3]. The main result of this note, Theorem 5.24, gives a classification of indecomposable vector bundles of fixed rank and degree on a complex torus in terms of factors of automorphy. Its statement coincides with the statement of Proposition 1 from [12], which was given without any proof.

The author thanks Igor Burban, Bernd Kreußler, and Günther Trautmann, who motivated him to prepare this manuscript.

## 1.2. Structure of the Paper

In Section 2 we establish a correspondence between vector bundles and factors of automorphy. Section 3 deals with properties of factors of automorphy, in particular we discuss a correspondence between operations on vector bundles and operations on factors of automorphy. From Section 4 on we restrict ourselves to the case of vector bundles on complex tori. It is shown in Theorem 4.11 that to define a vector bundle of rank  $r$  on a complex one-dimensional torus is the same as to fix a holomorphic function  $\mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$ . In Section 5 we first present in Theorem 5.13 a classification of indecomposable vector bundles of degree zero, using this we give then in Theorem 5.24 a complete classification of indecomposable vector bundles of fixed rank and degree in terms of factors of automorphy.

## 1.3. Notations and Conventions

Following Atiyah's paper [1] we denote by  $\mathcal{E}(r, d) = \mathcal{E}_X(r, d)$  the set of isomorphism classes of indecomposable vector bundles over  $X$  of rank  $r$  and degree  $d$ . For a vector bundle  $E$  we usually denote the corresponding locally free sheaf of its sections by  $\mathcal{E}$ . By  $\mathbf{Vect}$  we denote the category of finite dimensional vector spaces. For a divisor  $D$  we denote by  $[D]$  the corresponding line bundle.

## 2. Correspondence between Vector Bundles and Factors of Automorphy

Let  $X$  be a complex manifold and let  $p : Y \rightarrow X$  be a covering of  $X$ . Let  $\Gamma \subset \mathrm{Deck}(Y/X)$  be a subgroup in the group of deck transformations  $\mathrm{Deck}(Y/X)$  such that for any two points  $y_1$  and  $y_2$  with  $p(y_1) = p(y_2)$  there exists an element  $\gamma \in \Gamma$  such that  $\gamma(y_1) = y_2$ . In other words,  $\Gamma$  acts transitively in each fiber. We call this property **(T)**.

REMARK 2.1. Note that for any two points  $y_1$  and  $y_2$  there can be only one  $\gamma \in \text{Deck}(Y/X)$  with  $\gamma(y_1) = y_2$  (see [5], Satz 4.8). Therefore,  $\Gamma = \text{Deck}(Y/X)$  and the property **(T)** simply means that  $p : Y \rightarrow X$  is a normal (Galois) covering.

We have an action of  $\Gamma$  on  $Y$ :

$$\Gamma \times Y \rightarrow Y, \quad y \mapsto \gamma(y) =: \gamma y.$$

DEFINITION 2.2. A holomorphic function  $f : \Gamma \times Y \rightarrow \text{GL}_r(\mathbb{C})$ ,  $r \in \mathbb{N}$  is called an  $r$ -dimensional factor of automorphy if it satisfies the relation

$$f(\lambda\mu, y) = f(\lambda, \mu y)f(\mu, y).$$

Denote by  $Z^1(\Gamma, r)$  the set of all  $r$ -dimensional factors of automorphy.

We introduce the relation  $\sim$  on  $Z^1(\Gamma, r)$ . We say that  $f$  is equivalent to  $f'$  if there exists a holomorphic function  $h : Y \rightarrow \text{GL}_r(\mathbb{C})$  such that

$$h(\lambda y)f(\lambda, y) = f'(\lambda, y)h(y).$$

We write in this case  $f \sim f'$ .

LEMMA 2.3. The relation  $\sim$  is an equivalence relation on  $Z^1(\Gamma, r)$ .

*Proof.* Straightforward verifications. □

We denote the set of equivalence classes of  $Z^1(\Gamma, r)$  with respect to  $\sim$  by  $H^1(\Gamma, r)$ .

Consider  $f \in Z^1(\Gamma, r)$  and a trivial vector bundle  $Y \times \mathbb{C}^r \rightarrow Y$ . Define a holomorphic action of  $\Gamma$  on  $Y \times \mathbb{C}^r$ :

$$\Gamma \times Y \times \mathbb{C}^r \rightarrow Y \times \mathbb{C}^r, \quad (\lambda, y, v) \mapsto (\lambda y, f(\lambda, y)v) =: \lambda(y, v).$$

Denote  $E(f) = Y \times \mathbb{C}^r / \Gamma$  and note that for two equivalent points  $(y, v) \sim_\Gamma (y', v')$  with respect to the action of  $\Gamma$  on  $Y \times \mathbb{C}^r$  it follows that  $p(y) = p(y')$ . In fact,  $(y, v) \sim_\Gamma (y', v')$  implies in particular that  $y = \gamma y'$  for some  $\gamma \in \Gamma$  and by the definition of deck transformations  $p(y) = p(\gamma y') = p(y')$ . Hence the projection  $Y \times \mathbb{C}^r \rightarrow Y$  induces the map

$$\pi : E(f) \rightarrow X, \quad [y, v] \mapsto p(y).$$

We equip  $E(f)$  with the quotient topology.

THEOREM 2.4.  $E(f)$  inherits a complex structure from  $Y \times \mathbb{C}^r$  and the map  $\pi : E(f) \rightarrow X$  is a holomorphic vector bundle on  $X$ .

*Proof.* First we prove that  $\pi$  is a topological vector bundle. Clearly  $\pi$  is a continuous map. Consider the commutative diagram

$$\begin{array}{ccc} Y \times \mathbb{C}^r & \longrightarrow & E(f) \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow{p} & X. \end{array}$$

Let  $x$  be a point of  $X$ . Since  $p$  is a covering, one can choose an open neighbourhood  $U$  of  $x$  such that its preimage is a disjoint union of open sets biholomorphic to  $U$ , i. e.,  $p^{-1}(U) = \bigsqcup_{i \in \mathcal{I}} V_i$ ,  $p_i := p|_{V_i} : V_i \rightarrow U$  is a biholomorphism for each  $i \in \mathcal{I}$ . For each pair  $(i, j) \in \mathcal{I} \times \mathcal{I}$  there exists a unique  $\lambda_{ij} \in \Gamma$  such that  $\lambda_{ij} p_j^{-1}(x) = p_i^{-1}(x)$  for all  $x \in U$ . This follows from the property **(T)**.

We have  $\pi^{-1}(U) = ((\bigsqcup_{i \in \mathcal{I}} V_i) \times \mathbb{C}^r) / \Gamma$ .

Choose some  $i_U \in \mathcal{I}$ . Consider the holomorphic map

$$\varphi'_U : \left( \bigsqcup_{i \in \mathcal{I}} V_i \right) \times \mathbb{C}^r \rightarrow U \times \mathbb{C}^r, \quad (y_i, v) \mapsto (p(y_i), f(\lambda_{i_U i}, y_i)v), \quad y_i \in V_i.$$

Suppose that  $(y_i, v') \sim_{\Gamma} (y_j, v)$ . This means

$$(y_i, v') = \lambda_{ij}(y_j, v) = (\lambda_{ij}y_j, f(\lambda_{ij}, y_j)v).$$

Therefore,

$$\begin{aligned} \varphi'_U(y_i, v') &= (p(y_i), f(\lambda_{i_U i}, y_i)v') = (p(\lambda_{ij}y_j), f(\lambda_{i_U i}, \lambda_{ij}y_j)f(\lambda_{ij}, y_j)v) \\ &= (p(y_j), f(\lambda_{i_U j}, y_j)v) = \varphi'_U(y_j, v). \end{aligned}$$

Thus  $\varphi'_U$  factorizes through  $((\bigsqcup_{i \in \mathcal{I}} V_i) \times \mathbb{C}^r) / \Gamma$ , i. e., the map

$$\varphi_U : \left( \left( \bigsqcup_{i \in \mathcal{I}} V_i \right) \times \mathbb{C}^r \right) / \Gamma \rightarrow U \times \mathbb{C}^r, \quad [(y_i, v)] \mapsto (p(y_i), f(\lambda_{i_U i}, y_i)v), \quad y_i \in V_i$$

is well-defined and continuous. We claim that  $\varphi_U$  is bijective.

Suppose  $\varphi_U([(y_i, v')]) = \varphi_U([(y_j, v)])$ , where  $y_i \in V_i$ ,  $y_j \in V_j$ . By definition this is equivalent to  $(p(y_i), f(\lambda_{i_U i}, y_i)v') = (p(y_j), f(\lambda_{i_U j}, y_j)v)$ , which means  $y_i = \lambda_{ij}y_j$  and

$$\begin{aligned} f(\lambda_{i_U i}, \lambda_{ij}y_j)v' &= f(\lambda_{i_U i}, y_i)v' = f(\lambda_{i_U j}, y_j)v \\ &= f(\lambda_{i_U i}\lambda_{ij}, y_j)v = f(\lambda_{i_U i}, \lambda_{ij}y_j)f(\lambda_{ij}, y_j)v. \end{aligned}$$

We conclude  $v' = f(\lambda_{ij}, y_j)v$  and  $[(y_i, v')] = [(y_j, v)]$ , which means injectivity of  $\varphi_U$ .

At the same time for each element  $(y, v) \in U \times \mathbb{C}^r$  one has

$$\begin{aligned} & \varphi_U([(p_i^{-1}(y), f(\lambda_{i_U i}, p_i^{-1}(y))^{-1}v)]) \\ &= (pp_i^{-1}(y), f(\lambda_{i_U i}, p_i^{-1}(y))f(\lambda_{i_U i}, p_i^{-1}(y))^{-1}v) = (y, v), \end{aligned}$$

i.e.,  $\varphi_U$  is surjective and we obtain that  $\varphi_U$  is a bijective map.

This means, that  $\varphi_U$  is a trivialization for  $U$  and that  $\pi : E(f) \rightarrow X$  is a (continuous) vector bundle. If  $U$  and  $V$  are two neighbourhoods of  $X$  defined as above for which  $E(f)|_U, E(f)|_V$  are trivial, then the corresponding transition function is

$$\varphi_U \varphi_V^{-1} : (U \cap V) \times \mathbb{C}^r \rightarrow (U \cap V) \times \mathbb{C}^r, \quad (x, v) \mapsto (x, g_{UV}(x)v),$$

where  $g_{UV} : U \cap V \rightarrow \mathrm{GL}_r(\mathbb{C})$  is a cocycle defining  $E(f)$ . But from the construction of  $\varphi_U$  it follows that

$$g_{UV}(x) = f(\lambda_{i_U i_V}, p_{i_V}^{-1}(x)).$$

Therefore,  $g_{UV}$  is a holomorphic map, hence  $\varphi_U \varphi_V^{-1}$  is also a holomorphic map. Thus the maps  $\varphi_U$  give  $E(f)$  a complex structure. Since  $\pi$  is locally a projection, one sees that  $\pi$  is a holomorphic map.  $\square$

REMARK 2.5. *Note that  $p^*E(f)$  is isomorphic to  $Y \times \mathbb{C}^r$ . An isomorphism can be given by the map*

$$p^*E(f) \rightarrow Y \times \mathbb{C}^r, \quad (y, [\tilde{y}, v]) \mapsto (y, f(\lambda, \tilde{y})v), \quad \lambda \tilde{y} = y.$$

Now we have the map from  $Z^1(\Gamma, r)$  to the set  $K_r = \{[E] \mid p^*(E) \simeq Y \times \mathbb{C}^r\}$  of isomorphism classes of vector bundles of rank  $r$  over  $X$  with trivial pull back with respect to  $p$ .

$$\phi' : Z^1(\Gamma, r) \rightarrow K_r; \quad f \mapsto [E(f)].$$

THEOREM 2.6. *Let  $K_r$  denote the set of isomorphism classes of vector bundles of rank  $r$  on  $X$  with trivial pull back with respect to  $p$ . Then the map*

$$H^1(\Gamma, r) \rightarrow K_r, \quad [f] \mapsto [E(f)]$$

*is a bijection.*

*Proof.* This proof generalizes the proof from [2, Appendix B] given only for line bundles.

