A FIXED POINT THEOREM
FOR METRIC SPACES (*)

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SOMMARIO. - Oggetto della presente nota è la dimostrazione di un teorema di punto fisso tramite espressioni razionali, e di dedurne alcuni risultati che non sembrano ancora noti. 

SUMMARY. - The object of this paper is to prove a fixed point theorem using rational expression and to study related results which are believed to be new. 

1. - Let \((X, d)\) be a complete metric space, and let \(T: X \to X\) satisfy \(d(Tx, Ty) \leq K d(x, y)\)

where \(0 \leq K < 1\) and \(x, y \in X\). Then, by Banach's fixed point theorem \(T\) has a unique fixed point.

Many extensions and generalizations of Banach's theorem were derived in recent years. For related results see [1], [2], [3], [4], [5], [6]. In this note, we shall prove a fixed point theorem using symmetric rational expression and study the continuity of fixed point. 

THEOREM 1. Let \((X, d)\) be a complete metric space and \(T: X \to X\) satisfy

\[
\tag{A} d(Tx, Ty) \leq K \frac{d(x, Tx) d(x, Ty) + d(y, Ty) d(y, Tx)}{d(x, Ty) + d(y, Tx)}
\]

where \(0 \leq K < 1\) and \(x, y \in X\). Then \(T\) has a unique fixed point.

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PROOF. Let \( x_0 \in X \). Put \( x_n = T(x_{n-1}), \ n = 1, 2, 3, \ldots \) then we have

\[
\begin{align*}
\hat{d}(x_1, x_2) &= \hat{d}(Tx_0, Tx_1) \\
\leq K \frac{\hat{d}(x_0, Tx_0) \cdot \hat{d}(x_0, Tx_1) + \hat{d}(x_1, Tx_1) \cdot \hat{d}(x_1, Tx_0)}{\hat{d}(x_0, Tx_1) + \hat{d}(x_1, Tx_0)} \\
&= K \frac{\hat{d}(x_0, x_1) \cdot \hat{d}(x_0, x_2) + \hat{d}(x_1, x_2) \cdot \hat{d}(x_1, x_1)}{\hat{d}(x_0, x_2) + \hat{d}(x_1, x_1)}. 
\end{align*}
\]

Hence \( \hat{d}(x_1, x_2) \leq K \hat{d}(x_0, x_1) \).

Similarly, we have

\[
\begin{align*}
\hat{d}(x_2, x_3) &= \hat{d}(Tx_1, Tx_2) \\
\leq K \frac{\hat{d}(x_1, x_2) \cdot \hat{d}(x_1, x_3) + \hat{d}(x_2, x_3) \cdot \hat{d}(x_2, x_2)}{\hat{d}(x_1, x_3) + \hat{d}(x_2, x_2)}. 
\end{align*}
\]

Therefore, \( \hat{d}(x_2, x_3) \leq K \hat{d}(x_1, x_2) \leq K^2 \hat{d}(x_0, x_1) \).

In general, we have

\[
\hat{d}(x_n, x_{n+1}) \leq K^n \hat{d}(x_0, x_1) 
\]

This means that \( \{x_n\} \) is a Cauchy-sequence which, by the completeness of \( X \), converges to some point \( x \in X \). For the point \( x \),

\[
\hat{d}(x, Tx) \leq \hat{d}(x, x_{n+1}) + \hat{d}(Tx_n, Tx) \\
\leq \hat{d}(x, x_{n+1}) + K \frac{\hat{d}(x_n, Tx_n) \cdot \hat{d}(x_n, Tx) + \hat{d}(x, Tx) \cdot \hat{d}(x, Tx_n)}{\hat{d}(x_n, Tx) + \hat{d}(x, Tx_n)} \\
= \hat{d}(x, x_{n+1}) + K \frac{\hat{d}(x_n, x_{n+1}) \cdot \hat{d}(x_n, Tx) + \hat{d}(x, Tx) \cdot \hat{d}(x, x_{n+1})}{\hat{d}(x_n, Tx) + \hat{d}(x, x_{n+1})}. 
\]

Letting \( n \to \infty \), we get \( \hat{d}(x, Tx) = 0 \). Hence \( x \) is a fixed point of \( T \).

