SOME REMARKS ON CATEGORIES WITH CONSTANT MAPS AND MONOIDAL CLOSEDNESS (*)

by M. CRISTINA PEDICCHIO and F. ROSSI (**)

SUMMARY. - We study the «canonicity» of monoidal closed structures on categories with constant maps. The main results regard monoidal closed structures obtained by a «normal reflective embedding» on opportune reflective subcategories.

Introduction. There is a great difference, concerning monoidal closed structures, between algebraic and topological categories. In the algebraic case we have at most one monoidal closed structure \( (\square, I, r, l, a, [-, -]) \), such that \( \mathcal{U}(A \square B) \cong \mathcal{U}A \times \mathcal{U}B \), where \( \mathcal{U} \) is the forgetful functor on \( \textbf{Set} \), while in the topological case there is at least one structure of this kind [15], [12], and in many cases even a proper class [4], [5], [6]. Observe that, in the latter context, the forgetful functor \( \mathcal{U} \) is strict monoidal with respect to the cartesian closed structure of \( \textbf{Set} \), and all the natural isomorphisms of monoidal closed structure have canonical underlying maps. In this paper, we shall always refer to such a structure as to a canonical structure. In [10] it has been proved that there exists a large class

(*) Pervenuto in Redazione il 28 novembre 1983.
This work has been supported by a contribution (60% funds) from the M.P.I.
(**) Indirizzo degli Autori: Istituto di Matematica dell'Università degli Studi di Trieste - Piazzale Europa, 1 - 34100 Trieste.
of initially structured (in the sense of Nel and Wyler [11], [16]) but non-topological categories, where all the monoidal closed structures are canonical. Other examples are given in [12], [13] and [1].

Our present aim is to investigate the following problem: Let us consider a symmetric monoidal closed (necessarily canonical) structure on any topological category \( \mathcal{B} \). Let \( \mathcal{A} \) be a full reflective subcategory of \( \mathcal{B} \) such that the embedding \( E \) admits an enrichment to a normal reflective one; is the reflected structure still canonical?

First of all, we give an example of a non-canonical reflected structure. Then, in the more general context of concrete categories with constant maps, we give a sufficient condition in order that the reflected structure is canonical and discuss it in the topological case. Finally we show by another example that the previous condition is not necessary.

1. Preliminaries. The aim of this Section is to recall some results about monoidal closed structures on concrete categories with constant maps. For more details, see [10].

Let \( \mathcal{A} \) be a concrete category (i.e. there exists a faithful functor \( \mathcal{A} : \mathcal{A} \rightarrow \text{Set} \)) with the following proprieties:

a) for every constant map \( f : \mathcal{A} \rightarrow \mathcal{B} \) there exists an \( \mathcal{A} \)-morphism \( \bar{f} : A \rightarrow B \) with \( \mathcal{A}(\bar{f}) = f \);

b) \( \mathcal{A} \) transports structures;

c) there exists an \( \mathcal{A} \)-object \( A \) with card(\( \mathcal{A}A \)) \( \geq 2 \).

Let \( (-\square-, I, r, l, a, [-, -]) \) be a monoidal closed structure on \( \mathcal{A} \).

**Proposition 1.1:** The structure \( (-\square-, I, r, l, a, [-, -]) \) is (up to natural isomorphisms) as follows:

1) \( \mathcal{A}1 = 1 \), terminal object of Set; \( I \) is also terminal in \( \mathcal{A} \) and it is a representing object for \( \mathcal{A} \);

2) \( \mathcal{A}A \times \mathcal{A}B \subseteq \mathcal{A}(A \square B) \), where the inclusion is natural in \( A, B \);

3) \( \mathcal{A}[B, C] = A(B, C) \);

4) For every \( u, v : A \square B \rightarrow C \) such that \( \mathcal{A}u \mid \mathcal{A}A \times \mathcal{A}B = \mathcal{A}v \mid \mathcal{A}A \times \mathcal{A}B \), it follows \( u = v \);

5) If \( \pi : A(A \square B, C) \cong A(A, [B, C]) \) is the adjunction, then \( \mathcal{A}[(\mathcal{A}f)(x)] = (y \mapsto \mathcal{A}f(x, y)) \) and \( (\mathcal{A}g^{-1})(x, y) = (\mathcal{A}g)(x)(y) \), for every \( f : A \square B \rightarrow C \), \( g : A \rightarrow [B, C] \), \( (x, y) \in \mathcal{A}A \times \mathcal{A}B \);