Consider the map  $\phi' : Z^1(\Gamma, r) \rightarrow K_r$  and let  $f$  and  $f'$  be two equivalent  $r$ -dimensional factors of automorphy. It means that there exists a holomorphic function  $h : Y \rightarrow \mathrm{GL}_r(\mathbb{C})$  such that

$$f'(\lambda, y) = h(\lambda y)f(\lambda, y)h(y)^{-1}.$$

Therefore, for two neighbourhoods  $U, V$  constructed as above we have the following relation for cocycles corresponding to  $f$  and  $f'$ .

$$\begin{aligned} g'_{UV}(x) &= f'(\lambda_{UV}, p_{i_V}^{-1}(x)) = h(\lambda_{UV}, p_{i_V}^{-1}(x))f(\lambda_{UV}, p_{i_V}^{-1}(x))h(p_{i_V}^{-1}(x))^{-1} \\ &= h(p_{i_U}^{-1}(x))g_{UV}(x)h(p_{i_V}^{-1}(x))^{-1} = h_U(x)g_{UV}(x)h_V(x)^{-1}, \end{aligned}$$

where  $\lambda_{UV} = \lambda_{i_U i_V}$ ,  $h_U(x) = h(p_{i_U}^{-1}(x))$  and  $h_V(x) = h(p_{i_V}^{-1}(x))$ . We obtained

$$g'_{UV} = h_U g_{UV} h_V^{-1},$$

which is exactly the condition for two cocycles to define isomorphic vector bundles. Therefore,  $E(f) \simeq E(f')$  and it means that  $\phi'$  factorizes through  $H^1(\Gamma, r)$ , i.e., the map

$$\phi : H^1(\Gamma, r) \rightarrow K_r; \quad [f] \mapsto [E(f)]$$

is well-defined.

It remains to construct the inverse map. Suppose  $E \in K_r$ , in other words  $p^*(E)$  is the trivial bundle of rank  $r$  over  $Y$ . Let  $\alpha : p^*E \rightarrow Y \times \mathbb{C}^r$  be a trivialization. The action of  $\Gamma$  on  $Y$  induces a holomorphic action of  $\Gamma$  on  $p^*E$  :

$$\lambda(y, e) := (\lambda y, e) \text{ for } (y, e) \in p^*E = Y \times_X E.$$

Via  $\alpha$  we get for every  $\lambda \in \Gamma$  an automorphism  $\psi_\lambda$  of the trivial bundle  $Y \times \mathbb{C}^r$ . Clearly  $\psi_\lambda$  should be of the form

$$\psi_\lambda(y, v) = (\lambda y, f(\lambda, y)v),$$

where  $f : \Gamma \times Y \rightarrow \text{GL}_r(\mathbb{C})$  is a holomorphic map. The equation for the action  $\psi_{\lambda\mu} = \psi_\lambda \psi_\mu$  implies that  $f$  should be an  $r$ -dimensional factor of automorphy.

Suppose  $\alpha'$  is another trivialization of  $p^*E$ . Then there exists a holomorphic map  $h : Y \rightarrow \text{GL}_r(\mathbb{C})$  such that  $\alpha'\alpha^{-1}(y, v) = (y, h(y)v)$ . Let  $f'$  be a factor of automorphy corresponding to  $\alpha'$ . From

$$\begin{aligned} (\lambda y, f'(\lambda, y)v) &= \psi'_\lambda(y, v) = \alpha'\lambda\alpha'^{-1}(y, v) = \alpha'\alpha^{-1}\alpha\lambda\alpha^{-1}\alpha\alpha'^{-1}(y, v) \\ &= \alpha'\alpha^{-1}\psi_\lambda(\alpha'\alpha^{-1})^{-1}(y, v) = \alpha'\alpha^{-1}\psi_\lambda(y, h(y)^{-1}v) \\ &= \alpha'\alpha^{-1}(\lambda y, f(\lambda, y)h(y)^{-1}v) = (\lambda y, h(\lambda y)f(\lambda, y)h(y)^{-1}v), \end{aligned}$$

we obtain  $f'(\lambda, y) = h(\lambda y)f(\lambda, y)h(y)^{-1}$ . The last means that  $[f] = [f']$ , in other words, the class of a factor of automorphy in  $H^1(\Gamma, r)$  does not depend on the trivialization and we get a map  $K_r \rightarrow H^1(\Gamma, r)$ . This map is the inverse of  $\phi$ .  $\square$

Let  $X$  be a connected complex manifold, let  $p : \tilde{X} \rightarrow X$  be a universal covering of  $X$ , and let  $\Gamma = \text{Deck}(\tilde{X}/X)$ . Since universal coverings are normal coverings,  $\Gamma$  satisfies the property **(T)** (see [5, Satz 5.6]). Moreover,  $\Gamma$  is isomorphic to the fundamental group  $\pi_1(X)$  of  $X$  (see [5, Satz 5.6]). An isomorphism is given as follows.

Fix  $x_0 \in X$  and  $\tilde{x}_0 \in \tilde{X}$  with  $p(\tilde{x}_0) = x_0$ . We define a map

$$\Phi : \text{Deck}(\tilde{X}/X) \rightarrow \pi_1(X, x_0)$$

as follows. Let  $\sigma \in \text{Deck}(\tilde{X}/X)$  and  $v : [0; 1] \rightarrow \tilde{X}$  be a curve with  $v(0) = \tilde{x}_0$  and  $v(1) = \sigma(\tilde{x}_0)$ . Then a curve

$$pv : [0; 1] \rightarrow X, \quad t \mapsto pv(t)$$

is such that  $pv(0) = pv(1) = x_0$ . Define  $\Phi(\sigma) := [pv]$ , where  $[pv]$  denotes a homotopy class of  $pv$ . The map  $\Phi$  is well defined and is an isomorphism of groups.

So we can identify  $\Gamma$  with  $\pi_1(X)$ . Therefore, we have an action of  $\pi_1(X)$  on  $\tilde{X}$  by deck transformations.

Consider an element  $[w] \in \pi_1(X, x_0)$  represented by a path  $w : [0; 1] \rightarrow X$ . We denote  $\sigma = \Phi^{-1}([w])$ . Consider any  $\tilde{x}_0 \in \tilde{X}$  such that  $p(\tilde{x}_0) = w(0) = w(1)$ , then the path  $w$  can be uniquely lifted to the path

$$v : [0; 1] \rightarrow \tilde{X}$$

with  $v(0) = \tilde{x}_0$  (see [5], Satz 4.14). Denote  $\tilde{x}_1 = v(1)$ . Then  $\sigma$  is a unique element in  $\text{Deck}(\tilde{X}/X)$  such that  $\sigma(\tilde{x}_0) = \tilde{x}_1$ . This gives a description of the action of  $\pi_1(X, x_0)$  on  $\tilde{X}$ .

Now we have a corollary to Theorem 2.6.

**COROLLARY 2.7.** *Let  $X$  be a connected complex manifold, let  $p : \tilde{X} \rightarrow X$  be the universal covering, let  $\Gamma$  be the fundamental group of  $X$  naturally acting on  $\tilde{X}$  by deck transformations. As above,  $H^1(\Gamma, r)$  denotes the set of equivalence classes of  $r$ -dimensional factors of automorphy*

$$\Gamma \times \tilde{X} \rightarrow \text{GL}_r(\mathbb{C}).$$

*Then there is a bijection*

$$H^1(\Gamma, r) \rightarrow K_r, \quad [f] \mapsto E(f),$$

*where  $K_r$  denotes the set of isomorphism classes of vector bundles of rank  $r$  on  $X$  with trivial pull back with respect to  $p$ .*

### 3. Properties of Factors of Automorphy

DEFINITION 3.1. Let  $f : \Gamma \times Y \rightarrow \mathrm{GL}_r(\mathbb{C})$  be an  $r$ -dimensional factor of automorphy. A holomorphic function  $s : Y \rightarrow \mathbb{C}^r$  is called an  $f$ -theta function if it satisfies

$$s(\gamma y) = f(\gamma, y)s(y) \text{ for all } \gamma \in \Gamma, y \in Y.$$

THEOREM 3.2. Let  $f : \Gamma \times Y \rightarrow \mathrm{GL}_r(\mathbb{C})$  be an  $r$ -dimensional factor of automorphy. Then there is a one-to-one correspondence between sections of  $E(f)$  and  $f$ -theta functions.

*Proof.* Let  $\{V_i\}_{i \in \mathcal{I}}$  be a covering of  $Y$  such that  $p$  restricted to  $V_i$  is a homeomorphism. Denote  $\varphi_i := (p|_{V_i})^{-1}$ ,  $U_i := p(V_i)$ . Then  $\{U_i\}$  is a covering of  $X$  such that  $E(f)$  is trivial over each  $U_i$ .

Consider a section of  $E(f)$  given by functions  $s_i : U_i \rightarrow \mathbb{C}^r$  satisfying

$$s_i(x) = g_{ij}(x)s_j(x) \text{ for } x \in U_i \cap U_j,$$

where

$$g_{ij}(x) = f(\lambda_{U_i U_j}, \varphi_j(x)), \quad x \in U_i \cap U_j$$

is a cocycle defining  $E(f)$  (see the proof of Theorem 2.6). Define  $s : Y \rightarrow \mathbb{C}^r$  by  $s(\varphi_i(x)) := s_i(x)$ . To prove that this is well-defined we need to show that  $s_i(x) = s_j(x)$  when  $\varphi_i(x) = \varphi_j(x)$ . But since  $\varphi_i(x) = \varphi_j(x)$  we obtain  $\lambda_{U_i U_j} = 1$ . Therefore,

$$s_i(x) = g_{ij}(x)s_j(x) = f(\lambda_{U_i U_j}, \varphi_j(x))s_j(x) = f(1, \varphi_j(x))s_j(x) = s_j(x).$$

For any  $\gamma \in \Gamma$  for any point  $y \in Y$  take  $i, j \in \mathcal{I}$  and  $x \in X$  such that  $y = \varphi_j(x)$  and  $\gamma y = \gamma \varphi_j(x) = \varphi_i(x)$ . Thus  $\gamma = \lambda_{U_i U_j}$  and one obtains

$$\begin{aligned} s(\gamma y) &= s(\varphi_i(x)) = s_i(x) = g_{ij}(x)s_j(x) \\ &= f(\lambda_{U_i U_j}, \varphi_j(x))s_j(x) = f(\gamma, y)s(\varphi_j(x)) = f(\gamma, z)s(y). \end{aligned}$$

In other words,  $s$  is an  $f$ -theta function.

Vice versa, let  $s : Y \rightarrow \mathbb{C}^r$  be an  $f$ -theta function. We define  $s_i : U_i \rightarrow \mathbb{C}^r$  by  $s_i(x) := s(\varphi_i(x))$ . Then for a point  $x \in U_i \cap U_j$  we have

$$\begin{aligned} s_i(x) &= s(\varphi_i(x)) = s(\lambda_{U_i U_j} \varphi_j(x)) \\ &= f(\lambda_{U_i U_j}, \varphi_j(x))s(\varphi_j(x)) = g_{ij}(x)s_j(x), \end{aligned}$$

which means that the functions  $s_i$  define a section of  $E(f)$ . The described correspondences are clearly inverse to each other.  $\square$

The following statement will be useful in the sequel.



THEOREM 3.3. *Let*

$$f(\lambda, y) = \begin{pmatrix} f'(\lambda, y) & \tilde{f}(\lambda, y) \\ 0 & f''(\lambda, y) \end{pmatrix}$$

be an  $r' + r''$ -dimensional factor of automorphy, where  $f'(\lambda, y) \in \mathrm{GL}_{r'}(\mathbb{C})$ ,  $f''(\lambda, y) \in \mathrm{GL}_{r''}(\mathbb{C})$ . Then

- (a)  $f' : \Gamma \times Y \rightarrow \mathrm{GL}_{r'}(\mathbb{C})$  and  $f'' : \Gamma \times Y \rightarrow \mathrm{GL}_{r''}(\mathbb{C})$  are  $r'$  and  $r''$ -dimensional factors of automorphy respectively;
- (b) there is an extension of vector bundles

$$0 \longrightarrow E(f') \xrightarrow{i} E(f) \xrightarrow{\pi} E(f'') \longrightarrow 0.$$

*Proof.* The statement of (a) follows from straightforward verification. To prove (b) we define maps  $i$  and  $\pi$  as follows.

$$\begin{aligned} i : E(f') \rightarrow E(f), \quad [y, v] &\mapsto [y, \begin{pmatrix} v \\ 0 \end{pmatrix}], \quad v \in \mathbb{C}^{r'}, \quad \begin{pmatrix} v \\ 0 \end{pmatrix} \in \mathbb{C}^{r'+r''} \\ \pi : E(f) \rightarrow E(f''), \quad [y, \begin{pmatrix} v \\ w \end{pmatrix}] &\mapsto [y, w], \quad v \in \mathbb{C}^{r'}, \quad w \in \mathbb{C}^{r''} \end{aligned}$$

Since  $[\lambda y, f'(\lambda, y)v]$  is mapped via  $i$  to

$$\left[ \lambda y, \begin{pmatrix} f'(\lambda, y)v \\ 0 \end{pmatrix} \right] = \left[ \lambda y, f(\lambda, y) \begin{pmatrix} v \\ 0 \end{pmatrix} \right],$$

one concludes that  $i$  is well-defined. Analogously, since  $[\lambda y, f''(\lambda, y)w] = [y, w]$  one sees that  $\pi$  is well-defined. Using the charts from the proof of (2.4) one easily sees that the defined maps are holomorphic.

