A CLASS OF LINEAR OPERATORS IN PERIODIC FUNCTION SPACES INCLUDING DIFFERENCE-DIFFERENTIAL OPERATORS (*)

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SOMMARIO. - Si fa uno studio degli operatori lineari definiti negli spazi di Banach $C^p_T$ delle funzioni $T$-periodiche e di classe $C^n$, $u: \mathbb{R} \to \mathbb{C}$, $n \geq 0$, per i quali la composizione con gli operatori di traslazione $u \to u(.+\xi)$, $\xi \in \mathbb{R}$, è commutativa. Si trovano gli autovalori e si dà una rappresentazione del tipo $Lu = \int_0^T u(x+.) \, dG(x)$ per mezzo di funzioni a variazione limitata. I risultati teorici sono applicati ad operatori definiti da equazioni differenziali alle differenze.

SUMMARY. - This is a study of linear operators for which composition with shift operators $u \to u(.+\xi)$, $\xi \in \mathbb{R}$, on Banach spaces $C^p_T$ of $T$-periodic functions $u: \mathbb{R} \to \mathbb{C}$, $n \geq 0$, is commutative. Eigenvalues are found and representations of the type $Lu = \int_0^T u(x+.) \, dG(x)$ by functions of bounded variation are given. The abstract results are applied to operators given by difference-differential equations.

1. Introduction.

For $T \in \mathbb{R}^+$, the space

$$C^n_T := \{u \in C^n(\mathbb{R}, \mathbb{C}) \mid u^{(i)}(t+T) = u^{(i)}(t) \forall t \in \mathbb{R}, i = 0, \ldots, n\}$$

with the norm $\|u\|_{C^n_T} := \sum_{k=0}^{n} \max_{t} |u^{(k)}(t)|$ is a Banach space.

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Besides consider the space

\[ L_{T}^{2} = \{ u : \mathbb{R} \to \mathbb{C} \mid u(t+T) = u(t) \ \forall t \in \mathbb{R} \text{ and } u \in L^{2}([0,T],\mathbb{C}) \} \]

which, equipped with the usual inner product \( (u,v) = \int_{0}^{T} u(t) \overline{v(t)} \, dt \), is a Hilbert space of equivalence classes.

For every \( n \geq 0 \) the space \( C_{T}^{n} \) is continuously embeddable in the space \( L_{T}^{2} \) and, moreover, for every \( m \geq n \) the embedding operator \( I : C_{T}^{m} \to C_{T}^{n} \) is completely continuous.

For each real number \( \tau \) let us define the following shift operator acting on \( T \)-periodic functions

\[ S_{\tau} u(t) : = u(t+\tau). \]

It is easy to see that \( S_{\tau} \) is an isometry (isomorphism which preserves the norm) on the spaces \( C_{T}^{n} \) for every \( n \geq 0 \). Since \( S_{\tau} S_{\tau'} = S_{\tau+\tau'} \) for every \( \tau, \tau' \in \mathbb{R} \), the set \( \mathcal{S} \) of all the shifts turns out to be a group.

Next consider the space \( \mathcal{L}^{n,m} : = \mathcal{L}(C_{T}^{n},C_{T}^{m}) \) of the continuous linear operators mapping \( C_{T}^{n} \) into \( C_{T}^{m} \) and define an action of the group \( \mathcal{S} \) onto \( \mathcal{L}^{n,m} \) as follows

\[ S_{\tau} x L : = S_{-\tau} \circ L \circ S_{\tau} \quad \forall L \in \mathcal{L}^{n,m} \text{ and } \forall \tau \in \mathbb{R}. \]

Therefore the set

\[ \mathcal{LS}^{n,m} : = \{ L \in \mathcal{L}^{n,m} \mid S_{\tau} x L = L \forall \tau \in \mathbb{R} \} \]

is a closed linear subspace of \( \mathcal{L}^{n,m} \), and hence it is a Banach space.

Observe that the operators of \( \mathcal{LS}^{n,m} \) are characterized by the property that they commute with the shifts, i.e.

\[ L \circ S_{\tau} = S_{\tau} \circ L \quad \forall \tau \in \mathbb{R}. \]

The spaces \( \mathcal{LS}^{n,m} \) include various kinds of operators given, for example, by linear difference-differential equations (DDE’s) with constant coefficients or by integral equations such as \( u(t) = \int_{0}^{T} K(t-x) u(x) \, dx + f(t) \) with a \( T \)-periodic convolution kernel.

We give an integral representation theorem for the operators of \( \mathcal{LS}^{n,m} \) and hence, in particular, for the \( T \)-periodic solutions of DDE’s. Many representation theorems are known for initial value problems to DDE’s (see for example L.El’sgol’ts-S. B. Norkin [3] and J. Hale [4]) and for ordinary differential equations in Banach spaces with many kinds of lateral conditions (see for example C. S. Hönig [5]). On the contrary similar results do not seem to exist for \( T \)-periodic solutions to DDE’s.
We also obtain again some known results on the spectral theory and on the solvability of DDE's with constant coefficients (see for example L.E. El'sgol'ts-S. B. Norkin [2] and S. Invernizzi-F. Zanolin [6]). Furthermore the representation theorems provide a straightforward estimate of the rate of uniform convergence for the Fourier expansion of the solutions.

This research was suggested by a paper of A. Bellen [1] in which he studies an iterative monotone method for the numerical solution of nonlinear delay differential equations of the type

\[ u^{(n)}(t) = f(t, u(t), u(t-\tau)) \quad n = 1, 2, \]

in spaces of \( T \)-periodic functions. An iteration requires the solution of a linear difference-differential equation.

Moreover a maximum principle and the knowledge of upper and lower solutions are needed. The results of this paper are fully used in M. Zennaro [7], where some maximum principles are proved, and in A. Bellen-M. Zennaro [2], where a method for finding upper and lower solutions is given.

2. The spaces \( \mathcal{L}^n \). Eigenvalues and eigenspaces.

Define for every \( k \in \mathbb{Z} \) the function \( e_k(t) := \exp \left( \frac{2k\pi i t}{T} \right) \). The set \( E := \{e_k\}_{k \in \mathbb{Z}} \) is an orthogonal system in \( L^2_T \), and is a fundamental set in \( C^n_T \), since \( \text{span} E \) is dense in \( C^n_T \) for every \( n \geq 0 \).

The spaces \( \mathcal{L}^n \) can be characterized as follows.