6) \( \mathcal{A}I \times \mathcal{A}A = \mathcal{A}(I \square A) \), \( \mathcal{A}A \times \mathcal{A}I = \mathcal{A}(A \square I) \), \( (\mathcal{A}I)(\cdot, x) = x \),
\((\mathfrak{U} \mathfrak{r})(x, \cdot) = x, \mathfrak{M} a | \mathfrak{M} A \times \mathfrak{M} B \times \mathfrak{M} C((x, y), z) = (x(y, z)). \) If the monoidal structure is symmetric, then \(\mathfrak{M} A \times \mathfrak{M} B(x, y) = (y, x).\)

**Definition 1.2:** Let \( A \) be a concrete category. A monoidal closed structure on \( A \) such that verifies the conditions 1), 3), 5), 6) of 1.1 and, instead of 2), the stronger:

\[ 2') \mathfrak{M} A \times \mathfrak{M} B = \mathfrak{M} (A \square B) \text{ for all } A, B \in A, \]

is called canonical.

Of course, for every canonical monoidal closed structure on \( A \), the natural isomorphisms \( a, r, l, (c) \) and the adjunction \( \pi \) have the obvious underlying bijections.

**Theorem 1.3:** If \( A \) is a concrete category that verifies the previous conditions a), b), c) and also:

\[ d) \text{ for every } X \subseteq \mathfrak{M} A \text{ there exists a morphism } j : B \rightarrow A \text{ such that } \mathfrak{M} B = X \text{ and } \mathfrak{M} j \text{ is the inclusion;} \]

\[ e) \mathfrak{M} \text{ preserves epimorphisms,} \]

then all monoidal closed structures on \( A \) are canonical (up to natural isomorphisms).

2. **Reflected structures. Examples.**

A class of categories satisfying Theorem 1.3 is given in [10, Remark 1.5]. Now, let \( B \) be a concrete category which verifies the hypotheses of 1.1, and let \( A \) be a full (replete) and reflective subcategory of \( B \) with embedding functor \( E \). Furthermore, let \( (\square, I, r, l, a, c, \square, I, r, l, a, c) \) be a canonical symmetric monoidal closed structure on \( B \), such that \( E \) admits an enrichment to a normal reflective embedding (in the sense of [2]). Our aim is to study the canonicity of the reflected structure in such a situation, that we shall call basic situation.

First we give an example of a non-canonical reflected structure. Let \( S \) be the category of semi-lattices and preserving binary suprema maps; and let \( P : S \rightarrow \text{Set} \) be the natural forgetful functor. Since \( S \) is an equational variety of algebras for a commutative theory, then \( S \) admits a symmetric monoidal closed structure \( (\square, I, r, l, a, c, \square, I, r, l, a, c) \), where the tensor product represents bimorphisms (see [9] and [3]). Since \( P \) is monadic over \( \text{Set} \), then \( P \) is semi-topological in the sense of Tholen [14]; furthermore \( P \) is functional in the sense of Porst-Wischnewsky [12] and \( \square, I, r, l, a, c, \square, I, r, l, a, c \).
are, respectively, the $P$-semifinal tensor product (generated by the cartesian closed structure on $\textbf{Set}$) and the semiinitial Hom-functor on $\mathbf{S}$ (see [12]). Denote by $Q = \text{Quot}(P)$ the set $\{(e, A) : (e, A) \text{ is a } P\text{-quotient, } e : X \to PA\}$ (see [14] p. 56), and by $\mathbf{K}$ the category whose objects are elements of $Q$ and whose morphisms are pairs $(f, \tilde{f}) : (p, A) \to (q, B)$ with $f : X \to Y$ in $\textbf{Set}$ and $\tilde{f} : A \to B$ in $\mathbf{S}$ such that

\[
\begin{array}{c}
X \xrightarrow{p} PA \\
\downarrow{f} \quad \quad \quad \quad \downarrow{P\tilde{f}} \\
Y \quad \quad \quad \quad PB
\end{array}
\]