Notice that  $i$  and  $\pi$  respect fibers,  $i$  is injective and  $\pi$  is surjective in each fiber. This proves the statement.  $\square$

Now we recall one standard construction from linear algebra. Let  $A$  be an  $m \times n$  matrix. It represents some morphism  $\mathbb{C}^n \rightarrow \mathbb{C}^m$  for fixed standard bases in  $\mathbb{C}^n$  and  $\mathbb{C}^m$ .

Let  $\mathcal{F} : \mathbf{Vect}^p \rightarrow \mathbf{Vect}$  be a covariant functor. Let  $A_1, \dots, A_p$  be the matrices representing morphisms  $\mathbb{C}_1^n \xrightarrow{f_1} \mathbb{C}_1^m, \dots, \mathbb{C}_p^n \xrightarrow{f_p} \mathbb{C}_p^m$  in standard bases.

If for each object  $\mathcal{F}(\mathbb{C}^m)$  we fix some basis, then the matrix corresponding to the morphism  $\mathcal{F}(f_1, \dots, f_p)$  is denoted by  $\mathcal{F}(A_1, \dots, A_p)$ . Clearly it satisfies

$$\mathcal{F}(A_1 B_1, \dots, A_p B_p) = \mathcal{F}(A_1, \dots, A_p) \mathcal{F}(B_1, \dots, B_p).$$

In this way  $A \otimes B$ ,  $S^q(A)$ ,  $\Lambda^q(A)$  can be defined. As  $\mathcal{F}$  one considers the functors

$$-\otimes -: \mathbf{Vect}^2 \rightarrow \mathbf{Vect}, \quad S^n : \mathbf{Vect} \rightarrow \mathbf{Vect}, \quad \Lambda : \mathbf{Vect} \rightarrow \mathbf{Vect}$$

respectively.

Recall that every holomorphic functor  $\mathcal{F} : \mathbf{Vect}^n \rightarrow \mathbf{Vect}$  can be canonically extended to the category of vector bundles of finite rank over  $X$ . By abuse of notation we will denote the extended functor by  $\mathcal{F}$  as well.

**THEOREM 3.4.** *Let  $\mathcal{F} : \mathbf{Vect}^n \rightarrow \mathbf{Vect}$  be a covariant holomorphic functor. Let  $f_1, \dots, f_n$  be  $r_i$ -dimensional factors of automorphy. Then  $f = \mathcal{F}(f_1, \dots, f_n)$  is a factor of automorphy defining  $\mathcal{F}(E(f_1), \dots, E(f_n))$ .*

*Proof.* One clearly has

$$\begin{aligned} \mathcal{F}(f_1, \dots, f_n)(\lambda\mu, y) &= \mathcal{F}(f_1(\lambda\mu, y), \dots, f_n(\lambda\mu, y)) \\ &= \mathcal{F}(f_1(\lambda, \mu y)f_1(\mu, y), \dots, f_n(\lambda, \mu y)f_n(\mu, y)) \\ &= F(f_1(\lambda, \mu y), \dots, f_n(\lambda, \mu y))F(f_1(\mu, y), \dots, f_n(\mu, y)) \\ &= \mathcal{F}(f_1, \dots, f_n)(\lambda, \mu y)\mathcal{F}(f_1, \dots, f_n)(\mu, y). \end{aligned}$$

Since  $(f_1, \dots, f_n)$  represents an isomorphism in  $\mathbf{Vect}^n$ ,  $\mathcal{F}(f_1, \dots, f_n)$  also represents an isomorphism  $\mathbb{C}^r \rightarrow \mathbb{C}^r$  for some  $r \in \mathbb{N}$ . Therefore,  $f$  is an  $r$ -dimensional factor of automorphy.

Since  $f = \mathcal{F}(f_1, \dots, f_n)$ , for cocycles defining the corresponding vector bundles the equality  $g_{U_1 U_2} = \mathcal{F}(g_{1U_1 U_2}, \dots, g_{nU_1 U_2})$  holds true, where  $g_{iU_1 U_2}$  is a cocycle defining  $E(f_i)$ . This shows that  $E(f) = \mathcal{F}(E(f_1), \dots, E(f_n))$  and proves the required statement.  $\square$

For example for  $\mathcal{F} = \_ \otimes \_ : \mathbf{Vect}^2 \rightarrow \mathbf{Vect}$  we get the following obvious corollary.

**COROLLARY 3.5.** *Let  $f' : \Gamma \times Y \rightarrow \mathrm{GL}_{r'}(\mathbb{C})$  and  $f'' : \Gamma \times Y \rightarrow \mathrm{GL}_{r''}(\mathbb{C})$  be two factors of automorphy. Then  $f = f' \otimes f'' : \Gamma \times Y \rightarrow \mathrm{GL}_{r'r''}(\mathbb{C})$  is also a factor of automorphy. Moreover,  $E(f) \simeq E(f') \otimes E(f'')$ .*

It is not essential that the functor in Theorem 3.4 is covariant. The following theorem is a generalization of Theorem 3.4.

**THEOREM 3.6.** *Let  $\mathcal{F} : \mathbf{Vect}^n \rightarrow \mathbf{Vect}$  be a holomorphic functor. Let  $\mathcal{F}$  be covariant in  $k$  first variables and contravariant in  $n - k$  last variables. Let  $f_1, \dots, f_n$  be  $r_i$ -dimensional factors of automorphy. Then*

$$f = \mathcal{F}(f_1, \dots, f_k, f_{k+1}^{-1}, \dots, f_n^{-1})$$

*is a factor of automorphy defining  $\mathcal{F}(E(f_1), \dots, E(f_n))$ .*

*Proof.* The proof is analogous to the proof of Theorem 3.4.  $\square$

## 4. Vector Bundles on Complex Tori

### 4.1. One Dimensional Complex Tori

Let  $X$  be a complex torus, i.e.,  $X = \mathbb{C}/\Gamma$ ,  $\Gamma = \mathbb{Z}\tau + \mathbb{Z}$ ,  $\text{Im } \tau > 0$ . Then the universal covering is  $\tilde{X} = \mathbb{C}$ , namely

$$\text{pr} : \mathbb{C} \rightarrow \mathbb{C}/\Gamma, \quad x \mapsto [x].$$

We have an action of  $\Gamma$  on  $\mathbb{C}$ :

$$\Gamma \times \mathbb{C} \rightarrow \mathbb{C}, \quad (\gamma, y) \mapsto \gamma + y.$$

Clearly  $\Gamma$  acts on  $\mathbb{C}$  by deck transformations and satisfies the property **(T)**.

Since  $\mathbb{C}$  is a non-compact Riemann surface, by [5, Theorem 30.4, p. 204], there are only trivial bundles on  $\mathbb{C}$ . Therefore, we have a one-to-one correspondence between classes of isomorphism of vector bundles of rank  $r$  on  $X$  and equivalence classes of factors of automorphy

$$f : \Gamma \times \mathbb{C} \rightarrow \text{GL}_r(\mathbb{C}).$$

As usually,  $V_a$  denotes the standard parallelogram constructed at point  $a$ ,  $U_a$  is the image of  $V_a$  under the projection,  $\varphi_a : U_a \rightarrow V_a$  is the local inverse of the projection.

**REMARK 4.1.** *Let  $f$  be an  $r$ -dimensional factor of automorphy. Then*

$$g_{ab}(x) = f(\varphi_a(x) - \varphi_b(x), \varphi_b(x))$$

*is a cocycle defining  $E(f)$ . This follows from the construction of the cocycle in the proof of Theorem 2.6.*

**EXAMPLE 4.2.** *There are factors of automorphy corresponding to classical theta functions. For any theta-characteristic  $\xi = a\tau + b$ , where  $a, b \in \mathbb{R}$ , there is a holomorphic function  $\theta_\xi : \mathbb{C} \rightarrow \mathbb{C}$  defined by*

$$\theta_\xi(z) = \sum_{n \in \mathbb{Z}} \exp(\pi i(n+a)^2 \tau) \exp(2\pi i(n+a)(z+b)),$$

*which satisfies*

$$\theta_\xi(\gamma + z) = \exp(2\pi i a \gamma - \pi i p^2 \tau - 2\pi i p(z + \xi)) \theta_\xi(z) = e_\xi(\gamma, z) \theta_\xi(z),$$

*where  $\gamma = p\tau + q$  and  $e_\xi(\gamma, z) = \exp(2\pi i a \gamma - \pi i p^2 \tau - 2\pi i p(z + \xi))$ . Since*

$$e_\xi(\gamma_1 + \gamma_2, z) = e_\xi(\gamma_1, \gamma_2 + z) e_\xi(\gamma_2, z),$$

*we conclude that  $e_\xi(\gamma, z)$  is a factor of automorphy.*

*By Theorem 3.2  $\theta_\xi(z)$  defines a section of  $E(e_\xi(\gamma, z))$ .*

*For more information on classical theta functions see [8, 9, 10].*

THEOREM 4.3.  $\deg E(e_\xi) = 1$ .

*Proof.* We know that sections of  $E(e_\xi)$  correspond to  $e_\xi$ -theta functions. The classical  $e_\xi$ -theta function  $\theta_\xi(z)$  defines a section  $s_\xi$  of  $E(e_\xi)$ . Since  $\theta_\xi$  has only simple zeros and the set of zeros of  $\theta_\xi(z)$  is  $\frac{1}{2} + \frac{\tau}{2} + \xi + \Gamma$ , we conclude that  $s_\xi$  has exactly one zero at point  $p = [\frac{1}{2} + \frac{\tau}{2} + \xi] \in X$ . Hence by [6, p. 136] we get  $E(e_\xi) \simeq [p]$  and thus  $\deg E(e_\xi) = 1$ .  $\square$

THEOREM 4.4. *Let  $\xi$  and  $\eta$  be two theta-characteristics. Then*

$$E(e_\xi) \simeq t_{[\eta-\xi]}^* E(e_\eta),$$

where  $t_{[\eta-\xi]} : X \rightarrow X$ ,  $x \mapsto x + [\eta - \xi]$  is the translation by  $[\eta - \xi]$ .

*Proof.* As in the proof of Theorem 4.3  $E(e_\xi) \simeq [p]$  and  $E(e_\eta) = [q]$  for  $p = [\frac{1}{2} + \frac{\tau}{2} + \xi]$  and  $q = [\frac{1}{2} + \frac{\tau}{2} + \eta]$ . Since  $t_{[\eta-\xi]}p = q$ , we get

$$E(e_\xi) \simeq [p] \simeq t_{[\eta-\xi]}^*[q] \simeq t_{[\eta-\xi]}^* E(e_\eta),$$

which completes the proof.  $\square$

Now we are going to investigate the extensions of the type

$$0 \rightarrow X \times \mathbb{C} \rightarrow E \rightarrow X \times \mathbb{C} \rightarrow 0.$$

In this case the transition functions are given by matrices of the type

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

and  $E$  is isomorphic to  $E(f)$  for some factor of automorphy  $f$  of the form

$$f(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}.$$

Note that the condition for  $f$  to be a factor of automorphy in this case is equivalent to the condition

$$\mu(\lambda + \lambda', \tilde{x}) = \mu(\lambda, \lambda' + \tilde{x}) + \mu(\lambda', \tilde{x}),$$

where we use the additive notation for the group operation since  $\Gamma$  is commutative.