**Theorem 2.1** - For every \( L \in \mathcal{L}^n \) the statements

(i) \( L \in \mathcal{L}^n \);

(ii) - For every \( k \in \mathbb{Z} \) there exists \( \lambda_k \in \mathbb{C} \) such that \( L e_k = \lambda_k e_k \);

are equivalent.

**Proof.** Let (i) be true. Since \( S_\tau e_k = e_k(\tau) e_k \) for every \( \tau \in \mathbb{R} \), we have that \( S_\tau L e_k = L S_\tau e_k = e_k(\tau) L e_k \) and then \( L e_k(t + \tau) = e_k(\tau) L e_k(t) \) for every \( t, \tau \). For \( t = 0 \) we have \( L e_k(\tau) = L e_k(0) e_k(\tau) \) for every \( \tau \in \mathbb{R} \). Therefore (ii) is proved, with \( \lambda_k = L e_k(0) \).

Conversely, assume (ii) to be true. It follows that if \( p \) is a trigonometric polynomial, i.e. \( p = \sum_{i=-s}^{s} a_i e_i \), then

\[ L \cdot S_\tau p = \sum_i a_i L \cdot S_\tau e_i = \sum_i a_i e_i(\tau) L e_i = \sum_i a_i e_i(\tau) \lambda_i e_i = \]
\[ S(\sum_i a_i\lambda_i e_i) = S \circ Lp. \]

Since \( \text{span} \, E \) is dense in \( C_n \) and \( L \) is continuous, (i) holds, too. ■

Throughout the paper we shall mark the dependence of the numbers \( \lambda_k \) on \( L \) by \( \lambda_k \).

It is easy to prove the following three corollaries.

**Corollary 2.2** - Let \( L, M \in \mathcal{L} \mathbb{S}^{n,m} \); then we have \( \lambda_k^{L+M} = \lambda_k^L + \lambda_k^M \) for every \( k \in \mathbb{Z} \).

**Corollary 2.3** - Let \( L \in \mathcal{L} \mathbb{S}^{n,m} \); if there exists \( L^{-1} \in \mathcal{L}^m, n \), then \( L^{-1} \in \mathcal{L} \mathbb{S}^{m,n} \) and \( \lambda_k^{-1} = (\lambda_k^{-1})^{-1} \) for every \( k \in \mathbb{Z} \).

**Corollary 2.4** - Let \( L \in \mathcal{L} \mathbb{S}^{n,m} \) and \( M \in \mathcal{L} \mathbb{S}^{m,p} \); then \( M \circ L \in \mathcal{L} \mathbb{S}^{n,p} \) and \( \lambda_k^{M \circ L} = \lambda_k^M \cdot \lambda_k^L \) for every \( k \in \mathbb{Z} \).

When the continuous operator \( L \), acting from \( C^n_T \) into \( C^n_T \), will be regarded as a continuous operator acting from \( C^q_T \) into \( C^p_T \), we shall still denote it by \( L \).

**Theorem 2.5** - Let \( L \) belong both to \( \mathcal{L} \mathbb{S}^{n,m} \) and \( \mathcal{L} \mathbb{S}^{q,p} \) and let \( M \) belong both to \( \mathcal{L} \mathbb{S}^{m,p} \) and \( \mathcal{L} \mathbb{S}^{n,q} \); then they commute, i.e. \( L \circ M = M \circ L \).

**Proof.** By Corollary 2.4 we have that \( L \circ M \) and \( M \circ L \) belong to \( \mathcal{L} \mathbb{S}^{n,p} \) and \( \lambda_k^{M \circ L} = \lambda_k^{M \circ M} = \lambda_k^M \cdot \lambda_k^L \) for every \( k \in \mathbb{Z} \). Therefore, if \( p \) is a trigonometric polynomial, i.e. \( p = \sum_{i=-k}^k a_i e_i \), then

\[
M \circ Lp = \sum_i a_i M \circ L e_i = \sum_i a_i \lambda_i^{M \circ L} e_i = \sum_i a_i \lambda_i^{M \circ L} e_i = \sum_i a_i \lambda_i^{M \circ M} e_i = \sum_i a_i L \circ M e_i = L \circ Mp.
\]

Since \( \text{span} \, E \) is dense in \( C^n_T \) and \( L \circ M, M \circ L \) are continuous, it follows that \( L \circ M = M \circ L \). ■

**Theorem 2.6** - \( L \in \mathcal{L} \mathbb{S}^{n,m} \) implies \( L \in \mathcal{L} \mathbb{S}^{n+k,m+k} \) for every \( k \geq 1 \).

**Proof.** For \( n = 0 \) and \( m \geq 0 \) this is a consequence of the representation theorems for the spaces \( \mathcal{L} \mathbb{S}^{0,m} \) given in Section 3. In fact we shall see that for every \( L \in \mathcal{L} \mathbb{S}^{0,m} \) there exists a function \( G \), \( [G] \in \mathcal{S}^m \) (see Theorem 3.6), such that \( Lf(t) = \int_0^T f(x+t) \, dG(x) \) for every \( f \in C^n_T \). Now, if \( f \in C^n_T \), it is easily seen that the following equalities hold for \( i = 1, \ldots, k \)

\[(Lf)^{(i)}(t) = \int_0^T f^{(i)}(x+t) \, dG(x) = Lf^{(i)}(t)\]

and hence, since \( f^{(k)} \) is continuous, we have that \( Lf \in C^{m+k}_n \).
Moreover let \( ||L||_{0,m} \) be the norm of \( L \) as operator from \( C^0_T \) into \( C^m_T \); then

\[
||L||_{C^m_T}^{m+k} = \sum_{i=0}^{k-1} ||(Lf)^{(i)}||_\infty + \sum_{i=k}^{m+k} ||(Lf)^{(i)}||_\infty = \sum_{i=0}^{k-1} ||Lf^{(i)}||_\infty + \\
+ \sum_{i=0}^{m} ||Lf^{(k)}||_\infty \leq \sum_{i=0}^{k-1} ||L||_{0,m} ||f^{(i)}||_\infty + ||Lf^{(k)}||_{C^m_T} \leq \\
\leq ||L||_{0,m} \sum_{i=0}^{k} ||f^{(i)}||_\infty = ||L||_{0,m} ||f||_{C^k_T}
\]

and therefore \( L \) is continuous also from \( C^k_T \) into \( C^{m+k}_T \).