is commutative. From [14] it follows that $P$ admits a factorization

\[
\begin{array}{c}
\mathbf{K} \\
\downarrow{E} \quad \quad \quad \quad \downarrow{Q} \\
\mathbf{S} \quad \quad \quad \quad \downarrow{P} \\
\textbf{Set}
\end{array}
\]

where $\mathcal{Q}$ is (properly) topological, and $E$ is a full reflective embedding; furthermore $\mathcal{Q}(p, A) = X, E(A) = (1_{PA}, A)$ and the reflection $R : \mathbf{K} \to \mathbf{S}$ is such that $R(p, A) = A$. So we can consider $\mathbf{S}$ as a full, reflective subcategory of $\mathbf{K}$. $\mathcal{Q}$ is topological, then $\mathbf{K}$ is symmetric monoidal closed with structure $(-\Box_{\mathbf{K}} - , I', r', l', a', c', [- , -]_{\mathbf{K}})$, where $-\Box_{\mathbf{K}} -$ is the $\mathcal{Q}$-(semi) final tensor product, and $[- , -]_{\mathbf{K}}$ is the (semi) initial Hom-functor.

**Proposition 2.1.** i) $\mathbf{K}$ is topological in the sense of Herrlich [7], so $\mathbf{K}$ verifies the hypotheses of 1.3;

ii) the embedding $E$ is a normal reflective embedding with respect to $(-\Box_{\mathbf{K}} - , I', r', l', a', c', [- , -]_{\mathbf{K}})$ and to $(-\Box_{\mathbf{S}} - , I, r, l, a, c, [- , -]_{\mathbf{S}})$;

iii) $(-\Box_{\mathbf{S}} - , I, r, l, a, c, [- , -]_{\mathbf{S}})$ is not canonical.
Proof. Let us consider any $P$-quotient $p : X \to PA$ as a composite

\[ PFX \\ \downarrow i \\ X \\
\downarrow p \\
PA = P(FX/R) \\
\]

where $FX$ is the free semi-lattice on $X$, $\mathcal{R}$ is a congruence on $FX$, and $p$ is the canonical epimorphism; since $F(1) = 1$, then $K$ is well-fibred. $\mathcal{Q}$ is amnestic; if $\psi : Y \cong X$ and $(p, A) \in Q$, then also $(p \cdot \psi, A) \in Q$, so $\mathcal{Q}$ is transportable and $i)$ follows.

$ii)$ Consider $(p, A) \in K$ and $(1_{PA'}, A') \in S$; we shall prove that $[(p, A), (1_{PA'}, A')]_K \in S$.

Denote by $K((p, A), (1_{PA'}, A'))$ the set $K((p, A), (1_{PA'}, A'))$ with the semi-lattice structure defined by $(f, \tilde{f}) \vee (\varphi, \tilde{\varphi}) = (f \vee \varphi, \tilde{f} \vee \tilde{\varphi})$, where $\vee$ is the pointwise composition. The $\mathcal{Q}$-cone

\[ K((p, A), (1_{PA'}, A')) \to \text{Set}(X, PA') \to \mathcal{Q}(1_{PA'}, A'_x) = PA'_x \]

where $A'_x = A'$ and $ev_x$ is the evaluation in $x$, for any $x \in X$, admits a lifting in $K$, for $ev_x \cdot \mathcal{Q} : K((p, A), (1_{PA'}, A')) \to PA'_x$ is a $S$-morphism (for any $x$). Let us prove that such a lifting is initial in $K$.

Consider a $\mathcal{Q}$-cone in $K$, $(\xi_x, \tilde{\xi}_x) : (\tilde{p}, \tilde{A}) \to (1_{PA'}, A'_x)$, such that the following diagram

is commutative, for any $x \in X$. Now it suffices to give a morphism $\tau : \tilde{A} \to K((p, A), (1_{PA'}, A'))$ in $S$, such that $P \tau \cdot \tilde{p} = h$. Since $(\tilde{p}, \tilde{A}) \in Q$, $\tilde{A}$ is a quotient $FX/\mathcal{R}$ with respect to a congruence $\mathcal{R}$ on the free semi-lattice $FX$, and $\tilde{p} = p \cdot i$, where $p : FX \to \tilde{A}$ is the canonical epimorphism.
morphism and \( i : \bar{X} \to F\bar{X} \) is the inclusion (we omit the symbol \( P \) for the sake of brevity). Then, in the following diagram

