SEMISIMPLENESS, COMPLETENESS, 
AND DIMENSION OF A BANACH ALGEBRA

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SOMMARIO. - In questa nota si caratterizzano la semisemplicità, la dimensione e certe proprietà di completezza di un'algebra di Banach combinando e facendo intervenire contemporaneamente proprietà algebriche, topologiche e di teoria della misura le quali sono naturalmente associate ad una tale algebra. La caratterizzazione ottenuta per le algebre semisemplici in termini di normalità, continuità e proprietà $T_2$ completano alcuni risultati precedenti di [3]. La caratterizzazione della completezza e della dimensione estende considerevolmente il lavoro di Cohen e risponde ad alcune delle questioni da lui poste in [1].

SUMMARY. - This note characterizes semisimpleness, dimension, and certain completeness properties of a Banach algebra by combining and interlacing algebraic, topological, and measure-theoretic properties naturally associated with such an algebra. Our characterization of semisimpleness in terms of normability, continuity, the $T_2$ property, and denseness nicely rounds out some earlier results in [3]. Our characterizations of completeness and dimension considerably extend Cohen's work and answer some of the questions raised by him in [1].

1. Preliminaries.

Let $\mathcal{K}$ and $\mathfrak{M}$ respectively denote the non zero homomorphisms and the maximal ideals of a complex, commutative Banach algebra $(X, ||||)$ with identity $e$ and continuous dual $X^\prime$. If $M \in \mathfrak{M}$, then

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$X/M = C$ and $x + M = x(M) (e + M)$ for each $x \in X$. Therefore each fixed $x \in X$ determines a mapping $\hat{x} : \mathfrak{M} \rightarrow C$, and $\mathfrak{M}$ is assumed to carry the weakest (compact, $T_2$) topology under which every member of $\mathcal{C} = \{\hat{x} : x \in X\}$ is continuous. As is well known, the bijection $\mathcal{H} \rightarrow \mathfrak{M}$ is bicontinuous when $\mathcal{H}$ carries the weakest topology $\sigma(\mathcal{H}, X)$ under which every $\xi : \mathcal{H} \rightarrow C \ (x \in X)$ is continuous.

It is assumed throughout that $\nu$ is a probability measure on $\mathfrak{M}$ which is positive on non-empty open subsets of $\mathfrak{M}$. For example, if $\mathfrak{M}$ is separable with $\{M_n : n \in N\} = \mathfrak{M}$ [as will be the case ([2], 426) when $X$ is separable or $\mathfrak{M}$ is metrizable], then $\nu = \sum_{n=1}^{\infty} 2^{-n} \chi_{M_n}$ meets our requirements (here $\chi_{M_n}$ denotes the characteristic function of $M_n \in \mathfrak{M}$).

2. Semisimpleness Property.

By definition, $X$ is semisimple if $\cap M = \{0\}$. One can readily show that the following are also equivalent: (1) $X$ is semisimple; (2) the seminorm $r_\sigma(x) = \sup |\hat{x}(M)| = \sup |h(x)|$ is a norm on $\mathfrak{M}$; (3) the mapping $\psi : X \rightarrow \hat{X}$ is injective (in fact, an $r_\sigma - \|\|_0$ congruence since $r_\sigma(x)$ coincides with the sup norm $\|\hat{x}\|_0$ on $\mathfrak{M}$ for each $x \in X$).

Semisimpleness may be characterized in terms of normability, continuity, the $T_2$ property, and denseness once we prove the following.

**Lemma. Distinct members of $\mathcal{H}$ are linearly independent.**

**Proof.** By induction. The result is trivially true for $n = 1$. If $a_1 h_1 + a_2 h_2 = 0$ and $h_2(x_o) \neq h_1(x_o)$ for some $x_o \in X$, subtract $h_2(x_o) \{a_1 h_1(x) + a_2 h_2(x)\} = 0$ from $a_1 h_1(x) h_1(x_o) + a_2 h_2(x) h_2(x_o) = 0$ to obtain $a_1 \{h_1(x) - h_2(x_o)\} h_1(x) = 0$ for each $x \in X$. Thus, $a_1 = a_2 = 0$. If every set of $n$ distinct members of $\mathcal{H}$ is linearly independent and $\sum_{i=1}^{n+1} a_i h_i = 0$, then $h_{n+1}(x_o) \neq h_1(x_o)$ for some $x_o \in X$. By hypothesis, $\alpha_1 = 0$
and \( \sum_{i=2}^{n+1} \alpha_i h_i = 0 \) implies that \( \alpha_i = 0 \) for all \( i = 2, 3, ..., n + 1 \).