Assume the theorem true for \( n-1 \) and \( m \geq 0 \). It is easy to see that the operator \( J: u \rightarrow u' - u \) belongs to \( \mathfrak{L}^p \) for every \( p \geq 1 \) and that there exists \( J^{-1} \in \mathfrak{L}^p \). By Corollary 2.4 we have \( L_0J^{-1} \in \mathfrak{L}^{n-1,m} \), since \( J^{-1} \in \mathfrak{L}^{n-1,n} \).

By the inductive hypothesis \( L_0J^{-1} \in \mathfrak{L}^{n-1+k,m+k} \) for every \( k \geq 1 \); since \( J \in \mathfrak{L}^{n+k,n-1+k} \), we have that \( L = L_0J^{-1}J \in \mathfrak{L}^{n+k,m+k} \). So the proof is complete.

For every \( L \in \mathfrak{L}^{n,m} \) let us call eigenvalue of \( L \) each complex number \( \lambda \) such that \( Lu = \lambda u \) for some \( u \in C^n_T, u \neq 0 \).

For every \( \lambda \in \mathbb{C} \) and \( L \in \mathfrak{L}^{n,m} \) define the set

\[
K_{L,\lambda} := \{ k \in \mathbb{Z} \mid \lambda^L_k = \lambda \}
\]

which, obviously, may be empty. Besides, define

\[
E_{L,\lambda} := \{ e_k \}_{k \in K_{L,\lambda}}
\]

and denote the sets \( E_{L,0} \) and \( K_{L,0} \) by \( E_L \) and \( K_L \) respectively.

Let \( N_{L,\lambda} \) be the linear manifold of the functions \( u \in C^n_T \) such that \( Lu = \lambda u \). Note that \( N_{L,0} = \text{ker} L \) and \( N_{L,\lambda} = \{0\} \) if and only if \( \lambda \) is not an eigenvalue of \( L \).

**Lemma 2.7** - If \( E = E_1 \cup E_2 \) and \( E_1 \cap E_2 = \emptyset \), then we have that

\[
C^n_T = \overline{\text{span } E_1 \oplus \text{span } E_2} \quad \text{for every } n \geq 0.
\]

The proof is standard and is omitted for the sake of brevity.

Now we are able to prove the following theorem concerning the structure of the linear manifold \( N_{L,\lambda} \).

**Theorem 2.8** - Let \( L \in \mathfrak{L}^{n,m} \) and let \( \lambda \) be a complex number; then we have \( N_{L,\lambda} = \overline{\text{span } E_{L,\lambda}} \), where \( \text{span } \emptyset = \{0\} \).
Proof. Since \( L \) is continuous, we have \( \text{span} \, E_{L,\lambda}^T \subseteq N_{L,\lambda} \). Conversely, let \( u \in N_{L,\lambda} \); then, by Lemma 2.7, we have that \( u = v + w \), where \( v \in \text{span} \, E_{L,\lambda}^T \) and \( w \in \text{span} \, (E-E_{L,\lambda})^T \).

Consider the operator \( J \) defined in the proof of Theorem 2.6; by Theorem 2.5 we have \( J^{-1} \circ L = L \circ J^{-1} \) and then \( L(J^{-1} u) = J^{-1}(Lu) = \lambda J^{-1} u \). Since \( J^{-1} u \in C_T^{n+1} \), its Fourier expansion \( \sum_{k} a_k e_k \) converges uniformly with all its derivatives up to the \( n \)-th, i.e. in \( C_T^n \), to \( J^{-1} u \). Therefore, since \( L \) is continuous,

\[
L(J^{-1} u) - \lambda J^{-1} u = \sum_{k} a_k (\lambda_k^L - \lambda) \, e_k = 0
\]

and we have \( \lambda = \lambda_k^L \) for every \( a_k \neq 0 \), i.e. \( J^{-1} u \in \text{span} \, E_{L,\lambda}^T \). On the other hand \( J^{-1} \) is an isomorphism of \( C_T^n \) onto \( C_T^{n+1} \) and \( J^{-1} \) maps \( \text{span} \, E_{L,\lambda} \) into itself. Thus we have

\[
J^{-1} v \in \text{span} \, E_{L,\lambda}^T \subset \text{span} \, E_{L,\lambda}^n
\]

and

\[
J^{-1} w \in \text{span} \, (E-E_{L,\lambda})^T \subset \text{span} \, (E-E_{L,\lambda})^n.
\]

Hence, by Lemma 2.7, \( J^{-1} w = 0 \), i.e. \( w = 0 \) and \( u = v \). \( \Box \)

This theorem yields, as a corollary, the following result on the set of the eigenvalues of \( L \).

**Corollary 2.9.** If \( L \in \mathcal{L}^n \), its eigenvalues are exactly \( \{\lambda_k^L\}_{k \in \mathbb{Z}} \).

**Theorem 2.10.** Let \( L \in \mathcal{L}^n \); then \( L = 0 \) if and only if \( \lambda_k^L = 0 \) for every \( k \in \mathbb{Z} \).

Proof. If \( L = 0 \), it obviously follows that \( \lambda_k^L = 0 \) for every \( k \in \mathbb{Z} \). Conversely, if \( \lambda_k^L = 0 \) for every \( k \in \mathbb{Z} \), we have \( Lp = 0 \) for every \( p \in \text{span} \, E \) and therefore, since \( \text{span} \, E \) is dense in \( C_T^n \) and \( L \) is continuous, it follows that \( L = 0 \). \( \Box \)

The following theorem can be proved by the same arguments of Theorem 2.8.

**Theorem 2.11.** Let \( L \in \mathcal{L}^n \) and let \( R(L) \) be the range of \( L \); then \( \overline{R(L)} = \text{span} \, (E-E_L) \).

Since \( \text{span} \, (E-E_L)^T = C_T^n \cap \text{span} \, E_L^2 \), we have immediately the following corollary.

**Corollary 2.12.** Let \( L \in \mathcal{L}^n \) and let \( R(L) \) be closed; then the equation \( Lu = f \) has a solution in \( C_T^n \) if and only if \( (f, e_k) = 0 \)
for every \( k \in K_L \).
3. **Representation theorems for the spaces $\mathcal{S}^{0,n}$**.

First consider the case $\mathcal{S}^{0,0}$.

**Lemma 3.1** - The space $\mathcal{S}^{0,0}$ is isometrically isomorphic to $C_T^0$, the dual space of $C_T^0$.

**Proof.** Indeed one can prove by direct arguments that the operator

$$K : \mathcal{S}^{0,0} \to C_T^0$$

such that $K(L) = P_0^* L$,

where $P_0^*$ is the evaluation functional defined by $P_0^* u = u(0)$, is linear and preserves the norm.