For the unicity of \( x \), consider a \( y \neq x \) such that \( Ty = y \). Then

\[
\hat{d}(x, y) = \hat{d}(Tx, Ty) \leq K \frac{\hat{d}(x, Tx) \cdot \hat{d}(x, Ty) + \hat{d}(y, Ty) \cdot \hat{d}(y, Tx)}{\hat{d}(x, Ty) + \hat{d}(y, Tx)} \\
= K \frac{\hat{d}(x, x) \cdot \hat{d}(x, y) + \hat{d}(y, y) \cdot \hat{d}(y, x)}{\hat{d}(x, y) + \hat{d}(x, y)}. 
\]

Hence \( \hat{d}(x, y) \leq 0 \) or \( x = y \). This completes the proof.
A fixed point theorem for metric spaces

As simple consequence we state the following theorems.

THEOREM 2. Let \( T \) and \( S \) be self mappings of a complete metric space \((X, d)\) such that \( T \) satisfies (A) and \( TS = ST \), then \( S \) and \( T \) have a unique common fixed point.

PROOF. If \( x_0 \) is the unique fixed point of \( T \), then \( T(x_0) = x_0 \) implies \( TS(x_0) = ST(x_0) = S(x_0) \), which gives \( S(x_0) = x_0 \), that is, \( S \) and \( T \) have a unique common fixed point.

THEOREM 3. If \( T \) be a self mapping of a complete metric space \((X, d)\) such that for positive integer \( n \), \( T^n \) satisfies (A). Then \( T \) has a unique fixed point in \( X \).

PROOF. Let \( x_0 \) be the unique fixed point of \( T^n \). Then
\[
T(T^n x_0) = T x_0
\]
or
\[
T^n (T x_0) = T x_0.
\]
This gives \( T x_0 = x_0 \).

2. - In this section, we prove a convergence theorem concerning fixed points.

THEOREM 4. Suppose \((X, d_0)\) is a metric space and \( \{d_n\} \) is a sequence of metrics converging uniformly to \( d_0 \). Let \( \{T_n\} \) be a sequence of mappings converging \( d_0 \)-pointwise to a map \( T_0 \) with fixed point \( x_0 \) and let each \( T_n \) having fixed points \( x_n \) satisfy

\[
d_n(T_n x, T_n y) \leq K \left( \frac{d_n(x, T_n x) d_n(x, T_n y) + d_n(y, T_n y) d_n(y, T_n x)}{d_n(x, T_n y) + d_n(y, T_n x)} \right)
\]

where \( 0 < K < 1 \) and \( x, y \in X \). Then \( \{x_n\} \) converges to \( x_0 \).

PROOF. For any \( \varepsilon > 0 \), the conditions of the theorem give

\[
|d_n(x, y) - d_0(x, y)| < \frac{(1 - K) \varepsilon}{2 + K}
\]
and

\[
d_0(T_n x_0, T_0 x_0) < \frac{(1 - K) \varepsilon}{2 + K}
\]
whenever \( n \geq N \) for some natural number \( N \).
Now for $n \geq N$ we get,

\[
\begin{align*}
    d_0(x_n, x_0) &= d_0(T_n x_n, T_0 x_0) 
    \leq d_0(T_n x_n, T_n x_0) + d_0(T_n x_0, T_0 x_0) \\
    &\leq d_n(T_n x_n, T_n x_0) + \frac{(1 - K) \varepsilon}{2 + K} + \frac{(1 - K) \varepsilon}{2 + K} \\
    &\leq K d_n(x_n, x_0) d_n(x_n, T_n x_0) + d_n(x_0, T_n x_0) d_n(x_0, T_n x_n) \\
    &\leq K d_n(x_n, x_0) + d_n(x_0, x_n) + d_n(x_0, T_n x_0) + 2 \frac{(1 - K) \varepsilon}{2 + K} \\
    &\leq K d_0(x_0, x_n) + \frac{2 (1 - K) \varepsilon}{2 + K} \leq K d_0(x_0, x_n) \\
    &\quad + \frac{K (1 - K) \varepsilon}{2 + K} + \frac{2 (1 - K) \varepsilon}{2 + K}.
\end{align*}
\]

Hence

\[
    d(x_n, x_0) \leq \varepsilon \quad \text{for} \quad n \geq N.
\]

This shows that $\{x_n\}$ converges to $x_0$.

**BIBLIOGRAPHY**