**Theorem 1.** The following are equivalent for \( X \):

(i) \( X \) is semisimple

(ii) The seminorm \( p_\alpha(x) = \left\{ \int |x|^\alpha d\nu \right\}^{1/\alpha} \) (\( \alpha \geq 1 \)) is a norm

(iii) There is a Hausdorff TVS \( Y \) and a collection \( \mathcal{A} \): \( X \rightarrow \bigcap_{A \in \mathcal{A}} A^{-1}(0) = \{0\} \)

(iv) \( \sigma(X, X') = T_2 \)

(v) \( [\mathcal{K}] \sigma(X', X) = X' \)

**Proof.** By definition, \( x \in M \) iff \( x(M) = 0 \) and \( x \in \bigcap M \) iff \( \hat{x} = 0 \).

If \( X \) is semisimple and \( p_\alpha(x) = 0 \), then \( \hat{x} = 0 \) a.e. relative \( \nu \). Since \( u = \{ M \in \mathcal{K} : x(M) \neq 0 \} \) is open, \( u = \emptyset \) and \( \hat{x} = 0 \) yields \( x = 0 \). Conversely, if \( p_\alpha(x) = 0 \) and \( x \in \bigcap M \), then \( \hat{x} = 0 \) and \( p_\alpha(x) = 0 \) gives \( x = 0 \).

Thus, (i) \( \iff \) (ii). Since (ii) is precisely the requirement that the \( p_\alpha \) topology be \( T_2 \), one has (i) \( \Rightarrow \) (iii) by taking \( Y = (X, p_\alpha) \) and \( \mathcal{A} = \{1\} \).

If \( \mathcal{A} \) satisfies (iii), the weakest topology on \( X \) making all \( A \in \mathcal{A} \) continuous is \( T_2 \) and (by definition) weaker than \( p_\alpha \). Therefore, \( p_\alpha \) is \( T_2 \) and (ii) holds. Surely (i) \( \iff \) (iv) since (i) means \( \bigcap \{0\} = \{0\} \)

and \( \sigma(X, X') \) is determined by the seminorms \( \{p_h(x) = |h(x)| : h \in \mathcal{K}\} \). Finally, the polar \( [\mathcal{K}]^o \) always contains \( \{0\} \subset X \). The reverse inclusion holds iff \( \bigcap \{0\} = \{0\} \). Therefore, \( X \) is semisimple iff \( \{0\} = [\mathcal{K}]^o \)

which is equivalent to \( \{0\}^o = X' \) being equal to \( [\mathcal{K}]^o = [\mathcal{K}] \sigma(X', X) \)

[See [5], 274, for example].

Remark. If \( X \) is semisimple, \( (x, y) = \int x \bar{y} d\nu \) defines an inner product on \( X \).

**Corollary 1.1.** A finite dim Banach algebra \( X \) is semisimple iff \( \mathfrak{N}(\mathcal{K}) = \dim X \).

**Proof.** \( \dim X < \infty \) assures that \( \dim X' = \dim X \) and \( [\mathcal{K}] = [\mathcal{K}] \sigma(X', X) \).
Therefore, $X$ is semisimple iff $[\mathcal{H}] = X'$.

3. Dimension and Completeness.

Throughout this section, $X$ is assumed to be semisimple. For convenience, we let $X'_\tau$ abbreviate the continuous dual of $(X, \tau)$ and we let $\| \|_1 \sim \| \|_2$ denote norm equivalence.