On the other hand there exists the inverse

$$K^{-1} : C_T^0 \to \mathcal{S}^{0,0}$$

defined as follows:

for every $F \in C_T^0$ and $u \in C_T^0$, $K^{-1}(F) u(t) = F \ast S_T u$. ■

Let us consider the space $BV_0([0,T], C)$ of the complex functions $G$ defined in $[0,T]$ which are of bounded variation and are such that $G(x + 0) = G(x)$ for every $x \in (0,T)$ and $G(0) = 0$ (see C. S. Hönig [5]).

Let $\Phi(x) = \begin{cases} 0 & \text{if } x = 0 \text{ and } x = T \\ 1 & \text{if } 0 < x < T \end{cases}$

$$S_T^0 := \frac{BV_0([0,T], C)}{span\{\Phi\}}.$$  

The following lemma is a trivial consequence of the Riesz theorem.

**Lemma 3.2** - The space $C_T^0$ is isometrically isomorphic to the space $S_T^0$, and for every $F \in C_T^0$ we have that

$$F^* u = \int_0^T u(x)\,dG(x)$$

for every $u \in C_T^0$,

where $[G]$ is the element of $S_T^0$ corresponding to the linear functional $F^*$ in the isometry.

Combining the results of Lemmata 3.1 - 3.2, we easily obtain the representation theorem for the operators of the space $\mathcal{S}^{0,0}$.

**Theorem 3.3** - There exists an isometry $\mathcal{S}_0$ between the space $\mathcal{S}^{0,0}$ and the space $S_T^0$, and for every $L \in \mathcal{S}^{0,0}$ we have that
\[ Lu(t) = \int_0^T u(x + t) \, dG(x) \]

for every \( u \in \mathcal{C}_T^0 \) and for every real number \( t \), where \([G] = \mathfrak{S}_0(L)\).

Each function \( G \in \mathfrak{S}_0(L) \) will be called \textit{representative function} of the operator \( L \).

In the space \( S_T^0 \) the norm is given by \( \| [G] \|_0 = \inf_{\lambda \in \mathbb{C}} V(G + \lambda \Phi) \), where \( V(G + \lambda \Phi) \) is the variation of \( G + \lambda \Phi \).

Using Corollary 2.9 and Theorem 3.3 we can derive a result on the representation of all the eigenvalues of the operators which belong to the space \( \mathcal{L}S_{0,0} \).

**Corollary 3.4** - All the eigenvalues of \( L \in \mathcal{L}S_{0,0} \) are given by

\[ \lambda^L_k = \int_0^T e_k(x) \, dG(x), \text{ where } [G] = \mathfrak{S}_0(L). \]

**Proof.** Let \( e_k(t) = \int_0^T e_k(x + t) \, dG(x) = \int_0^T e_k(t) \, e_k(x) \, dG(x) = \int_0^T e_k(x) \, dG(x) \, e_k(t) \).

Now consider \( n \geq 1 \) and observe that \( L \in \mathcal{L}S_{0,n} \) implies \( L \in \mathcal{L}S_{0,m} \) for every \( m \leq n \).

Let us denote by \( S_T^n \) the subspace of \( S_T^0 \) of the classes of the representative functions of the operators \( L \in \mathcal{L}S_{0,n} \) for \( n \geq 1 \). In order to characterize the classes of \( S_T^n \), we begin with the case \( n = 1 \).

**Theorem 3.5** - The space \( S_T^1 \) is made up as follows:

\[ S_T^1 = \{ [G] \in S_T^0 \mid G(x) = \int_0^x F(\xi) \, d\xi \} \]

for some \( F \) such that \( F - F(0) \in BV_0([0,T], \mathbb{C}) \) and \( F(0) = F(T) \).

**Proof.** If such an \( F \) exists, we have immediately that \([G] \in S_T^1\). Conversely let us consider \( L \in \mathcal{L}S_{0,1} \). The derivative operator \( D \) belongs to \( \mathcal{L}S_{1,0} \) and then \( D \cdot L \in \mathcal{L}S_{0,0} \). According to Theorem 3.3, let \([G] = \mathfrak{S}_0(L)\) and \([H] = \mathfrak{S}_0(D \cdot L)\).

Let \( g(t) = 1 \); it follows that

\[ D \cdot Lg(t) = \frac{d}{dt} \int_0^T dG(x) = 0 \]
and also
\[ D_* L g(t) = \int_0^T dH(x) = H(T) - H(0) \]
and then \( H(0) = H(T) \).

Now consider the function
\[ K(x) := \int_0^x H(\xi) \, d\xi \]
and let \( f \in C^0_T \); it follows that
\[ \int_0^T D_* L f(\xi) \, d\xi = \int_0^T \left( \frac{d}{d\xi} \int_0^T f(x+\xi) \, dG(x) \right) \, d\xi = \]
\[ = \int_0^T f(x+t) \, dG(x) - \int_0^T f(x) \, dG(x) \]
and also
\[ \int_0^T D_* L f(\xi) \, d\xi = \int_0^T \left( \int_0^T f(x+\xi) \, dH(x) \right) \, d\xi = \]
\[ = \int_0^T \left( \int_0^T f(x+\xi) \, dH(x) \right) \, d\xi = H(T) \int_0^T f(T+\xi) \, d\xi - H(0) \int_0^T f(\xi) \, d\xi - \]
\[ \int_0^T H(x) \left( \frac{d}{dx} \int_0^T f(x+\xi) \, d\xi \right) \, dx = \]
\[ = [H(T) - H(0)] \int_0^T f(\xi) \, d\xi - \int_0^T H(x) \left[ f(x+t) - f(x) \right] \, dx = \]
\[ = - \int_0^T f(x+t) \, dK(x) + \int_0^T f(x) \, dK(x) \]
and then
\[ \int_0^T f(x+t) \, d(G+K)(x) = \int_0^T f(x) \, d(G+K)(x) \quad \forall \, t \in \mathbb{R}. \]

By integrating we obtain
\[ \int_0^T \left( \int_0^T f(x+t) \, d(G+K)(x) \right) \, dt = \int_0^T \left( \int_0^T f(x+t) \, dt \right) \, d(G+K)(x) = \]
\[ T \bar{f} \int_0^T d(G+K)(x), \text{ where } \bar{f} = \frac{1}{T} \int_0^T f(\xi) \, d\xi \text{ is the mean of } f. \]

Moreover
\[ \int_0^T \left( \int_0^T f(x+t) \, d(G+K)(x) \right) \, dt = T \int_0^T f(x) \, d(G+K)(x) \]
and therefore we can conclude that
\[ \int_0^T f(x+t) \, d(G+K)(x) = \bar{f} \int_0^T d(G+K)(x). \]

If we put
\[ \rho := \frac{1}{T} \int_0^T d(G+K)(x), \]
we have that \([G(x)] = [\rho x - K(x)]\) in \(S^0_T\) and hence we can suppose that \(G(x) = \rho x - K(x)\).