THEOREM 4.5.  *$f$  defines the trivial bundle if and only if  $\mu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$  for some holomorphic function  $\xi : \mathbb{C} \rightarrow \mathbb{C}$ .*

*Proof.* We know that  $E$  is trivial if and only if  $h(\lambda\tilde{x}) = f(\lambda, \tilde{x})h(\tilde{x})$  for some holomorphic function  $h : \tilde{X} \rightarrow \mathrm{GL}_2(\mathbb{C})$ . Let

$$h = \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix},$$

then the last condition is

$$\begin{aligned} \begin{pmatrix} a(\lambda\tilde{x}) & b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & d(\lambda\tilde{x}) \end{pmatrix} &= \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} \\ &= \begin{pmatrix} a(\tilde{x}) + c(\tilde{x})\mu(\lambda, \tilde{x}) & b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}. \end{aligned}$$

In particular it means  $c(\lambda\tilde{x}) = c(\tilde{x})$  and  $d(\lambda\tilde{x}) = d(\tilde{x})$ , i.e.,  $c$  and  $d$  are doubly periodic functions on  $\tilde{X} = \mathbb{C}$ , so they should be constant, i.e.,  $c(\lambda, \tilde{x}) = c \in \mathbb{C}$ ,  $d(\lambda, \tilde{x}) = d \in \mathbb{C}$ .

Now we have

$$\begin{aligned} a(\tilde{x}) + c\mu(\lambda, \tilde{x}) &= a(\lambda\tilde{x}) \\ b(\tilde{x}) + d\mu(\lambda, \tilde{x}) &= b(\lambda\tilde{x}) \end{aligned}$$

which implies

$$\begin{aligned} c\mu(\lambda, \tilde{x}) &= a(\lambda\tilde{x}) - a(\tilde{x}) \\ d\mu(\lambda, \tilde{x}) &= b(\lambda\tilde{x}) - b(\tilde{x}). \end{aligned}$$

Since  $\det h(\tilde{x}) \neq 0$  for all  $\tilde{x} \in \tilde{X} = \mathbb{C}$  one of the numbers  $c$  and  $d$  is not equal to zero. Therefore, one concludes that  $\mu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$  for some holomorphic function  $\xi : \tilde{X} = \mathbb{C} \rightarrow \mathbb{C}$ .

Now suppose  $\mu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x})$  for some holomorphic function  $\xi : \mathbb{C} \rightarrow \mathbb{C}$ . Clearly for  $h(\tilde{x}) = \begin{pmatrix} 1 & \xi(\tilde{x}) \\ 0 & 1 \end{pmatrix}$  one has that  $\det h(\tilde{x}) = 1 \neq 0$  and

$$\begin{aligned} f(\lambda, \tilde{x})h(\tilde{x}) &= \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi(\tilde{x}) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \xi(\tilde{x}) + \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \xi(\lambda\tilde{x}) \\ 0 & 1 \end{pmatrix} = h(\lambda\tilde{x}). \end{aligned}$$

We have shown, that  $f$  defines the trivial bundle. This proves the statement of the theorem.  $\square$

**THEOREM 4.6.** *Two factors of automorphy*

$$f(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f'(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$$

defining non-trivial bundles are equivalent if and only if

$$\mu(\lambda, \tilde{x}) - k\nu(\lambda, \tilde{x}) = \xi(\lambda\tilde{x}) - \xi(\tilde{x}), \quad k \in \mathbb{C}, \quad k \neq 0$$

for some holomorphic function  $\xi : \mathbb{C} = \tilde{X} \rightarrow \mathbb{C}$ .

*Proof.* Suppose that the factors of automorphy

$$f(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$$

and

$$f'(\lambda, \tilde{x}) = \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$$

are equivalent. Then there is an equality  $f(\lambda, \tilde{x})h(\tilde{x}) = h(\lambda\tilde{x})f(\lambda, \tilde{x})$  for some holomorphic function  $h : \mathbb{C} = \tilde{X} \rightarrow \text{GL}_2(\mathbb{C})$ . Let us write  $h$  in the form

$$h(\tilde{x}) = \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix}.$$

Then the condition for equivalence of  $f$  and  $f'$  can be rewritten as follows:

$$\begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a(\tilde{x}) & b(\tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} = \begin{pmatrix} a(\lambda\tilde{x}) & b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & d(\lambda\tilde{x}) \end{pmatrix} \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix}$$

After multiplication one obtains

$$\begin{pmatrix} a(\tilde{x}) + c(\tilde{x})\mu(\lambda, \tilde{x}) & b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) \\ c(\tilde{x}) & d(\tilde{x}) \end{pmatrix} = \begin{pmatrix} a(\lambda\tilde{x}) & a(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + b(\lambda\tilde{x}) \\ c(\lambda\tilde{x}) & c(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + d(\lambda\tilde{x}) \end{pmatrix},$$

which leads to the system of equations

$$\begin{cases} a(\tilde{x}) + c(\tilde{x})\mu(\lambda, \tilde{x}) = a(\lambda\tilde{x}) \\ b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) = a(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + b(\lambda\tilde{x}) \\ c(\tilde{x}) = c(\lambda\tilde{x}) \\ d(\tilde{x}) = c(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + d(\lambda\tilde{x}). \end{cases}$$

The third equation means that  $c$  is a double periodic function. Therefore,  $c$  should be a constant function.

If  $c \neq 0$  from the first and the last equations using Theorem 4.5 one concludes that  $f$  and  $f'$  define the trivial bundle.

In the case  $c = 0$  one has

$$\begin{cases} a(\tilde{x}) = a(\lambda\tilde{x}) \\ b(\tilde{x}) + d(\tilde{x})\mu(\lambda, \tilde{x}) = a(\lambda\tilde{x})\nu(\lambda, \tilde{x}) + b(\lambda\tilde{x}) \\ d(\tilde{x}) = d(\lambda\tilde{x}), \end{cases}$$

i.e., as above,  $a$  and  $d$  are constant and both not equal to zero since  $\det(h) \neq 0$ . Finally one concludes that

$$d\mu(\lambda, \tilde{x}) - a\nu(\lambda, \tilde{x}) = b(\lambda\tilde{x}) - b(\tilde{x}), \quad a, d \in \mathbb{C}, \quad ad \neq 0 \quad (1)$$

Vice versa, if  $\mu$  and  $\nu$  satisfy (1) for

$$h(\tilde{x}) = \begin{pmatrix} a & b(\tilde{x}) \\ 0 & d \end{pmatrix}$$

we have

$$\begin{aligned} f(\lambda, \tilde{x})h(\tilde{x}) &= \begin{pmatrix} 1 & \mu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b(\tilde{x}) \\ 0 & d \end{pmatrix} \\ &= \begin{pmatrix} a & b(\tilde{x}) + d\mu(\lambda, \tilde{x}) \\ 0 & d \end{pmatrix} \\ &= \begin{pmatrix} a & b(\lambda\tilde{x}) + a\nu(\lambda, \tilde{x}) \\ 0 & d \end{pmatrix} \\ &= \begin{pmatrix} a & b(\lambda\tilde{x}) \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & \nu(\lambda, \tilde{x}) \\ 0 & 1 \end{pmatrix} = h(\lambda\tilde{x})f(\lambda, \tilde{x}). \end{aligned}$$

This means that  $f$  and  $f'$  are equivalent.  $\square$

## 4.2. Higher Dimensional Complex Tori

One can also consider higher dimensional complex tori. Let  $\Gamma \subset \mathbb{C}^g$  be a lattice,

$$\Gamma = \Gamma_1 \times \cdots \times \Gamma_g, \quad \Gamma_i = \mathbb{Z} + \mathbb{Z}\tau_i, \quad \text{Im } \tau > 0.$$

Then as for one dimensional complex tori we obtain that  $X = \mathbb{C}^g/\Gamma$  is a complex manifold. Clearly the map

$$\mathbb{C}^g \rightarrow \mathbb{C}^g/\Gamma = X, \quad x \mapsto [x]$$

is the universal covering of  $X$ . Since all vector bundles on  $\mathbb{C}^g$  are trivial, we obtain a one-to-one correspondence between equivalence classes of  $r$ -dimensional factors of automorphy

$$f : \Gamma \times \mathbb{C}^g \rightarrow \text{GL}_r(\mathbb{C})$$

and vector bundles of rank  $r$  on  $X$ .

Let  $\Gamma = \mathbb{Z}^g + \Omega\mathbb{Z}^g$ , where  $\Omega$  is a symmetric complex  $g \times g$  matrix with positive definite real part. Note that  $\Omega$  is a generalization of  $\tau$  from one dimensional case.

For any theta-characteristic  $\xi = \Omega a + b$ , where  $a \in \mathbb{R}^g$ ,  $b \in \mathbb{R}^g$  there is a holomorphic function  $\theta_\xi : \mathbb{C}^g \rightarrow \mathbb{C}$  defined by

$$\theta_\xi(z) = \theta_b^a(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i(n+a)^t \Omega(n+a) \tau) \exp(2\pi i(n+a)^t \Omega(z+b)),$$

which satisfies

$$\theta_\xi(\gamma + z) = \exp(2\pi i a^t \gamma - \pi i p^t \Omega p - 2\pi i p^t (z + \xi)) \theta_\xi(z) = e_\xi(\gamma, z) \theta_\xi(z),$$

where  $\gamma = \Omega p + q$  and  $e_\xi(\gamma, z) = \exp(2\pi i a^t \gamma - \pi i p^t \Omega p - 2\pi i p^t (z + \xi))$ . Since

$$e_\xi(\gamma_1 + \gamma_2, z) = e_\xi(\gamma_1, \gamma_2 + z) e_\xi(\gamma_2, z),$$

we conclude that  $e_\xi(\gamma, z)$  is a factor of automorphy.

As above  $\theta_\xi(z)$  defines a section of  $E(e_\xi(\gamma, z))$ .

For more detailed information on higher dimensional theta functions see [8, 9, 10].

### 4.3. Factors of Automorphy depending only on the $\tau$ -Direction of the Lattice $\Gamma$

Here  $X$  is a complex torus,  $X = \mathbb{C}/\Gamma$ ,  $\Gamma = \mathbb{Z}\tau + \mathbb{Z}$ ,  $\text{Im } \tau > 0$ . Denote  $q = e^{2\pi i \tau}$ . Consider the canonical projection

$$\text{pr} : \mathbb{C}^* \rightarrow \mathbb{C}^* / \langle q \rangle, \quad u \rightarrow [u] = u \langle q \rangle.$$

Clearly one can equip  $\mathbb{C}^* / \langle q \rangle$  with the quotient topology. Therefore, there is a natural complex structure on  $\mathbb{C}^* / \langle q \rangle$ .

Consider the homomorphism

$$\mathbb{C} \xrightarrow{\text{exp}} \mathbb{C}^* \xrightarrow{\text{pr}} \mathbb{C}^* / \langle q \rangle, \quad z \mapsto e^{2\pi i z} \mapsto [e^{2\pi i z}].$$

It is clearly surjective. An element  $z \in \mathbb{C}$  is in the kernel of this homomorphism if and only if  $e^{2\pi i z} = q^k = e^{2\pi i k \tau}$  for some integer  $k$ . But this holds if and only if  $z - k\tau \in \mathbb{Z}$  or, in other words, if  $z \in \Gamma$ . Therefore, the kernel of the map is exactly  $\Gamma$ , and we obtain an isomorphism of groups

$$\text{iso} : \mathbb{C}/\Gamma \rightarrow \mathbb{C}^* / \langle q \rangle = \mathbb{C}^* / \mathbb{Z}, \quad [z] \rightarrow [e^{2\pi i z}].$$

Since the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\text{pr}} & \mathbb{C}/\Gamma \\ \text{exp} \downarrow & & \uparrow \text{iso} \\ \mathbb{C}^* & \xrightarrow{\text{pr}} & \mathbb{C}^* / \mathbb{Z} = \mathbb{C}^* / \langle q \rangle \end{array}$$



is commutative, we conclude that the complex structure on  $\mathbb{C}^*/\langle q \rangle$  inherited from  $\mathbb{C}/\Gamma$  by the isomorphism  $iso$  coincides with the natural complex structure on  $\mathbb{C}^*/\langle q \rangle$ . Therefore,  $iso$  is an isomorphism of complex manifolds. Thus complex tori can be represented as  $\mathbb{C}^*/\langle q \rangle$ , where  $q = e^{2\pi i\tau}$ ,  $\tau \in \mathbb{C}$ ,  $\text{Im } \tau > 0$ .