\[
\begin{array}{c}
\bar{X} \\
\downarrow i \\
F\bar{X} \\
\downarrow \rho \\
\bar{A} \\
\end{array}
\quad \begin{array}{c}
\hat{K}((p, A), (1_{PA'}, A')) \\
\downarrow \lambda \\
\downarrow h \\
\end{array}
\quad \begin{array}{c}
\bar{A}'' \\
\downarrow \overline{\xi}_x \\
\end{array}
\quad \begin{array}{c}
ev \cdot Q \\
\end{array}
\quad \begin{array}{c}
A'' \tilde{x} \\
\end{array}
\]

there exists an unique \( S \)-morphism \( \lambda : F\bar{X} \to \hat{K}((p, A), (1_{PA'}, A')) \) with \( \lambda \cdot i = h \); furthermore \( \overline{\xi}_x \cdot \rho = ev_x \cdot Q \cdot \lambda \), for any \( x \in X \).

By the previous equality and since \( p : X \to PA \) is a \( P \)-epimorphism, it is possible to define a map \( \tau : \bar{A} \to \hat{K}((p, A), (1_{PA'}, A')) \) putting \( \tau(\rho(y)) = \lambda(y) \), for any \( y \in F\bar{X} \); since \( \lambda \) and \( \rho \) are \( S \)-morphism, then \( \tau \) is a \( S \)-morphism.

From the definition of internal Hom in \( K \), it follows that

\[
[(p, A), (1_{PA'}, A')]_K \cong (1_{\hat{K}((p, A), (1_{PA'}, A'))}, \hat{K}((p, A), (1_{PA'}, A')));
\]

so it is in \( S \). Since the map \( \psi : \hat{K}((p, A), (1_{PA'}, A')) \to S(A, A') \) defined by \( \psi(f, \overline{f}) = \overline{f} \) is an isomorphism between \( \hat{K}((p, A), (1_{PA'}, A')) \) and \( [A, A']_S \), then \( R[(p, A), (1_{PA'}, A')]_K \cong [(p, A), (1_{PA'}, A')]_K \cong [A, A']_S \) and \( R(A_1 \uplus_k A_2) \cong A_1 \uplus_S A_2 \), for any \( A_1, A_2 \in S \).

ii) follows from the reflection theorem of Day [2] by easy calculations.

iii) See [10] Section 2. \( \blacksquare \)

Now, we will give a Proposition to assure the canonicity of the reflected structure, in the basic situation. Let \( C \) be a concrete category and let \( \mathcal{L} : C \to Set \) be the forgetful functor.

**Definition 2.2.** A \( C \)-morphism \( f \) is called \( epi_{\mathcal{L}} \) iff \( \mathcal{L}f \) is epimorphism.

**Proposition 2.3.** In the basic situation, if \( A \) is \( \mathcal{L} \)-reflective in \( B \), then the reflected structure is canonical.
Proof. Denote by \((-\square',-I',r',l',a',c',[-,-])\) the symmetric monoidal closed structure obtained reflecting the B-structure \((-\square, -I, r, l, a, c, [\equiv, -])\) on A. By Day's reflection theorem, the following diagram

\[
\begin{array}{c}
\mathcal{B}(I, EA_1) \times \mathcal{B}(I, EA_2) \\
\downarrow \pi \times \pi \\
\mathcal{A}(I', A_1) \times \mathcal{A}(I', A_2) \\
\downarrow k \\
\mathcal{A}(I', A_1 \square A_2) = \mathcal{A}(I', R(\square_2 EA_1 \square EA_2)) \\
\end{array}
\]