Thus the function \(F(x) := \rho - H(x)\) is such that \(G(x) = \int_0^x F(\xi) \, d\xi\), \(F - F(0) \in BV_0([0,T], \mathbb{C})\) and \(F(0) = F(T)\).

The uniqueness of \(F\) is trivial. \(\blacksquare\)

The function \(F\) will be called \textit{main derivative} of the representative function \(G\).

Besides, we shall denote by \(\mathfrak{g}_i\) the restriction of \(\mathfrak{g}_0\) to \(\mathfrak{L}^{0.1}\) and renorm the space \(S^1_T\) by \(|[G]|_1 := ||[G]||_0 + V(F)\). It is easy to verify that \(\mathfrak{g}_i\) is an isometry between \(\mathfrak{L}^{0.1}\) and \(S^1_T\).

For \(n \geq 2\) the following characterization holds for \(S^n_T\).

**Theorem 3.6 -** The space \(S^n_T\) is made up as follows:

\[ S^n_T = \{ [G] \in S^0_T | G \in C^{n-1}([0,T], \mathbb{C}), \, [G^{(n-1)} - G^{(n-1)}(0)] \in S^1_T \} \]

and \(G^{(k)}(0) = G^{(k)}(T)\) for \(k = 1, \ldots, n - 1\).

\textit{Proof.} Let \(L \in \mathfrak{L}^{0.2}\); then \(D \ast L \in \mathfrak{L}^{0.1}\). Let \([G] = \mathfrak{g}_0(L)\) and \([H] = \mathfrak{g}_0(D \ast L)\); we have that \([H] \in S^1_{T'}\) so that \(H\) may be supposed to be continuous. By the same arguments of Theorem 3.5 we obtain \(H(0) = H(T)\) and we can suppose \(G(x) = \rho x - K(x)\), where we have put
\[ K(x) := \int_0^x H(\xi) \, d\xi \quad \text{and} \quad \rho := \frac{1}{T} \int_0^T d(G + K)(x). \]

It follows that \( G \in C^1([0,T], \mathbb{C}) \) and that \( G'(x) = \rho - H(x) \), so that \( G'(0) = G'(T) \) and \([G' - G'(0)] \in S^1_T\).

Conversely it is easy to see that a function \( G \) which fulfills these properties defines an operator \( L \in \mathcal{L}^0 \).

For \( n \geq 3 \) the proof can be easily carried out by induction. ■

Like before, \( \mathcal{S}_n \) will denote the restriction of \( \mathcal{S}_0 \) to the subspace \( \mathcal{L}^0_{n} \). The space \( S^n_T \) will be renormed by

\[ \| [G] \|_n := \| [G] \|_0 + V(G') + \ldots + V(G^{n-1}) + V(F), \]

where, according to the definitions, \( F \) is the main derivative of the function \( G^{(n-1)} - C^{(n-1)}(0) \). \( \mathcal{S}_n \) is an isometry between \( \mathcal{L}^0_{n} \) and \( S^n_T \). Remark that, if \( [G] \in S^n_T \) with \( n \geq 2 \), the main derivative of \( G \) is exactly the ordinary derivative \( G' \).

4. Smooth operators.

It is interesting to consider the subspace of \( \mathcal{L}^0_{n} \) which consists of those operators which have a representative function with certain smoothness properties.

We shall say that \( L \in \mathcal{L}^0_{n} \) is smooth if it has a representative function which is of class \( C^n \), and we shall denote by \( \mathcal{L}^0_{n} \) the set of such operators.

For the operators of \( \mathcal{L}^0_{n} \), the corresponding subspace of \( S^n_T \) is

\[ SS^n_T := \{ [G] \in S^n_T | [G] \cap C^n([0,T], \mathbb{C}) \neq \emptyset \}. \]

We have obviously that \( \mathcal{L}^0_{n} \subset \mathcal{L}^0_{m} \) for every \( m < n \).

**Theorem 4.1** - \( SS^n_T \) is closed in \( S^n_T \) for every \( n \geq 0 \).

**Proof.** If \( n = 0 \), it is a consequence of the fact that the convergence in \( BV_0 \) implies the uniform convergence; thus \( BV_0 \cap C^0 \) is closed in \( BV_0 \) and then \( SS^0_T \) is closed in \( S^0_T \).

If \( n = 1 \), assume \( \{ [G_k] \} \rightarrow [G] \) in \( S^1_T \) and \( [G_k] \in SS^1_T \). We can suppose \( G_k \in C^1, G \in C^0, G_k \rightarrow G \) uniformly and \( G'_k - G'_k(0) \in BV_0 \cap C^0 \). Let \( H_k \) and \( H \) be respectively the main derivatives of \( G_k \) and \( G \); it
is easy to see that \( H_k(x) = G'_k(x) + \rho_k [1 - \Theta(x)] \), where \( \rho_k := G'_k(T) - G'_k(0) \) and

\[
\Theta(x) := \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{if } 0 < x \leq T.
\end{cases}
\]

Since \( H_k - H_k(0) \to H - H(0) \) in \( BV_0 \) and \( G'_k(T) = H_k(0) \), we have that \( G'_k - G'_k(0) - \rho_k \Theta \to H - H(0) \). On the other hand \( \Theta \) is not a continuous function, and we recall that \( BV_0 \cap C^0 \) is closed in \( BV_0 \). Therefore \( G'_k - G'_k(0) \to F \in BV_0 \cap C^0 \) in \( BV_0 \) (and then uniformly, too) and \( \rho_k \to \rho \in C \). Thus, by integrating, we have that \( G_k(x) - G'_k(0) \to \int_0^x F(\xi) \, d\xi \) for every \( x \in [0, T] \).

Moreover, from the convergence of \( G_k(x) \) to \( G(x) \), it follows that \( G'_k(0) \to \eta \in C \) and so we have \( G(x) = \eta x + \int_0^x F(\xi) \, d\xi \), i.e. \( G \in C^1([0, T], C) \), i.e. \( [G] \in SS^1_T \).