So for any complex torus  $X = \mathbb{C}^*/\langle q \rangle$  we have a natural surjective holomorphic map

$$\mathbb{C}^* \rightarrow \mathbb{C}^*/\langle q \rangle = X, \quad u \rightarrow [u].$$

This map is moreover a covering of  $X$ . Consider the group  $\mathbb{Z}$ . It acts holomorphically on  $X = \mathbb{C}^*$ :

$$\mathbb{Z} \times \mathbb{C}^* \rightarrow \mathbb{C}^*, \quad (n, u) \mapsto q^n u.$$

Moreover, since  $\text{pr}(q^n u) = \text{pr}(u)$ ,  $\mathbb{Z}$  is naturally identified with a subgroup in the group of deck transformations  $\text{Deck}(X/\mathbb{C}^*)$ . It is easy to see that  $\mathbb{Z}$  satisfies the property **(T)**. We obtain that there is a one-to-one correspondence between classes of isomorphism of vector bundles over  $X$  and classes of equivalence of factors of automorphy

$$f : \mathbb{Z} \times \mathbb{C} \rightarrow \text{GL}_r(\mathbb{C}).$$

Consider the following action of  $\Gamma$  on  $\mathbb{C}^*$ :

$$\Gamma \times \mathbb{C}^* \rightarrow \mathbb{C}^*; \quad (\lambda, u) \mapsto \lambda u =: e^{2\pi i\lambda} u$$

Let  $A : \Gamma \times \mathbb{C}^* \rightarrow \text{GL}_r(\mathbb{C})$  be a holomorphic function satisfying

$$A(\lambda + \lambda', u) = A(\lambda, \lambda' u)A(\lambda', u) \quad (*)$$

for all  $\lambda, \lambda' \in \Gamma$ . We call such functions  $\mathbb{C}^*$ -factors of automorphy. Consider the map

$$\text{id}_\Gamma \times \exp : \Gamma \times \mathbb{C} \rightarrow \Gamma \times \mathbb{C}^*, \quad (\lambda, x) \rightarrow (\lambda, e^{2\pi i x})$$

Then the function

$$f_A = A \circ (\text{id}_\Gamma \times \exp) : \Gamma \times \mathbb{C} \rightarrow \text{GL}_r(\mathbb{C})$$

is an  $r$ -dimensional factor of automorphy, because

$$\begin{aligned} f_A(\lambda + \lambda', x) &= A(\lambda + \lambda', e^{2\pi i x}) = A(\lambda, e^{2\pi i \lambda'} e^{2\pi i x})A(\lambda', e^{2\pi i x}) \\ &= A(\lambda, e^{2\pi i(\lambda' + x)})A(\lambda', e^{2\pi i x}) \\ &= f_A(\lambda, \lambda' + x)f_A(\lambda', x). \end{aligned}$$

So, factors of automorphy on  $\mathbb{C}^*$  define factors of automorphy on  $\mathbb{C}$ .

We restrict ourselves to factors of automorphy  $f : \Gamma \times \mathbb{C} \rightarrow \mathrm{GL}_r(\mathbb{C})$  with the property

$$f(m\tau + n, x) = f(m\tau, x), \quad m, n \in \mathbb{Z}. \quad (2)$$

It follows from this property that  $f(n, x) = f(0, x) = \mathrm{id}_{\mathbb{C}^r}$ . Therefore,

$$f(\lambda + k, x) = f(\lambda, k + x)f(k, x) = f(\lambda, k + x) \text{ for all } \lambda \in \Gamma, k \in \mathbb{Z}$$

and it is possible to define the function

$$A_f : \Gamma \times \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C}), \quad (\lambda, e^{2\pi i x}) \mapsto f(\lambda, x),$$

which is well-defined because from  $e^{2\pi i x_1} = e^{2\pi i x_2}$  follows  $x_1 = x_2 + k$  for some  $k \in \mathbb{Z}$  and  $f(\lambda, x_1) = f(\lambda, x_2 + k) = f(\lambda, x_2)$ .

Consider  $A$  with the property  $A(m\tau + n, u) = A(m\tau, u) =: A(m, u)$ . Then clearly  $f_A(m\tau + n, u) = f_A(m\tau, u)$ . So for any  $\mathbb{C}^*$ -factor of automorphy  $A : \Gamma \times \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$  with the property  $A(m\tau + n, u) = A(m\tau, u)$  one obtains the factor of automorphy  $f_A$  satisfying (2). We proved the following

**THEOREM 4.7.** *Factors of automorphy  $f : \Gamma \times \mathbb{C} \rightarrow \mathrm{GL}_r(\mathbb{C})$  with the property (2) are in a one-to-one correspondence with  $\mathbb{C}^*$ -factors of automorphy with property  $A(m\tau + n, u) = A(m\tau, u)$ .*

Now we want to translate the conditions for factors of automorphy with the property (2) to be equivalent in the language of  $\mathbb{C}^*$ -factors of automorphy with the same property.

**THEOREM 4.8.** *Let  $f, f'$  be  $r$ -factors of automorphy with the property (2). Then  $f \sim f'$  if and only if there exists a holomorphic function  $B : \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$  such that*

$$A_f(m, u)B(u) = B(q^m u)A_{f'}(m, u)$$

for  $q := e^{2\pi i \tau}$ , where  $A(m, u) := A(m\tau, u)$ . In this case we also say  $A_f$  is equivalent to  $A_{f'}$  and write  $A_f \sim A_{f'}$ .

*Proof.* Let  $f \sim f'$ . By definition it means that there exists a holomorphic function  $h : \mathbb{C} \rightarrow \mathrm{GL}_r(\mathbb{C})$  such that  $f(\lambda, x)h(x) = h(\lambda x)f'(\lambda, x)$ . Therefore, from  $f(n, x)h(x) = h(n + x)f'(n, x)$  and  $f(n, x) = f'(n, x) = \mathrm{id}_{\mathbb{C}^r}$  it follows  $h(x) = h(n + x)$  for all  $n \in \mathbb{Z}$ . Therefore, the function

$$B : \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C}), \quad e^{2\pi i x} \mapsto h(x)$$

is well-defined. We have

$$\begin{aligned} A_f(m, e^{2\pi i x})B(e^{2\pi i x}) &= f(m\tau, x)h(x) = h(m\tau + x)f'(m\tau, x) = \\ &= B(e^{2\pi i(m\tau + x)})f'(m, e^{2\pi i x}) = B(q^m e^{2\pi i x})A_{f'}(m, e^{2\pi i x}). \end{aligned}$$

Vice versa, let  $B$  be such that  $A_f(m, u)B(u) = B(q^m A_{f'}(m, u))$ . Define  $h = B \circ \exp$ . We obtain

$$\begin{aligned} f(m\tau + n, x)h(x) &= A_f(m\tau + n, e^{2\pi ix})B(e^{2\pi ix}) \\ &= B(q^m e^{2\pi ix})A_{f'}(m\tau + n, e^{2\pi ix}) = B(e^{2\pi i(m\tau+x)})A_{f'}(m\tau + n, e^{2\pi ix}) \\ &= B(e^{2\pi i(m\tau+n+x)})A_{f'}(m\tau + n, e^{2\pi ix}) = h(m\tau + n + x)f'(m\tau + n, x), \end{aligned}$$

which means that  $f \sim f'$  and completes the proof.  $\square$

REMARK 4.9. *The last two theorems allow us to embed the set  $Z^1(\mathbb{Z}, r)$  of factors of automorphy  $\mathbb{Z} \times X \rightarrow \mathrm{GL}_r(\mathbb{C})$  to the set  $Z^1(\Gamma, r)$ . The embedding is*

$$\Psi : Z^1(\mathbb{Z}, r) \rightarrow Z^1(\Gamma, r), \quad f \mapsto g, \quad g(n\tau + m, x) := f(n, x).$$

*Two factors of automorphy from  $Z^1(\mathbb{Z}, r)$  are equivalent if and only if their images under  $\Psi$  are equivalent in  $Z^1(\Gamma, r)$ . That is why it is enough to consider only factors of automorphy*

$$\Gamma \times \mathbb{C} \rightarrow \mathrm{GL}_r(\mathbb{C})$$

*satisfying (2).*

COROLLARY 4.10. *A factor of automorphy  $f$  with property (2) is trivial if and only if  $A_f(m, u) = B(q^m u)B(u)^{-1}$  for some holomorphic function  $B : \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$ .*

THEOREM 4.11. *Let  $A$  be a  $\mathbb{C}^*$ -factor of automorphy.  $A(m, u)$  is uniquely determined by  $A(u) := A(1, u)$ .*

$$A(m, u) = A(q^{m-1}u) \dots A(qu)A(u), \quad m > 0 \quad (3)$$

$$A(-m, u) = A(q^{-m}u)^{-1} \dots A(q^{-1}u)^{-1}, \quad m > 0. \quad (4)$$

*$A(m, u)$  is equivalent to  $A'(m, u)$  if and only if*

$$A(u)B(u) = B(qu)A'(u) \quad (5)$$

*for some holomorphic function  $B : \mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$ . In particular  $A(m, u)$  is trivial iff  $A(u) = B(qu)B(u)^{-1}$ .*

*Proof.* Since  $A(1, u) = A(u)$  the first formula holds for  $m = 1$ . Therefore,

$$A(m+1, u) = A(1, q^m u)A(m, u) = A(q^m)A(m, u)$$

and we prove the first formula by induction.

Now  $\text{id} = A(0, u) = A(m - m, u) = A(m, q^{-m}u)A(-m, u)$  and hence

$$\begin{aligned} A(-m, u) &= A(m, q^{-m}u)^{-1} = (A(q^{m-1}q^{-m}u) \dots A(qq^{-m}u)A(q^{-m}u))^{-1} \\ &= A(-m, u) = A(q^{-m}u)^{-1} \dots A(q^{-1}u)^{-1} \end{aligned}$$

which proves the second formula.

If  $A(m, u) \sim A'(m, u)$  then clearly (5) holds.

Vice versa, suppose  $A(u)B(u) = B(qu)A'(u)$ . Then

$$\begin{aligned} A(m, u)B(u) &= A(q^{m-1}u) \dots A(qu)A(u)B(u) \\ &= A(q^{m-1}u) \dots A(qu)B(qu)A'(u) \\ &= \dots \\ &= B(q^m u)A'(q^{m-1}u) \dots A'(qu)A'(u) \\ &= B(q^m u)A'(m, u) \end{aligned}$$

for  $m > 0$ .

Since  $A(-m, u) = A(m, q^{-m}u)^{-1}$  we have

$$\begin{aligned} A(-m, u)B(u) &= A(m, q^{-m}u)^{-1}B(u) = (B(u)^{-1}A(m, q^{-m}u))^{-1} \\ &= (B(u)^{-1}A(m, q^{-m}u)B(q^{-m}u)B(q^{-m}u)^{-1})^{-1} \\ &= (B(u)^{-1}B(q^m q^{-m}u)A'(m, q^{-m}u)B(q^{-m}u)^{-1})^{-1} \\ &= (B(u)^{-1}B(u)A'(m, q^{-m}u)B(q^{-m}u)^{-1})^{-1} \\ &= B(q^{-m}u)A'(m, q^{-m}u)^{-1} = B(q^{-m}u)A'(-m, u), \end{aligned}$$

which completes the proof.  $\square$

**REMARK 4.12.** *Theorem 4.11 means that all the information about a vector bundle of rank  $r$  on a complex torus can be encoded by a holomorphic function  $\mathbb{C}^* \rightarrow \text{GL}_r(\mathbb{C})$ .*

For a holomorphic function  $A : \mathbb{C}^* \rightarrow \text{GL}_r(\mathbb{C})$ , let us denote by  $E(A)$  the corresponding vector bundle on  $X$ .

**THEOREM 4.13.** *Let  $A : \mathbb{C}^* \rightarrow \text{GL}_n(\mathbb{C})$ ,  $B : \mathbb{C}^* \rightarrow \text{GL}_m(\mathbb{C})$  be two holomorphic maps. Then  $E(A) \otimes E(B) \simeq E(A \otimes B)$ .*

*Proof.* By theorem 3.5 we have

$$E(A) \otimes E(B) \simeq E(A(n, u)) \otimes E(B(n, u)) \simeq E(A(n, u) \otimes B(n, u)).$$

Since  $A(1, u) \otimes B(1, u) = A(u) \otimes B(u)$ , we obtain  $E(A) \otimes E(B) \simeq E(A \otimes B)$ .  $\square$

## 5. Classification of Vector Bundles over a Complex Torus

Here we work with factors of automorphy depending only on  $\tau$ , i.e., with holomorphic functions  $\mathbb{C}^* \rightarrow \mathrm{GL}_r(\mathbb{C})$ .

### 5.1. Vector Bundles of Degree Zero

We return to extensions of the type  $0 \rightarrow I_1 \rightarrow E \rightarrow I_1 \rightarrow 0$ , where  $I_1$  denotes the trivial vector bundle of rank 1.

Theorem 4.5 can be rewritten as follows.

