APPROXIMATION OF FIXED POINTS BY CESARO'S MEANS OF ITERATES (*)

di Silvio Massa (a Milano) (**)

Sommario. - *Sia K un sottoinsieme chiuso e convesso di uno spazio di Banach uniformemente convesso, e sia T un'applicazione non espansiva di K in sé, dotata di punti fissi. In questa Nota si dimostra che, se $x \in K$ e $\{T^n(x)\}$ ammette punti limite, la successione delle medie secondo Cesaro di $\{T^n(x)\}$ converge a un punto fisso di T. Si osserva inoltre che il risultato precedente vale anche sotto condizioni più generali e si dànno controesempi per il caso di mappe quasi non espansive.*

Summary. - *Let K be a closed convex subset of a uniformly convex Banach space and let T a nonexpansive selfmapping of K which has at least one fixed point. In this Paper we prove that, if $x \in K$ and $\{T^n(x)\}$ has some limit point, then the sequence of the Cesaro's means of $\{T^n(x)\}$ converges to a fixed point of T. We remark moreover that the above result still holds under more general conditions and we give some counter-examples for quasi-nonexpansive mappings.*

1. Introduction.

Here and throughout the Paper, let X be a uniformly convex Banach space, K a closed convex subset of X, T a nonexpansive self-mapping of K, $F(T)$ the set of the fixed points of T.

For every x in K, let $0(x) = \bigcup_0^\infty T^n(x)$ the orbit of x, $L(x)$ the set of the limit points of $\{T^n(x)\}$, and $L(K) = \bigcup_{x \in K} L(x)$.

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(**) Indirizzo dell'Autore: Istituto di Matematica F. Enriques - Università degli Studi di Milano - Via C. Saldini, 50 - 20133 Milano.

Some Authors took an interest in the problem of constructing a sequence that converges to a fixed point of T: usually they give some sufficient conditions for the convergence of $\{T^n(x)\}$ or for the convergence of the sequence of the iterates of an asymptotically regular mapping constructed by T.

The purpose of this Paper is to prove that, if $F(T) \neq \emptyset$, the sequence of the Cesaro's means of $\{T^n(x)\}$ converges to a fixed point of T, for every x in K such that $L(x) \neq \emptyset$.

The advantage of this method is that, to locate the fixed points of T, it is not necessary to construct any mapping, but it is sufficient to know the sequence $\{T^n(x)\}$.

2. Results.

The following Theorem holds:

THEOREM. *Let $T: K \to K$ be nonexpansive, and $F(T) \neq \emptyset$. Then $\forall x \in K$ such that $L(x) \neq \emptyset$, the sequence*

$$M_n(x) = \frac{1}{n} \sum_{0}^{n-1} T^k(x)$$

converges to a fixed point of T.

REMARKS.

1. $L(x) \neq \emptyset \not\Rightarrow F(T) \neq \emptyset$ (see [3], Theorem 2.1); moreover $F(T) \neq \emptyset \not\Rightarrow \exists x \notin F(T)$: $L(x) \neq \emptyset$ (consider for instance a linear map $T: l^2 \to l^2$, $T: e_n \mapsto e_{n+1}$).

2. If K is compact, then $\{M_n(x)\}$ converges to a fixed point for every x in K. Thus the mapping $M: x \mapsto \lim_{n \to \infty} M_n(x)$ is a retraction of K onto $F(T)$. Remark that $M(y) = M(x)$ $\quad \forall y \in \overline{0(x)}$.

3. If $L(x)$ is finite, the process for the location of the fixed point $\bar{y}$ determined by x can be simplified, provided that we start from a point of $L(x)$.

Indeed let $L(x) = \{y_1, y_2, \dots, y_n\}$. We have

$$\bar{y} = \frac{1}{n}(y_1 + y_2 + \dots + y_n) = \frac{1}{n} \sum_{k=0}^{n-1} T^k(y_i).$$

4. In the Theorem, the assumption $F(T) \neq \emptyset$ can be replaced (if the other assumptions are still satisfied) by the assumption $\{M_n(y)\}$ bounded for some $y \in L(K)$ (1) or even by the assumption $\{M_n(x)\}$ bounded for some x such that $L(x) \neq \emptyset$ (2).

5. It is easy to prove, making use of the results of [4], that the Theorem still holds in the reflexive strictly convex spaces. More generally it holds in a strictly convex space for the points $x \in K$ such that, for some $y \in L(x)$, $\{M_n(y)\}$ has a weakly convergent subsequence.