If \( n \geq 2 \), the proof is carried out by induction. Assume that the theorem holds for \( n - 1 \). Consider a sequence \( \{G_k\} \) converging to \( [G] \) in \( S^n_T \) such that \( [G_k] \in SS^n_T \). We can suppose that \( G_k \in C^n \), \( G \in C^{n-1} \) and \( G'_k - G'_k(0) \in C^{n-1} \). Since \( n \geq 2 \), by Theorem 3.6 and by the definition of the norm in the space \( S^n_T \), we can conclude that \( [G'_k - G'_k(0)] \in SS^{n-1}_T \), \( [G' - G'(0)] \in S^{n-1}_T \) and

\[
\{ [G'_k - G'_k(0)] \to [G' - G'(0)] \}
\]

in \( S^{n-1}_T \); then, by the inductive hypothesis, \( [G' - G'(0)] \in SS^{n-1}_T \), i.e. \( G' \in C^{n-1} \) and so \( G \in C^n \), i.e. \( [G] \in SS^n_T \).

By virtue of the isometry \( \mathfrak{g}_n \), the subspace \( \mathfrak{g}_{S^0,n} \) is closed in \( \mathfrak{g}_{S^0,n} \).

For \( L \in \mathfrak{g}_{S^0,n} \) with \( n \geq 1 \), the representation given by Theorem 3.3 takes the particular form

\[
Lu(t) = \int_0^T \Gamma_L(x) u(x+t) \, dx \quad \forall u \in C^0_T,
\]

where \( \Gamma_L \) is uniquely determined in \( C^{n-1}([0, T], C) \).

Moreover \( \Gamma_L^{(n-1)} - \Gamma_L^{(n-1)}(0) \in BV_0 \cap C^0 \) and, for \( n \geq 2 \), \( \Gamma_L^{(k)}(0) = \Gamma_L^{(k)}(T) \) for \( k = 0, 1, \ldots, n - 2 \). This is a trivial consequence of Theorems 3.5-3.6 and of the smoothness of \( L \).

The function \( \Gamma_L \) will be called associated kernel to the smooth operator \( L \).

Conversely it is easily seen that every function \( \Gamma \) which fulfills these properties defines an operator \( L \in \mathfrak{g}_{S^0,n} \).
THEOREM 4.2 - Let \( \{L_k\} \) be a sequence of \( \mathcal{L}_2^{0,n}(n \geq 1) \) which converges to \( L \) in \( \mathcal{L}_2^{0,n} \); let \( \Gamma_k \) and \( \Gamma \) be respectively the associated kernels to \( L_k \) and \( L \). Then \( \{\Gamma_k\} \) converges uniformly to \( \Gamma \) with all the derivatives up to the \((n - 1) - \) th.

Proof. Let \( [G_k] = \mathcal{S}_n(L_k) \) and \( [G] = \mathcal{S}_n(L) \). We can suppose \( G_k, G \in C^n \) and hence we have \( \Gamma_k^{(m)} = G_k^{(m+1)} \) and \( \Gamma^{(m)} = G^{(m+1)} \) for \( m = 0,1,\ldots,n-1 \). Since \( \{G_k]\} \rightarrow [G] \) in \( \mathcal{S}_T^n \), we have that \( V(\Gamma_k^{(m)} - \Gamma^{(m)}) \rightarrow 0 \) for \( m = 0,1,\ldots,n-2 \) and that

\[
V(\Gamma_k^{(n-1)} + \rho_k [1 - \Theta] - \Gamma^{(n-1)} - \rho [1 - \Theta]) \rightarrow 0,
\]

where \( \rho_k := \Gamma_k^{(n-1)}(T) - \Gamma_k^{(n-1)}(0) \) and \( \rho := \Gamma^{(n-1)}(T) - \Gamma^{(n-1)}(0) \) (see the proof of Theorem 4.1, case \( n = 1 \)). It follows that \( \Gamma_k^{(m)} - \Gamma^{(m)}(0) \rightarrow \Gamma_k^{(m)} - \Gamma^{(m)}(0) \) uniformly for \( m = 0,1,\ldots,n-2 \) and that \( \Gamma_k^{(n-1)}(0) - \rho_k \Theta \rightarrow \Gamma^{(n-1)}(0) - \rho \Theta \) uniformly.

Since \( \Theta \notin C^0 \), while \( \Gamma_k^{(n-1)} \) and \( \Gamma^{(n-1)} \) are continuous, we have that \( \Gamma_k^{(n-1)}(0) \rightarrow \Gamma^{(n-1)}(0) \) uniformly, too. We can suppose that \( G_k \rightarrow G \) uniformly and then, since also \( \Gamma_k \rightarrow \Gamma(0) \) uniformly, it follows that

\[
\int_0^x [\Gamma_k(\xi) - \Gamma_k(0)] d\xi \rightarrow \int_0^x [\Gamma(\xi) - \Gamma(0)] d\xi
\]

uniformly, i.e. \( G_k(x) - \Gamma_k(0) x \rightarrow G(x) - \Gamma(0) x \) uniformly, and hence \( \Gamma_k(0) \rightarrow \Gamma(0) \) and \( \Gamma_k \rightarrow \Gamma \) uniformly.

In this way, by \( n-1 \) passages, we can show that \( \Gamma_k^{(m)}(0) \rightarrow \Gamma^{(m)}(0) \)

and that \( \Gamma_k^{(m)} \rightarrow \Gamma^{(m)} \) uniformly for \( m = 1,\ldots,n-1 \), too.

5. **Linear difference-differential equations with constant coefficients.**

In this last section we apply the foregoing theory to linear difference-differential equations with constant coefficients such as

\[
(DDE) \quad u^{(n)}(t) + \sum_{m_k}^{n-1} \sum_{j=1}^{m_k} a_{kj} u^{(k)}(t + \tau_{kj}) = f(t)
\]

where \( n \geq 1, f \in C^0_T, a_{kj} \in C, \tau_{kj} \in R \). We look for a solution in the space \( C_T^n \).

The equation DDE is of the form \( Nu = f \), where \( N \in \mathcal{L}_2^{n,0} \). If \( D^k \) denotes the \( k \) - th derivative operator, we have that
\[ N = D^n + \sum_{k=0}^{n-1} \sum_{j=1}^{m_k} a_{kj} D^k \cdot S_{kj}. \]

**Lemma 5.1** - The operator \( D^n - \omega I \) (which belongs to \( \mathcal{L}^n \)) is invertible for every \( \omega \in \mathbb{C} \) such that \( \omega \neq \left( \frac{2k\pi i}{T} \right)^n \) for every \( k \in \mathbb{Z} \), and the inverse operator \( J_{n,\omega} \) is smooth.