is commutative, for any \(A_1, A_2 \in A\), where \(h(f, g) = (f \square g) \cdot l^{-1}, f : I \to \to EA_1, g : I \to EA_2\), (the same for \(k\) in \(A\)) and \(\eta\) is the unit the adjunction \(\pi\). B verifies the hypotheses of 1.1, so \(I\) is terminal in \(B\) and \(\mathcal{B}(\equiv) \equiv \mathcal{B}(I, \equiv)\); then \(\mathcal{B}(I, \eta)\) is epimorphism and, since \(h\) is bijective, \(k\) is epimorphism. Since \(A\) is reflective in \(B, I \in A\), so \(I = I'\); if \(\text{card}(\mathcal{B}' A) \leq 1\), for any \(A \in A\), \((\mathcal{B}' = \mathcal{B} : E)\), then \(k\) is an isomorphism and the structure is canonical; if there exists an object \(A \in A\) with \(\text{card}(\mathcal{B}' A) \geq 2\), \(A\) verifies the hypotheses of 1.1 with respect to \(\mathcal{B}'\); so \(k\) is injective and then \(k\) is bijective. By 1.1 it follows that the structure is canonical. \(\Box\)

Examples. Many examples of the previous situation are determined by topological categories \(B\). Let \(B\) be topological and let \((-\square_B, -I, r, l, a, c, [\equiv, -])\) be the symmetric monoidal closed structure on \(B\), where \(- \square_B -\) and \([-,-]_B\) are, respectively, the \(\mathcal{B}\)-final tensor and the \(\mathcal{B}\)-initial Hom (see [15] section 3, example 3; \(\mathcal{B} : B \to \text{Set}\)); it is easily seen that \([A, B]_B\) is an extremal subobject of \(B^\mathcal{B}A\) (power of \(B\) in \(B\)). Then, the following hold.

Proposition 2.4. For any full, (replete) and epireflective subcategory \(A\) of \(B\), the embedding \(E\) admits an enrichment to a normal reflective embedding with respect to the structure \((-\square_B, -I, r, l, a, c, [\equiv, -])_B\); furthermore \(A\) verifies 2.3, so the reflected structure is canonical.

Proof. Day's theorem can be applied for \(A\) is epireflective. The
result follows from classical properties of topological categories. □

**Remark 2.5.** All the categories \( \mathbf{A} \) of 2.4 are initially structured, in the sense of Nel [11], and the reflected structures are just the structures introduced by Porst-Wischnewsky [12] and by Činčura [1]. See also [11], [7].

Condition 2.3 is not necessary; in fact, it is possible to give the following example of a basic situation where \( \mathbf{A} \) is reflective, but not epi-\( \mathcal{Q} \)-reflective in \( \mathbf{B} \) and the reflected structure is canonical.

Denote by \( \mathbf{B} = \mathcal{Q} \mathbf{Met} \) the category of quasi-metric spaces and non-expansive maps. The objects of \( \mathcal{Q} \mathbf{Met} \) are pairs \((X, d_X)\), where \( X \) is a set and \( d_X : X \times X \to [0, \infty] \) is a map satisfying the natural pseudo-metric properties.

The morphisms of \( \mathcal{Q} \mathbf{Met} \) are maps \( f : (X, d_X) \to (Y, d_Y) \) such that \( d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2) \), for any \( x_1, x_2 \in X \) (see [8] p. 321). Let \( \mathbf{Sep}^\mathcal{Q} \mathbf{Met} \) be the full subcategory of \( \mathcal{Q} \mathbf{Met} \) of separated spaces \((d(x, y) = 0 \Rightarrow x = y)\) and let \( \mathbf{A} = \mathcal{Q} \mathbf{Met}^\mathcal{Q} \) be the subcategory of Cauchy-complete separated \( q \)-metric spaces.

It is easily seen that \( \mathbf{A} \) is reflective in \( \mathbf{B} \), but not epi-\( \mathcal{Q} \)-reflective; in fact \( \mathcal{Q} \mathbf{Met} \) is a topological category with natural forgetful functor \( \mathcal{Q} \mathbf{f} : \mathcal{Q} \mathbf{Met} \to \mathbf{Set} \) and, if \((X, d_X) \in \mathbf{Sep}^\mathcal{Q} \mathbf{Met}\), the inclusion into its Cauchy-completion \((Y, d_Y)\) is not, generally, surjective (it is dense with respect to the topology induced by \( d_Y \), see [8]).

Let \((\square \mathbf{B}, -I, r, l, a, c, [-, -]_B)\) be the structure on \( \mathbf{B} \), where \( \square \mathbf{B} \) — and \([-, -]_B\) are the \( \mathcal{Q} \)-final tensor product and the \( \mathcal{Q} \)-initial Hom, respectively.