6. If T is quasi-nonexpansive (3) (but not necessarily nonexpansive), the Theorem fails, even if T is continuous and $F(T)$ is a singleton. Indeed let $X = \mathbb{C}$, $K = \{z : |z| \leqq 1\}$.

$$T: \rho \cdot \exp(i\theta) \mapsto \begin{cases} \rho \cdot \exp\left(2i\theta + i\dfrac{\pi}{2}\right) & \text{if } 0 \leqq \theta \leqq \dfrac{\pi}{2} \\ \rho \cdot \exp\left(i\dfrac{\theta}{2} + \dfrac{5}{4}\pi i\right) & \text{if } \dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2} \\ \rho \cdot \exp\left(i\theta - \dfrac{3}{2}\pi i\right) & \text{if } \dfrac{3\pi}{2} \leqq \theta < 2\pi. \end{cases}$$

$F(T) = \{0\}$, T is continuous and quasi-nonexpansive. We have $T^{3k}(1) = 1$, $T^{3k+1}(1) = i$, $T^{3k+2}(1) = -i$ and then $M_n(1) \to \dfrac{1}{3} \notin F(T)$.

7. If $X = \mathbb{R}$, the Theorem holds for quasi-nonexpansive and continuous mappings. Indeed let $K = [a, b]$, T quasi-nonexpansive and continuous. We consider the only two possible cases:

i) $F(T) = \{u\}$. $\{T^n(x)\}$ converges to u or has two limit points, equidistant from u. In any case $M_n(x) \to u$.

ii) $F(T) = [c, d]$. $\{T^n(x)\}$ is necessarily monotone, therefore it converges (to a fixed point, by the continuity of T) and this implies $\lim M_n(x) = \lim T^n(x)$.

(1) This assumption is equivalent to $F(T) \neq \emptyset$ (see [4]).
(2) It is sufficient to observe that $y \in L(x) \Rightarrow \|M_n(x) - M_n(y)\| \leqq \|x - y\|$.
(3) i. e. $F(T) \neq \emptyset \wedge \|T(x) - u\| \leqq \|x - u\| \; \forall x \in K \; \forall u \in F(T)$.

8. The Theorem of n. 7 fails if T is quasi-nonexpansive but not continuous, even if K is compact. Indeed let $K=[0,2]$, $T(x)=0$ if $0\leqq x\leqq 1$, $T(x)=\frac{x+1}{2}$ if $1<x\leqq 2$. $T^n(2)\to 1 \Rightarrow M_n(2)\to 1\notin F(T)$.

9. The Theorem of n. 7 does not hold under the only assumption that T is continuous, even if $F(T)\neq\emptyset$ (and K is compact). Indeed let $K=[0,3]$, $T(x)=3-\frac{x}{2}$ if $0\leqq x\leqq 2$, $T(x)=6-2x$ if $2\leqq x\leqq 3$. $F(T)=$ $=\{2\}$ and $M_n(x)\to\frac{6+x}{4}$ if $0\leqq x\leqq 2$, $M_n(x)\to\frac{6-x}{2}$ if $2\leqq x\leqq 3$.

3. Proof of the Theorem.

Let $y\in L(x)$; $\Lambda(y)$ the (real) linear variety spanned by $\{T^n(y)\}$; $C=\overline{co}(\{T^n(y)\})$. It is well-known (see [3]) that there exists an affine isometry S: $\overline{\Lambda(y)}\to\overline{\Lambda(y)}$ such that $S|_C=T|_C$. We set

$$W=\overline{\Lambda(y)}-y=\{w\in X:\ w=z-y,\ z\in\overline{\Lambda(y)}\}$$

and

$$U:\ w=z-y\mapsto S(z)-S(y)\quad \forall w\in W.$$

W is a (real) Banach space contained in X and U is a linear isometry of W into itself.

LEMMA 1. *For every* $w\in W$

$$\frac{1}{n}\sum_0^{n-1} U^k(w)\to 0. \tag{3.1}$$

Indeed, $\forall w\in W$ there exists (see [1])

$$\bar{w}=\lim\frac{1}{n}\sum_0^{n-1} U^k(w)$$

and then, for $n\geqq\bar{n}$, we have

$$\left\|\sum_0^{n-1} U^k(w)\right\|\geqq\frac{n}{2}\|\bar{w}\|.$$

CASE *a*) Let $w=S(y)-y$. For every $n\geqq 1$ it is

$$S^n(y)=y+\sum_{k=0}^{n-1} U^k(w). \tag{3.2}$$

If $\bar{w} \neq 0$, for $n \geq \bar{n}$ we have

$$\|S^n(y) - y\| = \left\| \sum_0^{n-1} U^k(w) \right\| \geq \frac{n}{2} \|\bar{w}\|$$

which is absurd, since $\{S^n(y)\} = \{T^n(y)\}$ is bounded (4).