**Proof.** Let \( \omega \neq \left( \frac{2k\pi i}{T} \right)^n \) for every \( k \in \mathbb{Z} \). By Theorem 2.8 it follows that \( \text{kern}(D^n - \omega I) = \{0\} \).

Consider the following equation with boundary conditions:

\[
\begin{align*}
  & y^{(n)}(x) - (-1)^n \omega y(x) = 0 \\
  & y^{(k)}(0) = y^{(k)}(T) \quad \text{for } k = 0, 1, \ldots, n - 2 \\
  & y^{(n-1)}(T) - y^{(n-1)}(0) = (-1)^{n-1}
\end{align*}
\]

It is easy to see that this equation has a unique solution \( \Gamma \in C^\infty(\mathbb{R}, \mathbb{C}) \). Since \( \Gamma \), restricted to \([0, T]\), fulfils the properties of the associated kernels, it defines an operator, say \( J_{n,\omega} \), which belongs to \( \mathcal{L}^0 \).

Let \( f \in C^0_T \); then we have

\[
(D^n - \omega I) \cdot J_{n,\omega} f(t) = \frac{d^n}{dt^n} \int_0^T \Gamma(x) f(x + t) \, dx - \omega \int_0^T \Gamma(x) f(x + t) \, dx =
\]

\[
= - \omega \int_0^T \Gamma(x) f(x + t) \, dx + (-1)^{n-1} [\Gamma^{(n-1)}(T) - \Gamma^{(n-1)}(0)] f(t) -
\]

\[
- \int_0^T [\Gamma(x) - (-1)^n \omega \Gamma^{(n)}(x)] f(x + t) \, dx =
\]

\[
= (-1)^{n-1} [\Gamma^{(n-1)}(T) - \Gamma^{(n-1)}(0)] f(t) -
\]

\[
- (-1)^n \int_0^T \omega \Gamma^{(n)}(x) f(x + t) \, dx = f(t).
\]

Hence \( (D^n - \omega I) \cdot J_{n,\omega} = I \) (the identity operator) and therefore we can conclude that \( D^n - \omega I \) is invertible and that its inverse \( J_{n,\omega} \) is smooth. \( \square \)

**Lemma 5.2** - Let \( L \in \mathcal{L}^{n,m}, M \in \mathcal{L}^{n,m+1} \) and let there exist the inverse \( L^{-1} \in \mathcal{L}^{m,n} \). Then the following properties hold:

(i) - \( \text{kern}(L+M) \) is a finite dimension subspace of \( C^0_T \);
(ii) - \( R(L) \) is closed in \( C^{m}_{T} \);

(iii) - \( L + M \) is invertible if and only if \( \text{ker} (L + M) = \{0\} \);

(iv) - If \( m = 0 \) and \( L^{-1} \) is smooth, then also \( (L + M)^{-1} \) is smooth, if it exists.

Proof. The properties (i), (ii), (iii) easily follow from the equality \( L + M = L \circ (I + L^{-1} \cdot M) \) and the complete continuity of \( M \) as operator from \( C^{n}_{T} \) into \( C^{m}_{T} \).

In order to prove (iv), assume \( m = 0 \), \( L^{-1} \) to be smooth and \( L + M \) to be invertible. Since

\[
(L + M)^{-1} = (I + L^{-1} \cdot M)^{-1} \cdot L^{-1} = \\
= (I + L^{-1} \cdot M - L^{-1} \cdot M) \cdot (I + L^{-1} \cdot M)^{-1} \cdot L^{-1} = \\
= L^{-1} - L^{-1} \cdot M \cdot (I + L^{-1} \cdot M)^{-1} \cdot L^{-1},
\]

\( L^{-1} \in \mathcal{S} \mathcal{S}^{0,n} \) and \( L^{-1} \cdot M \cdot (I + L^{-1} \cdot M)^{-1} \cdot L^{-1} \in \mathcal{S} \mathcal{S}^{0,n+1} \subseteq \mathcal{S} \mathcal{S}^{0,n} \), we have that \( (L + M)^{-1} \in \mathcal{S} \mathcal{S}^{0,n} \). ■

If we put \( L := D^{n} - \omega I \) and \( M := \omega I + \sum_{k=0}^{n-1} \sum_{j=1}^{m_{k}} a_{kj} D^{k} \cdot S_{\tau_{kj}}, \)

\( \omega \neq (\frac{2k\pi i}{T})^{n} \), equation DDE takes the form \( (L + M)u = f \); the hypotheses of Lemma 5.2 are fulfilled for \( m = 0 \) and therefore properties (i), (ii), (iii), (iv) hold.

In particular, by (i) and (ii), we have that \( \dim \ker N = d < \infty \) and that \( R(N) \) is closed in \( C_{T}^{0} \). Moreover, by (iv) and Lemma 5.1, the operator \( N^{-1} = (L + M)^{-1} \) is smooth, if it exists. In this case the solution of DDE has the following form:

\[ u(t) = \int_{0}^{T} \Gamma(x) f(x+t) \, dx \]

where \( \Gamma \) is the associated kernel to \( N^{-1} \), which will be called resolvent kernel of DDE.

Now consider the following complex function of complex variable:

\[ \varphi(z) := z^{n} + \sum_{k=0}^{n-1} \sum_{j=1}^{m_{k}} a_{kj} z^{k} \exp(\tau_{kj} z) \]

which is called, according to L.E.El’sgol’ts-S. B. Norkin [3], the characteristic quasipolynomial of DDE.

Observe that the eigenvalues of \( N \) are \( \lambda_{k}^{N} := \varphi(\frac{2k\pi i}{T}), k \in \mathbb{Z}, \) and
then the set \( B := \left\{ \exp \left( \frac{2k\pi it}{T} \right) \right\} \) of complex exponential functions is a basis of \( \text{ker} \, N \).

By Corollary 2.12 and Lemma 5.2, equation DDE has a solution \( u \in C_T^n \) if and only if the function \( f \) is \( L^2_T \)-orthogonal to \( B \).

Finally, by Lemma 5.2-(iii), we have that \( N \) is invertible if and only if \( \phi \left( \frac{2k\pi i}{T} \right) \neq 0 \) for every \( k \in \mathbb{Z} \).