**THEOREM 5.1.** *A function*

$$A(u) = \begin{pmatrix} 1 & a(u) \\ 0 & 1 \end{pmatrix}$$

*defines the trivial bundle if and only if  $a(u) = b(qu) - b(u)$  for some holomorphic function  $b : \mathbb{C}^* \rightarrow \mathbb{C}$ .*

**COROLLARY 5.2.**

$$A(u) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

*defines a non-trivial vector bundle.*

*Proof.* Suppose  $A$  defines the trivial bundle. Then  $1 = b(qu) - b(u)$  for some holomorphic function  $b : \mathbb{C}^* \rightarrow \mathbb{C}$ . Considering the Laurent series expansion  $\sum_{-\infty}^{+\infty} b_k u^k$  of  $b$  we obtain  $1 = b_0 - b_0 = 0$  which shows that our assumption was false.  $\square$

Let  $a : \mathbb{C}^* \rightarrow \mathbb{C}$  be a holomorphic function such that

$$A_2(u) = \begin{pmatrix} 1 & a(u) \\ 0 & 1 \end{pmatrix}$$

defines non-trivial bundle, i.e., by Theorem 5.1, there is no holomorphic function  $b : \mathbb{C}^* \rightarrow \mathbb{C}$  such that

$$a(u) = b(qu) - b(u).$$

Let  $F_2$  be the bundle defined by  $A_2$ . Then by Theorem 3.3 there exists an exact sequence

$$0 \rightarrow I_1 \rightarrow F_2 \rightarrow I_1 \rightarrow 0.$$

For  $n \geq 3$  we define  $A_n : \mathbb{C}^* \rightarrow \mathrm{GL}_n(\mathbb{C})$ ,

$$A_n = \begin{pmatrix} 1 & a & & \\ & \ddots & \ddots & \\ & & 1 & a \\ & & & 1 \end{pmatrix},$$

where empty entries stay for zeros.

Let  $F_n$  be the bundle defined by  $A_n$ . By (3.3) one sees that  $A_n$  defines the extension

$$0 \rightarrow I_1 \rightarrow F_n \rightarrow F_{n-1} \rightarrow 0.$$

**THEOREM 5.3.**  *$F_n$  is not the trivial bundle. The extension*

$$0 \rightarrow I_1 \rightarrow F_n \rightarrow F_{n-1} \rightarrow 0.$$

*is non-trivial for all  $n \geq 2$ .*

*Proof.* Suppose  $F_n$  is trivial. Then  $A_n(u)B(u) = B(qu)$  for some  $B = (b_{ij})_{ij}^n$ . In particular it means  $b_{ni}(u) = b_{ni}(qu)$  for  $i = \overline{1, n}$ . Let  $b_{ni} = \sum_{-\infty}^{+\infty} b_k^{(ni)} u^k$  be the expansion of  $b_{ni}$  in Laurent series. Then  $b_{ni}(u) = b_{ni}(qu)$  implies  $b_k^{(ni)} = q^k b_k^{(ni)}$  for all  $k$ .

Note that  $|q| < 1$  because  $\tau = \xi + i\eta$ ,  $\eta > 0$  and

$$|q| = |e^{2\pi i\tau}| = |e^{2\pi i(\xi+i\eta)}| = |e^{2\pi i\xi} e^{-2\pi\eta}| = e^{-2\pi\eta} < 1.$$

Therefore,  $b_k^{(ni)} = 0$  for  $k \neq 0$  and we conclude that  $b_{ni}$  should be constant functions.

We also have

$$b_{n-1i}(u) + b_{ni}a(n) = b_{n-1i}(qu).$$

Since at least one of  $b_{ni}$  is not equal to zero because of invertibility of  $B$ , we obtain

$$a(u) = \frac{1}{b_{ni}}(b_{n-1i}(qu) - b_{n-1i}(u))$$

for some  $i$ , which contradicts the choice of  $a$ . Therefore,  $F_n$  is not trivial.

Assume now, that for some  $n > 2$  the extension

$$0 \rightarrow I_1 \rightarrow F_n \rightarrow F_{n-1} \rightarrow 0$$

is trivial (for  $n = 2$  it is not trivial since  $F_2$  is not a trivial vector bundle). This means

$$A_n \sim \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix},$$

i.e., there exists a holomorphic function  $B : \mathbb{C}^* \rightarrow \text{GL}_n(\mathbb{C})$ ,  $B = (b_{ij})_{i,j}^n$  such that

$$A_n(u)B(u) = B(qu) \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix}.$$

Considering the elements of the first and second columns we obtain for the first column

$$b_{n1}(u) = b_{n1}(qu),$$

$$b_{i1}(u) + b_{i+11}(u)a(u) = b_{i1}(qu), \quad i < n$$

and for the second column

$$b_{n2}(u) = b_{n2}(qu),$$

$$b_{i2}(u) + b_{i+12}(u)a(u) = b_{i2}(qu), \quad i < n.$$

For the first column as above considering Laurent series we have that  $b_{n1}$  should be a constant function. If  $b_{n1} \neq 0$  it follows

$$a(u) = \frac{1}{b_{n1}}(b_{n-11}(qu) - b_{n-11}(u)),$$

which contradicts the choice of  $a$ . Therefore,  $b_{n1} = 0$  and  $b_{n-11}(qu) = b_{n-11}(u)$ , in other words  $b_{n-11}$  is a constant function. Proceeding by induction one obtains that  $b_{i1}$  is a constant function and  $b_{i1} = 0$  for  $i > 1$ .

For the second column absolutely analogously we obtain a similar result:  $b_{i2}$  is constant,  $b_{i2} = 0$  for  $i > 1$ . This contradicts the invertibility of  $B(u)$  and proves the statement.  $\square$

**COROLLARY 5.4.** *The vector bundle  $F_n$  is the only indecomposable vector bundle of rank  $n$  and degree 0 that has non-trivial sections.*

*Proof.* This follows from [1, Theorem 5].  $\square$

So we have that the vector bundles  $F_n = E(A_n)$  are exactly  $F_n$ 's defined by Atiyah in [1].

**REMARK 5.5.** *Note that constant matrices  $A$  and  $B$  having the same Jordan normal form are equivalent. This is clear because  $A = SBS^{-1}$  for some constant invertible matrix  $S$ , which means that  $A$  and  $B$  are equivalent.*

Consider an upper triangular matrix  $B = (b_{ij})_1^n$  of the following type:

$$b_{ii} = 1, \quad b_{ii+1} \neq 0. \tag{6}$$

It is easy to see that this matrix is equivalent to the upper triangular matrix  $A$ ,

$$a_{ii} = a_{ii+1} = 1, \quad a_{ij} = 0, \quad j \neq i + 1, \quad j \neq i. \tag{7}$$

In fact, these matrices have the same characteristic polynomial  $(t-1)^n$  and the dimension of the eigenspace corresponding to the eigenvalue 1 is equal to 1 for both matrices. Therefore,  $A$  and  $B$  have the same Jordan form. By Remark above we obtain that  $A$  and  $B$  are equivalent. We proved the following:

**LEMMA 5.6.** *A matrix satisfying (6) is equivalent to the matrix defined by (7). Moreover, two matrices of the type (6) are equivalent, i.e., they define two isomorphic vector bundles.*

THEOREM 5.7.  $F_n \simeq S^{n-1}(F_2)$ .

*Proof.* We know that  $F_2$  is defined by the constant matrix

$$A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We know by Theorem 3.4 that  $S^n(F_2)$  is defined by  $S^n(A_2)$ . We calculate  $S^n(f_2)$  for  $n \in \mathbb{N}_0$ . Since  $f_2$  is a constant matrix,  $S^n(f_2)$  is also a constant matrix defining a map  $S^n(\mathbb{C}^2) \rightarrow S^n(\mathbb{C}^2)$ . Let  $e_1, e_2$  be the standard basis of  $\mathbb{C}^2$ , then  $S^n(\mathbb{C})$  has a basis

$$\{e_1^k e_2^{n-k} \mid k = n, n-1, \dots, 0\}.$$

Since  $A_2(e_1) = e_1$  and  $A_2(e_2) = e_1 + e_2$ , we conclude that  $e_1^k e_2^{n-k}$  is mapped to

$$\begin{aligned} A_2(e_1)^k A_2(e_2)^{n-k} &= e_1^k (e_1 + e_2)^{n-k} \\ &= e_1^k \sum_{i=0}^{n-k} \binom{n-k}{i} e_1^{n-k-i} e_2^i = \sum_{i=0}^{n-k} \binom{n-k}{i} e_1^{n-i} e_2^i. \end{aligned}$$

Therefore,

$$S^n(A_2) = \begin{pmatrix} 1 & 1 & 1 & \dots & \binom{n}{0} \\ & 1 & 2 & \dots & \binom{n}{1} \\ & & 1 & \dots & \binom{n}{2} \\ & & & \ddots & \vdots \\ & & & & \binom{n}{n} \end{pmatrix},$$

where empty entries stay for zero. In other words, the columns of  $S^n(A_2)$  are columns of binomial coefficients. By Lemma 5.6 we conclude that  $S^n(A_2)$  is equivalent to  $A_{n+1}$ . This proves the statement of the theorem.  $\square$

Let  $E$  be a 2-dimensional vector bundle over a topological space  $X$ . Then there exists an isomorphism

$$S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus (\det E \otimes S^{p-1}(E) \otimes S^{q-1}(E)).$$

This is the Clebsch-Gordan formula. If  $\det E$  is the trivial line bundle, then we have  $S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus S^{p-1}(E) \otimes S^{q-1}(E)$ , and by iterating one gets

$$S^p(E) \otimes S^q(E) \simeq S^{p+q}(E) \oplus S^{p+q-2}(E) \oplus \dots \oplus S^{p-q}(E), \quad p \geq q. \quad (8)$$

THEOREM 5.8.  $F_p \otimes F_q \simeq F_{p+q-1} \oplus F_{p+q-3} \oplus \dots \oplus F_{p-q+1}$  for  $p \geq q$ .



*Proof.* Using Theorem 5.7 and (8) we obtain

$$\begin{aligned} F_p \otimes F_q &\simeq S^{p-1}(F_2) \otimes S^{q-1}(F_2) \\ &\simeq S^{p+q-2}(F_2) \oplus S^{p+q-4}(F_2) \oplus \cdots \oplus S^{p-q}(F_2) \\ &\simeq F_{p+q-1} \oplus F_{p+q-3} \oplus \cdots \oplus F_{p-q+1}. \end{aligned}$$

This completes the proof.  $\square$

REMARK 5.9. *The possibility of proving the last theorem using Theorem 5.7 is exactly what Atiyah states in remark 1) after Theorem 9 (see [1, p. 439]).*

We have already given (Corollary 5.4) a description of vector bundles of degree zero with non-trivial sections. We give now a description of all vector bundles of degree zero.

Consider the function  $\varphi_0(z) = \exp(-\pi i\tau - 2\pi iz) = q^{-1/2}u^{-1} = \varphi(u)$ , where  $u = e^{2\pi iz}$ . It defines the factor of automorphy

$$e_0(p\tau + q, z) = \exp(-\pi ip^2\tau - 2\pi izp) = q^{-\frac{p^2}{2}} u^{-p}$$

corresponding to the theta-characteristic  $\xi = 0$ .