CASE *b*) Let $w = S^p(y) - y$. (3.1) holds for $p=1$; let it hold for $p-1$ and remark that

$$S^p(y) - S(y) = S(S^{p-1}(y)) - S(y) = U(S^{p-1}(y) - y).$$

Then

$$\frac{1}{n} \sum_{k=0}^{n-1} U^k(S^p(y) - y) = \frac{1}{n} \sum_{k=0}^{n-1} \{U^k(S^p(y) - S(y)) + U^k(S(y) - y)\} =$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} U^{k+1}(S^{p-1}(y) - y) + o(1) =$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} U^k(S^{p-1}(y) - y) + o(1).$$

CASE *c*) w whatever. We may write $w = \bar{z} - y$, $\bar{z} \in \overline{\Lambda(y)}$. $\forall \varepsilon > 0\ \exists z = \sum_{i=1}^{p} \alpha_i S^{n_i}(y)$ such that $\|z - \bar{z}\| < \varepsilon$.

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} U^k(\bar{z} - y) \right\| = \frac{1}{n} \left\| \sum_{k=0}^{n-1} U^k(\bar{z} - z) + U^k(z - y) \right\| \leq$$

$$\leq \|\bar{z} - z\| + \frac{1}{n} \left\| \sum_{k=0}^{n-1} U^k \left\{ \sum_{i=1}^{p} \alpha_i (S^{n_i}(y) - y) \right\} \right\| \leq$$

$$\leq \|\bar{z} - z\| + \sum_{i=1}^{p} |\alpha_i| \cdot \left\| \frac{1}{n} \sum_{k=0}^{n-1} U^k(S^{n_i}(y) - y) \right\|.$$

LEMMA 2. *$\{M_n(y)\}$ converges to a fixed point of T.*

Indeed, if we set $w = S(y) - y$, by (3.2) we have

$$M_n(y) = \frac{1}{n} \sum_0^{n-1} S^k(y) = y + \frac{1}{n} \{(n-1)w + (n-2)U(w) + \ldots + U^{n-2}(w)\}.$$

(4) Recall that $F(T) \neq \emptyset \Rightarrow \{T^n(x)\}$ and so $\{M_n(x)\}$ bounded $\forall x \in K$.

Therefore

$$(3.3) \qquad (U-I)(M_n(y)-y)=-w+\frac{1}{n}\sum_{0}^{n-1} U^k(w) \rightarrow -w=y-S(y).$$

As $\{M_n(y)\}$ is bounded, $\exists \{n_i\}$: $M_{n_i}(y) \rightharpoonup \bar{y}$ (weakly).

As $U-I$ is linear and continuous in the strong topology of W, it is also continuous in the weak topology (5); therefore

$$(U-I)(M_{n_i}(y)-y) \rightharpoonup (U-I)(\bar{y}-y)=S(\bar{y})-S(y)-\bar{y}+y$$

and, from (3.3) $S(\bar{y})=\bar{y}=T(\bar{y})$ (since $\bar{y}\in C$).

We are left the proof that $M_n(y) \rightarrow \bar{y}$. Remark that

$$S(\bar{y})=U(\bar{y}-y)+S(y)$$

and, for every $k \geqq 1$

$$S^k(\bar{y})=U^k(\bar{y}-y)+S^k(y).$$

Therefore

$$\bar{y}=\frac{1}{n}\sum_{0}^{n-1} S^k(\bar{y})=\frac{1}{n}\sum_{0}^{n-1}\{U^k(\bar{y}-y)+S^k(y)\}=$$

$$=M_n(y)+\frac{1}{n}\sum_{0}^{n-1} U^k(\bar{y}-y)$$

and this complete the proof of Lemma 2.

For every $\varepsilon>0$ there exists $p=p(\varepsilon)$ such that $\|T^p(x)-y\|<\varepsilon$ and therefore $\|T^{p+k}(x)-T^k(y)\|<\varepsilon \ \forall k>0$.

We have

$$\|M_n(T^p(x))-M_n(y)\| = \frac{1}{n}\left\| \sum_{k=0}^{n-1} T^{p+k}(x)-T^k(y)\right\| < \varepsilon$$

and, moreover, for $n>p$

$$\|M_n(T^p(x))-M_n(x)\| = \frac{1}{n}\left\| \sum_{p}^{p+n-1} T^k(x) - \sum_{0}^{n-1} T^k(x)\right\| =$$

(5) See [2], V. 3. 15.

$$= \frac{1}{n} \left\| \sum_{n}^{p+n-1} T^k(x) - \sum_{0}^{p-1} T^k(x) \right\| \leqq \frac{1}{n} \sum_{0}^{p-1} \|T^{n+k}(x) - T^k(x)\| \leqq$$

$$\leqq \frac{p}{n} \operatorname{diam} 0(x).$$

Therefore $M_n(x) \to \bar{y}$ for $n \to \infty$ and the Theorem is proved.

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