If \( N \) is invertible, by Corollary 2.3, we have that \( \lambda_k^{-1} = \left[ \phi \left( \frac{2k\pi i}{T} \right) \right]^{-1} \) and then, by Corollary 3.4,

\[
\frac{1}{T} \int_0^T \Gamma(x) \exp\left(- \frac{2k\pi ix}{T} \right) \, dx = \frac{1}{T} \lambda_k^{-1} = \left[ T \phi \left( -\frac{2k\pi i}{T} \right) \right]^{-1}.
\]

So the Fourier expansion of the resolvent kernel \( \Gamma \) of DDE is

\[
\sum_{k \in \mathbb{Z}} \left[ T \phi \left( -\frac{2k\pi i}{T} \right) \right]^{-1} \exp\left( \frac{2k\pi ix}{T} \right).
\]

Since \( \Gamma^{(n-1)} \) is continuous and of bounded variation in \([0, T]\), the Fourier expansion converges uniformly to \( \Gamma \) with all its derivatives up to the \((n-1)\)-th in every closed subinterval \([a, b] \subset [0, T]\). Since for \( n \geq 2 \) we have \( \Gamma^{(k)}(0) = \Gamma^{(k)}(T) \) for \( k = 0, 1, \ldots, n-2 \), the first \( n-2 \) derivatives of the expansion converge uniformly in \([0, T]\).

In any case the solution \( u \) of equation DDE has the form:

\[
u(t) = \int_0^T \left( \sum_{k \in \mathbb{Z}} \left[ T \phi \left( -\frac{2k\pi i}{T} \right) \right]^{-1} \exp\left( \frac{2k\pi ix}{T} \right) \right) f(x+t) \, dx =
\]

\[
\sum_{k \in \mathbb{Z}} \left[ T \phi \left( -\frac{2k\pi i}{T} \right) \right]^{-1} \left( \int_t^{t+T} \exp\left( \frac{2k\pi ix}{T} \right) f(x) \, dx \right) \exp\left( -\frac{2k\pi it}{T} \right) =
\]

\[
\sum_{k \in \mathbb{Z}} \left[ T \phi \left( \frac{2k\pi i}{T} \right) \right]^{-1} \left( \int_0^T \exp\left( -\frac{2k\pi ix}{T} \right) f(x) \, dx \right) \exp\left( \frac{2k\pi it}{T} \right),
\]

which is nothing but the Fourier expansion of \( u \).

By truncating the expansion to \( 2m+1 \) terms, we have the approximation

\[
u_m(t) = \int_0^T \Gamma_m(x) \, f(x+t) \, dx,
\]

where \( \Gamma_m(x) := \sum_{k=-m}^{m} \left[ T \phi \left( -\frac{2k\pi i}{T} \right) \right]^{-1} \exp\left( \frac{2k\pi ix}{T} \right) \), which converges
uniformly to \( u \) as \( m \) tends to \( \infty \).

In order to give a bound to the error \( \| u - u_m \|_\infty \), note that

\[
\int_0^T \Gamma(x) p_m(x+t) \, dx = \int_0^T \Gamma_m(x) p_m(x+t) \, dx
\]

for every trigonometric polynomial \( p_m = \sum_{k=-m}^m a_k e_k \), and hence

\[
u(t) - u_m(t) = \int_0^T [\Gamma(x) - \Gamma_m(x)] [f(x+t) - p_m(x+t)] \, dx.
\]

We can choose \( p_m \) equal to \( p_m^* \), the best \( L^2 \)-approximation to \( f \) by trigonometric polynomials of degree \( \leq m \), and denote by \( e_m(f) \) the error \( \| f - p_m^* \|_{L^2} \).

By the Cauchy-Schwartz inequality we have

\[
\| u - u_m \|_\infty \leq \| \Gamma - \Gamma_m \|_{L^2_T} e_m(f) \leq \| \Gamma - \Gamma_m \|_{L^2_T} \| f - q_m^* \|_{L^2_T} \leq \\
\leq \sqrt{TE_m(f)} \| \Gamma - \Gamma_m \|_{L^2_T}
\]

where \( q_m^* \) is the best uniform approximation to \( f \) by trigonometric polynomials of degree \( \leq m \), and \( E_m(f) \) is the error \( \| f - q_m^* \|_\infty \).

We want to estimate \( \| \Gamma - \Gamma_m \|_{L^2_T} \).

To this aim we consider the function \( \Phi(y) := \frac{|\varphi(iy)|^2}{y^{2n}} \) which is defined and continuous in \( \mathbb{R} - \{0\} \). We have immediately that

\[
\lim_{y \to +\infty} \Phi(y) = \lim_{y \to -\infty} \Phi(y) = 1; \text{ therefore, since } \varphi \left( \frac{2k\pi i}{T} \right) \neq 0 \text{ for every } k \in \mathbb{Z}, \text{ there exists } \sigma > 0 \text{ such that } |\varphi \left( \frac{2k\pi i}{T} \right)|^2 > \sigma \left( \frac{2k\pi i}{T} \right)^{2n} \text{ for every } k \in \mathbb{Z}.
\]

Hence

\[
\| \Gamma - \Gamma_m \|_{L^2_T} = \left( \sum_{k=-m}^{m-1} \frac{1}{T} \left| \varphi \left( -\frac{2k\pi i}{T} \right) \right|^2 \right)^{1/2} + \\
+ \sum_{k=m+1}^{+\infty} \frac{1}{T} \left| \varphi \left( -\frac{2k\pi i}{T} \right) \right|^2 \right)^{1/2} \leq \frac{2T^n}{(2\pi)^n \sqrt{\sigma T}} \left( \sum_{k=m+1}^{+\infty} k^{-2n} \right)^{1/2}.
\]

Since \( k^{-2n} \leq \xi^{-2n} \) for every \( k \in [k-1,k] \), we obtain

\[
\sum_{k=m+1}^{+\infty} k^{-2n} \leq \int_m^{+\infty} \xi^{-2n} \, d\xi = \left[ (2n-1) m^{2n-1} \right]^{-1}.
\]

We can conclude that there exists a constant \( c > 0 \),
A CLASS OF LINEAR OPERATORS IN PERIODIC etc.

\[ c := \frac{2T^n}{(2\pi)^n \sqrt{(2n-1)\sigma}}, \]

depending on the operator \( N \) such that
\[
\| u - u_m \|_\infty \leq c E_m(f) m^{-(n-1/2)}.
\]

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