THEOREM 5.10. *deg  $E(\varphi_0) = 1$ , where as above  $\varphi_0(z) = \exp(-\pi i\tau - 2\pi iz) = q^{-1/2}u^{-1} = \varphi(u)$ .*

*Proof.* Follows from Theorem 4.3 for  $\xi = 0$ .  $\square$

THEOREM 5.11. *Let  $L' \in \mathcal{E}(1, d)$ . Then there exists  $x \in X$  such that  $L' \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{d-1}$ .*

*Proof.* Since  $E(\varphi_0)^d$  has degree  $d$ , we obtain that there exists  $\tilde{L} \in \mathcal{E}(1, 0)$  such that  $L' \simeq E(\varphi_0)^d \otimes \tilde{L}$ . We also know that  $\tilde{L} \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1}$  (cf. proof of Theorem 4.3 and Theorem 4.4) for some  $x \in X$ . Combining these one obtains

$$L' \simeq E(\varphi_0)^d \otimes t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1} \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{d-1}.$$

This proves the required statement.  $\square$

THEOREM 5.12. *The map*

$$\mathbb{C}^* / \langle q \rangle \rightarrow \text{Pic}^0(X), \quad \bar{a} \mapsto E(a).$$

*is well-defined and is an isomorphism of groups.*

*Proof.* Let  $\varphi_0(z) = \exp(-\pi i\tau - 2\pi iz)$  as above. For  $x \in X$  consider  $t_x^*E(\varphi_0)$ , where the map

$$t_x : X \rightarrow X, \quad y \mapsto y + x$$

is the translation by  $x$ . Let  $\xi \in \mathbb{C}$  be a representative of  $x$ . Clearly,  $t_x^*E(\varphi_0)$  is defined by

$$\varphi_{0\xi}(z) = t_x^*\varphi_0(z) = \varphi_0(z + \xi) = \exp(-\pi i\tau - 2\pi iz - 2\pi i\xi) = \varphi_0(z)\exp(-2\pi i\xi).$$

(Note that if  $\eta$  is another representative of  $x$ , then  $\varphi_{0\xi}$  and  $\varphi_{0\eta}$  are equivalent.) Therefore, the bundle  $t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1}$  is defined by

$$(\varphi_{0\xi}\varphi_0^{-1})(z) = \varphi_0(z)\exp(-2\pi i\xi)\varphi_0^{-1}(z) = \exp(-2\pi i\xi).$$

Since for any  $L \in \mathcal{E}(1, 0)$  there exists  $x \in X$  such that  $L \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1}$ , we obtain  $L \simeq E(a)$  for  $a = \exp(-2\pi i\xi) \in \mathbb{C}^*$ , where  $\xi \in \mathbb{C}$  is a representative of  $x$ . We proved that any line bundle of degree zero is defined by a constant function  $a \in \mathbb{C}^*$ .

Vice versa, let  $L = E(a)$  for  $a \in \mathbb{C}^*$ . Clearly, there exists  $\xi \in \mathbb{C}$  such that  $a = \exp(-2\pi i\xi)$ . Therefore,

$$L \simeq E(a) \simeq L(\varphi_{0\xi}\varphi_0^{-1}) \simeq t_x^*E(\varphi_0) \otimes E(\varphi_0)^{-1},$$

where  $x$  is the class of  $\xi$  in  $X$ , which implies that  $E(a)$  has degree zero. So we obtained that the line bundles of degree zero are exactly the line bundles defined by constant functions.

We have the map

$$\phi : \mathbb{C}^* \rightarrow \text{Pic}^0(X), \quad a \mapsto E(a),$$

which is surjective. By Theorem 4.13 it is moreover a homomorphism of groups. We are looking now for the kernel of this map.

Suppose  $E(a)$  is a trivial bundle. Then there exists a holomorphic function  $f : \mathbb{C}^* \rightarrow \mathbb{C}^*$  such that  $f(qu) = af(u)$ . Let  $f = \sum f_\nu a^\nu$  be the Laurent series expansion of  $f$ . Then from  $f(qu) = af(u)$  one obtains

$$af_\nu = f_\nu q^\nu \text{ for all } \nu \in \mathbb{Z}.$$

Therefore,  $f_\nu(a - q^\nu) = 0$  for all  $\nu \in \mathbb{Z}$ .

Since  $f \not\equiv 0$ , we obtain that there exists  $\nu \in \mathbb{Z}$  with  $f_\nu \neq 0$ . Hence  $a = q^\nu$  for some  $\nu \in \mathbb{Z}$ .

Vice versa, if  $a = q^\nu$ , for  $f(u) = u^\nu$  we get

$$f(qu) = q^\nu u^\nu = af(u).$$

This means that  $E(a)$  is the trivial bundle, which proves  $\text{Ker } \phi = \langle q \rangle$ . We obtain the required isomorphism

$$\mathbb{C}^* / \langle q \rangle \rightarrow \text{Pic}^0(X), \quad \bar{a} \mapsto E(a).$$

This completes the proof.  $\square$

**THEOREM 5.13.** *For any  $F \in \mathcal{E}(r, 0)$  there exists a unique  $\bar{a} \in \mathbb{C}^* / \langle q \rangle$  such that  $F \simeq E(A_r(a))$ , where*

$$A_r(a) = \begin{pmatrix} a & 1 & & & \\ & \ddots & \ddots & & \\ & & & a & 1 \\ & & & & a \end{pmatrix}.$$

*Proof.* By [1, Theorem 5]  $F \simeq F_r \otimes L$  for a unique  $L \in \mathcal{E}(1, 0)$ . Since  $F_r \simeq E(A_r)$  and  $L \simeq E(a)$  for a unique  $\bar{a} \in \mathbb{C}^* / \langle q \rangle$  we get  $F \simeq E(A_r \otimes a)$ . So  $F$  is defined by the matrix

$$\begin{pmatrix} a & a & & & \\ & \ddots & \ddots & & \\ & & & a & a \\ & & & & a \end{pmatrix},$$

where empty entries stay for zeros. It is easy to see that the Jordan normal form of this matrix is

$$\begin{pmatrix} a & 1 & & & \\ & \ddots & \ddots & & \\ & & & a & 1 \\ & & & & a \end{pmatrix}.$$

This proves the statement of the theorem.  $\square$

## 5.2. Vector Bundles of Arbitrary Degree

Denote by  $E_\tau = \mathbb{C}/\Gamma_\tau$ , where  $\Gamma_\tau = \mathbb{Z}\tau + \mathbb{Z}$ . Consider the  $r$ -covering

$$\pi_r : E_{r\tau} \rightarrow E_\tau, \quad [x] \mapsto [x].$$

**THEOREM 5.14.** *Let  $F$  be a vector bundle of rank  $n$  on  $E_\tau$  defined by  $A(u) = A(1, u) = A(\tau, u)$ . Then  $\pi_r^*(F)$  is defined by*

$$\tilde{A}(r\tau, u) = \tilde{A}(u) = \tilde{A}(1, u) := A(r\tau, u) = A(q^{r-1}u) \dots A(qu)A(u).$$

*Proof.* Consider the following commutative diagram.

$$\begin{array}{ccc} & \mathbb{C} & \\ p_{r\tau} \swarrow & & \searrow p_\tau \\ E_{r\tau} & \xrightarrow{\pi_r} & E_\tau \end{array}$$

Consider the map

$$\begin{aligned} E(\tilde{A}) \rightarrow \pi_r^*(E(A)) &= E_{r\tau} \times_{E_\tau} E(A) = \{([z]_{r\tau}, [z, v]_\tau) \in E_{r\tau} \times E(A)\}, \\ [z, v]_{r\tau} &\mapsto ([z]_{r\tau}, [z, v]_\tau). \end{aligned}$$

It is clearly bijective. It remains to prove that it is biholomorphic. From the construction of  $E(A)$  and  $E(\tilde{A})$  it follows that the diagram

$$\begin{array}{ccc} E(\tilde{A}) & & \\ \searrow & & \\ \pi_r^*(E(A)) & \longrightarrow & E(A) \\ \downarrow & & \downarrow \\ E_{r\tau} & \longrightarrow & E_\tau \end{array}$$

locally looks as

$$\begin{array}{ccc} U \times \mathbb{C}^n & & (z, v) \\ \searrow & & \searrow \\ \Delta(U \times U) \times \mathbb{C}^n & \longrightarrow & U \times \mathbb{C}^n, & ((z, z), v) & \longmapsto & (z, v), \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ U & \xlongequal{\quad} & U & z & \xlongequal{\quad} & z \end{array}$$

where  $\Delta(U \times U)$  denotes the diagonal of  $U \times U$ .

This proves the required statement.  $\square$

**THEOREM 5.15.** *Let  $F$  be a vector bundle of rank  $n$  on  $E_{r\tau}$  defined by  $\tilde{A}(u) = \tilde{A}(r\tau, u)$ . Then  $\pi_{r*}(F)$  is defined by*

$$A(u) = \begin{pmatrix} 0 & I_{(r-1)n} \\ \tilde{A}(u) & 0 \end{pmatrix}.$$

*Proof.* Consider the following commutative diagram.

$$\begin{array}{ccc} & \mathbb{C} & \\ p_{r\tau} \swarrow & & \searrow p_\tau \\ E_{r\tau} & \xrightarrow{\pi_r} & E_\tau \end{array}$$

Let  $z \in \mathbb{C}$ . Consider  $y = p_{r\tau}(z) \in E_{r\tau}$  and  $x = p_\tau(z) = \pi_\tau p_{r\tau}(z) \in E_\tau$ .

Choose a point  $b \in \mathbb{C}$  such that  $z \in V_b$ , where  $V_b$  is the standard parallelogram at point  $b$ . Clearly  $x \in U_b = p_r(V_b)$  and we have the isomorphism  $\varphi_b : U_b \rightarrow V_b$  with  $\varphi_b(x) = z$ .

Consider  $\pi_r^{-1}(U_b) = W_b \sqcup \cdots \sqcup W_{b+(r-1)\tau}$ , where  $y \in W_b$  and  $\pi_r|_{W_{b+i\tau}} : W_{b+i\tau} \rightarrow U_b$  is an isomorphism for each  $0 \leq i < r$ .

We have

$$\begin{aligned} \pi_{r*}(\mathcal{E}(\tilde{A}))(U_b) &= \mathcal{E}(\tilde{A})(\pi_r^{-1}(U_b)) = \mathcal{E}(\tilde{A})\left(W_b \sqcup \cdots \sqcup W_{b+(r-1)\tau}\right) \\ &= \mathcal{E}(\tilde{A})(W_b) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{b+(r-1)\tau}), \end{aligned}$$

where  $\mathcal{E}(\tilde{A})$  is the sheaf of sections of  $E(\tilde{A})$ .

Choose  $a \in \mathbb{C}$  such that  $z \notin V_a$ ,  $z \in V_{a+\tau}$ . We have  $\varphi_a(x) = z + \tau$ . As above,  $\pi_r^{-1}(U_a) = W_a \sqcup \cdots \sqcup W_{a+(r-1)\tau}$  and

$$\begin{aligned} \pi_{r*}(\mathcal{E}(\tilde{A}))(U_a) &= \mathcal{E}(\tilde{A})(\pi_r^{-1}(U_a)) = \mathcal{E}(\tilde{A})\left(W_a \sqcup \cdots \sqcup W_{a+(r-1)\tau}\right) \\ &= \mathcal{E}(\tilde{A})(W_a) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{a+(r-1)\tau}). \end{aligned}$$

Since  $g_{ab}(x) = A(\varphi_a(x) - \varphi_b(x), \varphi_b(x))$ , we obtain

$$g_{ab}(x) = A(\varphi_a(x) - \varphi_b(x), \varphi_b(x)) = A(z + \tau - z, z) = A(\tau, z).$$

Therefore, to obtain  $A(\tau, z)$  it is enough to compute  $g_{ab}(x)$ .