**Proposition 2.6.** i) \( E : \mathbf{A} \to \mathbf{B} \) admits an enrichment to a normal reflective embedding;

ii) The reflected structure is canonical for it is the restriction on \( \mathbf{A} \) of the structure \((\square \mathbf{B}, -I, r, l, a, c, [-, -]_B)\).

**Proof.** i) Let \((Y, d_Y) \in \mathcal{Q} \mathbf{Met} \); we shall prove that \([ (X, d_X), (Y, d_Y) ]_B \in \mathcal{Q} \mathbf{Met} \), for any \((X, d_X) \in \mathcal{Q} \mathbf{Met} \). By definition, \([ (X, d_X), (Y, d_Y) ]_B \) is the set \( \mathbf{B}((X, d_X), (Y, d_Y)) \)-that we shall denote by \( \mathbf{B}(X, Y) \)-with the following \( q \)-metric:

\[
d_{\mathbf{B}(X, Y)}(f, g) = \sup_{x \in X} \{ d_Y(f(x), g(x)) \}.
\]

Then, \((\mathbf{B}(X, Y), d_{\mathbf{B}(X, Y)}) \in \mathbf{Sep}^\mathcal{Q} \mathbf{Met} \); furthermore, if \( \{f_n\}_{n \in \mathbb{N}} \) is a Cauchy-sequence in \((\mathbf{B}(X, Y), d_{\mathbf{B}(X, Y)})\), then \( \{f_n(x)\}_{n \in \mathbb{N}} \) is an uniform-
ly Cauchy-sequence in \((Y, d_Y)\). Since \((Y, d_Y)\) is complete, \(\{f_n(x)\}_{n \in \mathbb{N}}\)

is pointwise convergent to a map \(f : X \to Y\). Defining a "genuine metric" \(d'_Y\) on \(Y\) by \(d'_Y(y_1, y_2) = \min(1, d_Y(y_1, y_2))\), it is easily seen that \(\{f_n(x)\}_{n \in \mathbb{N}}\) uniformly converges to \(f(x)\) in \((Y, d_Y)\), for such a convergence is uniform in \((Y, d'_Y)\); so \(f\) is not expansive and \(\{f_n\}_{n \in \mathbb{N}}\) converges to \(f\) in \((B(X, Y), d_B(X, Y))\).

i) Follows by Day's theorem.

ii) For any \((X, d_X), (Y, d_Y) \in \mathbb{M}\), \((X, d_X) \boxtimes_B (Y, d_Y) = (X \times Y, d_B)\),

where \(d_B((x, y), (x_1, y_1)) = \sup \{d_C(f(x, y), f(x_1, y_1))\}\), for any \((C, d_C) \in \mathbb{M}\) and for any \(f : X \times Y \to C\) such that \(f(x, -)\) and \(f(\cdot, y)\) are

non-expansive for every \(x \in X\) and \(y \in Y\). Since \(p_X : X \times Y \to X\) and \(p_Y : X \times Y \to Y\) satisfy such a condition, then \(d_B((x, y), (x_1, y_1)) \geq d_{\text{prod}}((x, y), (x_1, y_1))\), where \(d_{\text{prod}}\) is the \(q\)-metric of the \(\mathbb{M}\)-product. Since

\[
d_C(f(x, y), f(x_1, y_1)) \leq d_C(f(x, y), f(x, y_1)) + \\
+ d_C(f(x, y_1), f(x_1, y_1)) \leq d_Y(y, y_1) + \\
+ d_X(x, x_1) \leq 2 \max(d_X(x, x_1), d_Y(y, y_1)) = \\
= 2 d_{\text{prod}}((x, y), (x_1, y_1)); \text{ then } d_{\text{prod}}((x, y), (x_1, y_1)) \leq \\
\leq d_B((x, y), (x_1, y_1)) \leq 2 d_{\text{prod}}((x, y), (x_1, y_1)), \text{ for any } (x, y), (x_1, y_1).
\]

If \((X, d_X), (Y, d_Y) \in \mathbb{M}\), then \((X, d_X) \times (Y, d_Y) \in \mathbb{M}\), so ii) follows. ■

REFERENCES