**THEOREM 2.** The following are equivalent for $X$:

(i) \quad $\text{Dim } X < \infty$

(i)' \quad $\text{Dim } L (\mathcal{H}, \nu) < \infty$

(ii) \quad $(X, p_\alpha) \text{ is complete}$

(ii)' \quad $(\hat{X}, \| \|_a) \text{ is complete}$

(iii) \quad $p_\alpha \sim r_\tau$ on $X$

(iii)' \quad $\| \|_a \sim \| \|_\nu$ on $X$

**Proof.** Clearly (i) \Leftrightarrow (i)' by virtue of $X \subset C(\mathcal{H}) \subset L_\alpha (\mathcal{H}, \nu) \subset \pi C$, $\mathcal{H}$. Corollary 1.1 and the bijective nature of $\mathcal{H} \rightarrow \mathcal{H}$. Furthermore, (ii)' \Leftrightarrow (ii) and (iii)' \Leftrightarrow (iii) since $\psi: (X, p_\alpha) \rightarrow (\hat{X}, \| \|_a)$ and $\psi: (X, r_\tau) \rightarrow (\hat{X}, \| \|_\nu)$ are congruences. It suffices therefore to prove (i) \Leftrightarrow (iii) equivalent. First, (i) \Rightarrow (ii) since a finite dim vector space has exactly one $T_2$ topology compatible with its linear structure. Next, (ii) \Rightarrow (iii) by the uniqueness of complete norm on the semisimple algebra $X$ ([5], 262). Since $p_\alpha (x) \leq r_\tau (x) \forall x \in X$, condition (iii) is equivalent to the existence of some $\lambda > 0$ satisfying $r_\tau (x) \leq \lambda p_\alpha (x) \forall x \in X$. Therefore $\{p_\alpha (x)\}_{a} \leq \{r_\tau (x)\}_{a} p_1 (x) \leq \lambda^{-1} p_1 (x) \forall x \in X$ and the norm topologies on $X$ satisfy $r_\tau = p_\alpha \subset p_1$. At the same time, (Riesz-Markoff Theorem) there is a $\Phi_\nu \in C' (\mathcal{H})$ satisfying $\Phi_\nu (f) = \int f \, d\nu \forall f \in C \mathcal{H}$. Since $C \mathcal{H} \rightarrow C \mathcal{H}$ is continuous, $p_1: x \rightarrow \{\Phi_\nu | f | \} (x) = \int | f | (x) \, d\nu$ is $r_\tau$ -continuous. In particular, $p_1$ is $\sigma (X, X'_{r_\tau})$ -continuous and $p_1 \subset \sigma (X, X'_{r_\tau}) \subset r_\tau$. Therefore, $p_1 = \sigma (X, X'_{r_\tau})$ and $\text{dim } X < \infty$ ([5], 167).

**COROLLARY 2.1.** $X$ is finite dim iff $X'_{p_\alpha} = X'$.

**Proof.** $\text{Dim } X < \infty$ implies that $p_\alpha = \| \|$. Conversely, $p_\alpha = \tau (X, X'_{p_\alpha})$ (the Mackey topology on $X$) and $X'_{p_\alpha} = X'$ implies that $p_\alpha = \tau (X, X') = \| \| \supset r_\tau$. Thus, $p_\alpha = r_\tau$ on $X$.

As is well known, $(X, r_\tau)$ is complete if and only if $X$ is $\| \|_a$ - closed in $C \mathcal{H}$. By comparison, $(\mathcal{H}, \nu, \| \|_a) –$ completeness yields
COROLLARY 2.2. \((X, p_a)\) is complete if and only if \(X\) is \(|||_\alpha\) closed in \(L_\alpha(\mathcal{M}, \nu)\). The completion of \((X, p_a)\) is \(\overline{X}_{|||_\alpha}\), \(|||_\alpha\) \(\subset\) \(L_\alpha(\mathcal{M}, \nu)\).

The nature of \(\nu\) can be exploited to generalize the fact (Theorem 2) that \(p_a\) — completeness implies \(r_\sigma\) — completeness.

EXAMPLE 1. If \(\mathcal{C} \subset r_\sigma\) and \(x_n \rightarrow x\) implies that \(x_n \rightarrow x\), then \(\mathcal{C}\) — sequential completeness implies \(r_\sigma\) — completeness. Verification: If \(\mathcal{C}\) above is sequentially complete and \(f \in \overline{X}_{|||_\sigma} \subset C(\mathcal{M})\), there exist \(x_n \in X\) such that \(||x_n - f||_\sigma < 1/n\). Since \(\{x_n\}\) is \(|||_\sigma\)-convergent in \(\overline{X}\), the sequence \(x_n\) (being \(r_\sigma\), therefore \(\mathcal{C}\)-Cauchy) is \(\mathcal{C}\)-convergent to some \(x \in X\). In particular, \(x_n \rightarrow \hat{x}\) and (since \(\hat{x}_n \rightarrow f\)) \(\hat{x} = f\) a.e. on \(\mathcal{M}\). The reasoning in Theorem 1 confirms that \(\hat{x} = f\) and \(\overline{X} = \overline{X}_{|||_\sigma}\) in \(C(\mathcal{M})\).

4. Additional Comments.

The preceding results may be specialized [4] as well as extended [3]. In fact, all our results remain valid for complete \(LMCT_2\) \(Q\)-algebras with identity; that is, topological algebras with a nbd. base of absolutely convex idempotent sets (\(LMC\) algebras) whose set of units is open (\(Q\)-algebras). This generalization is non vacuous.

EXAMPLE 2. The algebra, under pointwise operations, of infinitely differentiable functions on \([a, b] \subset \mathbb{R}\) determined by \(q_n (f) = \sup |f^{(n)} (t)| : n \in \mathbb{N}\) is a semisimple, \(LMC\), Frechet \(Q\)-algebra which ([5], 278) is non normable.

REFERENCES