Note that  $\pi_{r*}(\mathcal{E}(\tilde{A}))_x = \mathcal{E}(\tilde{A})_y \oplus \cdots \oplus \mathcal{E}(\tilde{A})_{y+(r-1)\tau}$ . Note also that  $g_{ab}$  is a map from

$$\pi_{r*}(\mathcal{E}(\tilde{A}))(U_b) = \mathcal{E}(\tilde{A})(W_b) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{b+(r-1)\tau})$$

to

$$\pi_{r*}(\mathcal{E}(\tilde{A}))(U_a) = \mathcal{E}(\tilde{A})(W_a) \oplus \cdots \oplus \mathcal{E}(\tilde{A})(W_{a+(r-1)\tau}).$$

One easily sees that  $y \in W_b$ ,  $y \in W_{a+(r-1)\tau}$  and  $y + i\tau \in W_{b+i\tau}$ ,  $y + i\tau \in W_{a+(i-1)\tau}$  for  $0 < i < r$ . Therefore,  $g_{ab}(x)$  equals

$$\begin{pmatrix} 0 & \tilde{g}_{a+b\tau}(y+\tau) & & & \\ \vdots & & \ddots & & \\ 0 & & & \tilde{g}_{a+(r-2)\tau+b+(r-a)\tau}(y+(r-1)\tau) & \\ \tilde{g}_{a+(r-1)\tau+b}(y) & 0 & \dots & & 0 \end{pmatrix}.$$

It remains to compute the entries of this matrix. Since

$$\begin{aligned} \tilde{g}_{a+(r-1)\tau+b}(y) &= \tilde{A}(\tilde{\varphi}_{a+(r-1)\tau}(y) - \tilde{\varphi}_b(y), \tilde{\varphi}_b(y)) \\ &= \tilde{A}(z + r\tau - z, z) = \tilde{A}(r\tau, z) \end{aligned}$$

and

$$\begin{aligned}\tilde{g}_{a+(i-1)\tau, b+i\tau}(y+i\tau) &= \tilde{A}(\tilde{\varphi}_{a+(i-1)\tau}(y+i\tau) - \tilde{\varphi}_{b+i\tau}(y+i\tau), \tilde{\varphi}_{b+i\tau}(y+i\tau)) \\ &= \tilde{A}(z+i\tau - (z+i\tau)) = \tilde{A}(0, z+i\tau) = I_n,\end{aligned}$$

one obtains

$$g_{ab}(x) = \begin{pmatrix} 0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n \\ \tilde{A}(z) & 0 & \dots & 0 \end{pmatrix}.$$

Therefore,

$$A(z) = \begin{pmatrix} 0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n \\ \tilde{A}(z) & 0 & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_{(r-1)n} \\ \tilde{A}(u) & 0 \end{pmatrix},$$

which proves the required statement.  $\square$

LEMMA 5.16. *Let  $A_i \in \mathrm{GL}_n(\mathbb{R})$ ,  $i = 1, \dots, n$ . Then*

$$\prod_{i=1}^r \begin{pmatrix} 0 & I_{(r-1)n} \\ A_i & 0 \end{pmatrix} = \mathrm{diag}(A_r, \dots, A_1)$$

*Proof.* Straightforward calculation.  $\square$

From Theorem 5.14 and Theorem 5.15 one obtains the following:

COROLLARY 5.17. *Let  $E(A)$  be a vector bundle of rank  $n$  on  $E_{r\tau}$ , where  $A : \mathbb{C}^* \rightarrow \mathrm{GL}_n(\mathbb{C}V)$  is a holomorphic function. Then  $\pi_r^* \pi_{r*} E(A)$  is defined by*

$$\mathrm{diag}(A(q^{r-1}u), \dots, A(qu), A(u)).$$

*In other words  $\pi_r^* \pi_{r*} E(A)$  is isomorphic to the direct sum*

$$\bigoplus_{i=0}^{r-1} E(A(q^i u)).$$

*Proof.* We know that  $\pi_r^* \pi_{r*} E(A)$  is defined by  $B(r, u)$ , where

$$B(1, u) = \begin{pmatrix} 0 & I_{(r-1)n} \\ A & 0 \end{pmatrix}.$$

Therefore, using Lemma 5.16, one obtains

$$\begin{aligned} B(r, u) &= \begin{pmatrix} 0 & I_{(r-1)n} \\ A(q^{r-1}u) & 0 \end{pmatrix} \cdots \begin{pmatrix} 0 & I_{(r-1)n} \\ A(qu) & 0 \end{pmatrix} \begin{pmatrix} 0 & I_{(r-1)n} \\ A(u) & 0 \end{pmatrix} \\ &= \text{diag}(A(q^{r-1}u), \dots, A(qu), A(u)), \end{aligned}$$

which completes the proof.  $\square$

**COROLLARY 5.18.** *Let  $L \in \mathcal{E}(r, 0)$ , then  $\pi_r^* \pi_{r*} L = \bigoplus_1^r L$ .*

*Proof.* Clear, since  $L = E(A)$  for a constant matrix  $A$  by Theorem 5.13.  $\square$

Note that for a covering  $\pi_r : E_{r\tau} \rightarrow E_\tau$  the group of deck transformations  $\text{Deck}(E_{r\tau}/E_\tau)$  can be identified with the kernel  $\text{Ker}(\pi_r)$ . But  $\text{Ker} \pi_r$  is cyclic and equals  $\{1, [q], \dots, [q]^{r-1}\}$ , where  $[q]$  is a class of  $q = e^{2\pi i\tau}$  in  $E_{r\tau}$ . Clearly

$$[q]^*(E(A(u))) = E(A(qu)).$$

Therefore, we get one more corollary.

**COROLLARY 5.19.** *Let  $\epsilon$  be a generator of  $\text{Deck}(E_{r\tau}/E_\tau)$ . Then for a vector bundle  $E$  on  $E_{r\tau}$  we have*

$$\pi_r^* \pi_{r*} E = E \oplus \epsilon^* E \oplus \cdots \oplus (\epsilon^{r-1})^* E.$$

To proceed we need the following result.

**THEOREM 5.20** (Oda, [11, Theorem 1.2, (i)]). *Let  $\varphi : Y \rightarrow X$  be an isogeny of  $g$ -dimensional abelian varieties over a field  $k$ , and let  $L$  be a line bundle on  $Y$  such that the restriction of the map*

$$\Lambda(L) : Y \rightarrow \text{Pic}^0(Y), \quad y \mapsto t_y^* L \otimes L^{-1},$$

*to the kernel of  $\varphi$  is an isomorphism. Then  $\text{End}(\varphi_* L) = k$  and  $\varphi_* L$  is an indecomposable vector bundle on  $X$ .*

**THEOREM 5.21.** *Let  $L \in \mathcal{E}(1, d)$  and let  $(r, d) = 1$ . Then  $\pi_{r*}(L) \in \mathcal{E}(r, d)$ .*

*Proof.* It is clear that  $\pi_{r*} L$  has rank  $r$  and degree  $d$ . It remains to prove that  $\pi_{r*} L$  is indecomposable.

We have the isogeny  $\pi_r : E_{r\tau} \rightarrow E_\tau$ . Since  $Y = E_{r\tau}$  is a complex torus (elliptic curve),  $Y \simeq \text{Pic}^0(Y)$  with the identification  $y \leftrightarrow t_y^* E(\varphi_0) \otimes E(\varphi_0)^{-1}$ .

We know that  $L = E(\varphi_0)^d \otimes \tilde{L}$  for some  $\tilde{L} = E(a) \in \mathcal{E}(1, 0)$ ,  $a \in \mathbb{C}^*$ . Since  $t_y^*(\tilde{L}) = t_y^*(E(a)) = E(a) = \tilde{L}$ , as in the proof of Theorem 5.12 one gets

$$\begin{aligned} \Lambda(L)(y) &= t_y^*(L) \otimes L^{-1} = t_y^*(E(\varphi_0)^d \otimes \tilde{L}) \otimes (E(\varphi_0)^d \otimes \tilde{L})^{-1} \\ &= t_y^*(E(\varphi_0)^d) \otimes t_y^*(\tilde{L}) \otimes E(\varphi_0)^{-d} \otimes \tilde{L}^{-1} = t_y^*(E(\varphi_0)^d) \otimes E(\varphi_0)^{-d} \\ &= t_y^*(E(\varphi_0^d)(z)) \otimes E(\varphi_0^{-d}) = E(\varphi_0^d(z + \eta)) \otimes E(\varphi_0^{-d}) \\ &= E(\varphi_0^d(z + \eta)\varphi_0^{-d}(z)) = E(\exp(-2\pi i d\eta)) = t_{dy}^*(E(\varphi_0)) \otimes E(\varphi_0)^{-1}, \end{aligned}$$

where  $\eta \in \mathbb{C}$  is a representative of  $y$ . This means that the map  $\Lambda(L)$  corresponds to the map

$$d_Y : E_{r\tau} \rightarrow E_{r\tau}, \quad y \mapsto dy.$$

Since  $\text{Ker } \pi_r$  is isomorphic to  $\mathbb{Z}/r\mathbb{Z}$ , we conclude that the restriction of  $d_Y$  to  $\text{Ker } \pi_r$  is an isomorphism if and only if  $(r, d) = 1$ . Therefore, using Theorem 5.20, we obtain the required statement.  $\square$

Now we are able to prove the following main theorem:

**THEOREM 5.22.**

- (i) Every indecomposable vector bundle  $F \in \mathcal{E}_{E_\tau}(r, d)$  is of the form  $\pi_{r'}^*(L' \otimes F_h)$ , where  $(r, d) = h$ ,  $r = r'h$ ,  $d = d'h$ ,  $L' \in \mathcal{E}_{E_{r'\tau}}(1, d')$ .
- (ii) Every vector bundle of the form  $\pi_{r'}^*(L' \otimes F_h)$ , where  $L'$  and  $r'$  are as above, is an element of  $\mathcal{E}_{E_\tau}(r, d)$ .

*Proof*<sup>1</sup>.

- (i) By [1, Lemma 26] we obtain  $F \simeq E_A(r, d) \otimes L$  for some line bundle  $L \in \mathcal{E}(1, 0)$ . By [1, Lemma 24] we have  $E_A(r, d) \simeq E_A(r', d') \otimes F_h$ , hence  $F \simeq E_A(r', d') \otimes F_h \otimes L$ .

Consider any line bundle  $\tilde{L} \in \mathcal{E}_{E_{r'\tau}}(1, d')$ . Since by Theorem 5.21  $\pi_{r'}^*(\tilde{L}) \in \mathcal{E}(r', d')$ , it follows from [1, Lemma 26] that there exists a line bundle  $L''$  such that  $E_A(r', d') \otimes L \simeq \pi_{r'}^*(\tilde{L}) \otimes L''$ .

Using the projection formula, we get

$$\begin{aligned} F &\simeq \pi_{r'}^*(\tilde{L}) \otimes L'' \otimes F_h \\ &\simeq \pi_{r'}^*(\tilde{L} \otimes \pi_{r'}^*(L'')) \otimes \pi_{r'}^*(F_h) \\ &\simeq \pi_{r'}^*(L' \otimes \pi_{r'}^*(F_h)) \end{aligned}$$

for  $L' = \tilde{L} \otimes \pi_{r'}^*(L'')$ .

Since  $F_h$  is defined by a constant matrix we obtain by Theorem 5.14 that  $\pi_{r'}^*(F_h)$  is defined by  $f_h^{r'}$ , which has the same Jordan normal form as  $f_h$ . Therefore,  $\pi_{r'}^*(F_h) \simeq F_h$  and finally one gets  $F \simeq \pi_{r'}^*(L' \otimes F_h)$ .



- (ii) Consider  $F = \pi_{r'*}(L' \otimes F_h)$ . As above  $F_h = \pi_{r'}^*(F_h)$ . Using the projection formula we get

$$F = \pi_{r'*}(L' \otimes F_h) = \pi_{r'*}(L' \otimes \pi_{r'}^*(F_h)) = \pi_{r'*}(L') \otimes F_h.$$

By Theorem 5.21  $\pi_{r'*}(L')$  is an element from  $\mathcal{E}_{E_\tau}(r', d')$ . Therefore,  $\pi_{r'*}(L') = E_A(r', d') \otimes L$  for some line bundle  $L \in \mathcal{E}_{E_\tau}(1, 0)$ . Finally we obtain

$$\begin{aligned} F &= \pi_{r'*}(L') \otimes F_h \\ &= E_A(r', d') \otimes L \otimes F_h \\ &= E_A(r'h, d'h) \otimes L \\ &= E_A(r, d) \otimes L, \end{aligned}$$

which means that  $F$  is an element of  $\mathcal{E}_{E_\tau}(r, d)$ .  $\square$

REMARK 5.23. *Since any line bundle of degree  $d'$  is of the form  $t_x^*E(\varphi_0) \otimes E(\varphi_0)^{d'-1}$ , Theorem 5.22(i) takes exactly the form of Proposition 1 from [12], which was given without any proof.*

Any line bundle of degree  $d'$  over  $E_{r\tau}$  is of the form  $E(a) \otimes E(\varphi^{d'})$ , where  $a \in \mathbb{C}^*$ . Therefore,  $L' \otimes F_h = E(a) \otimes E(\varphi_0^{d'}) \otimes E(A_h) = E(\varphi_0^{d'} A_h(a))$ . Using Theorem 5.15 we obtain the following:

THEOREM 5.24. *Indecomposable vector bundles of rank  $r$  and degree  $d$  on  $E_\tau$  are exactly those defined by the matrices*

$$\begin{pmatrix} 0 & I_{(r'-1)h} \\ \varphi_0^{d'} A_h(a) & 0 \end{pmatrix},$$

where  $(r, d) = h$ ,  $r' = r/h$ ,  $d' = d/h$ ,  $\varphi_0(u) = q^{-\frac{r}{2}} u^{-1}$ ,  $q = e^{2\pi i \tau}$ ,  $a \in \mathbb{C}^*$ , and

$$A_h(a) = \begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & a & 1 \\ & & & a \end{pmatrix} \in \mathrm{GL}_h(\mathbb{C}).$$

Note that if  $d = 0$ , we get  $h = r$ ,  $r' = 1$ , and  $d' = 0$ . In this case the statement of Theorem 5.24 is exactly Theorem 5.13.

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Received May 13, 2011