SOME NEW SUMMATION FORMULAE FOR HYPERGEOMETRIC SERIES OF TWO VARIABLES (*)

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SOMMARIO. - *In questa nota dimostriamo alcune nuove formule di sommazione per serie ipergeometriche in due variabili. Questi tipi di formule trovano applicazione nella risoluzione di certi problemi di Fisica teorica e in problemi di ottimizzazione in scienza.*

SUMMARY. - *In this paper we prove some new summation formulae for hypergeometric series of two variables. These type of formulae have applications in the solutions of certain problems in theoretical physics and optimization problems in Management Sciences.*

1. Introduction.

This paper is a continuation of the previous paper [1]. In our previous paper we have proved some transformation formulae for hypergeometric series of two variables. Recently Professor Carlitz [5] has proved Saalschutzian theorem for double series and later on, in another paper [6] he gave the sum of another double series.

Further, Carlitz [7] proved a transformation formula for multiple hypergeometric series. Professor Carlitz's papers have created interest in the transformation and summation formulae for double series, because these type of formulae have applications in the solutions of certain problems in theoretical physics and optimization problems in Management Sciences. Following Carlitz's papers, Jain [9], Sharma [11-16], Sharma and Abiodun [17], Abiodun and Sharma [1] have

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proved various formulae associated with hypergeometric series of two variables. The following notation due to Chaundy [4] will be used to represent the hypergeometric series of higher order and of two variables.

$$(1)\qquad F\left[\begin{matrix}(a_p)\,;\ (b_q)\,;\ (c_r)\,;\ x,\ y\\ (d_s)\,;\ (e_h)\,;\ (f_k)\,;\end{matrix}\right]=\sum_{m,n=0}^{\infty}\frac{[(a_p)]_{m+n}\,[(b_q)]_m\,[(c_r)]_n\,x^m\,y^n}{[(d_s)]_{m+n}\,[(e_h)]_m\,[(f_k)]_n\,m\,!\,n\,!}$$

where (a_p) and $[(a_p)]_{m+n}$ will mean $a_1, \ldots, a_p$ and $(a_1)_{m+n}, \ldots, (a_p)_{m+n}$.

2. In this section we consider some new summation formulae for hypergeometric series of two variables. We consider.

$$\frac{\Gamma(\beta)\,\Gamma(\beta+\delta+\mu-\alpha-\varrho_1-\varrho_2)}{\Gamma(\beta-\alpha)\,\Gamma(\beta+\delta+\mu-\varrho_1-\varrho_2)}=F_1\,[\alpha\,;\,\varrho_1-\delta,\,\varrho_2-\mu\,;\,\beta\,;\,1,\,1]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\varrho_1)_m\,(\varrho_2)_n\,(\varrho_1-\delta)_m\,(\varrho_2-\mu)_n}{(\beta)_{m+n}\,m\,!\,n\,!\,(\varrho_1)_m\,(\varrho_2)_n},$$

now we make use of the result due to Bailey [3]

$$(2)\qquad {}_4F_3\left[\begin{matrix}\frac{1}{2}a,\ \frac{1}{2}+\frac{1}{2}a,\ b+n,\ -n\,;\ 1\\ \frac{1}{2}b,\ \frac{1}{2}+\frac{1}{2}b,\ 1+a\,;\end{matrix}\right]=\frac{(b-a)_n}{(b)_n}$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\varrho_1)_m\,(\varrho_2)_n}{(\beta)_{m+n}\,m\,!\,n!}\sum_{r=0}^{m}\frac{(-m)_r\,(\varrho_1+m)_r\left(\frac{1}{2}\delta\right)_r\left(\frac{1}{2}+\frac{1}{2}\delta\right)_r}{r!\left(\frac{1}{2}\varrho_1\right)_r\left(\frac{1}{2}+\frac{1}{2}\varrho_1\right)_r(1+\delta)_r}\,\cdot$$

$$\cdot\sum_{s=0}^{n}\frac{(-n)_s\,(\varrho_2+n)_s\left(\frac{1}{2}\mu\right)_s\left(\frac{1}{2}+\frac{1}{2}\mu\right)_s}{s\,!\left(\frac{1}{2}\varrho_2\right)_s\left(\frac{1}{2}+\frac{1}{2}\varrho_2\right)_s(1+\mu)_s}$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\alpha)_{r+s}\left(\frac{1}{2}\delta\right)_r\left(\frac{1}{2}+\frac{1}{2}\delta\right)_r\left(\frac{1}{2}\mu\right)_s\left(\frac{1}{2}+\frac{1}{2}\mu\right)_s(-1)^{r+s}2^{2r+2s}}{(\beta)_{r+s}\,(1+\delta)_r\,(1+\mu)_s\,r\,!\,s\,!}$$

$$F_1\,[\alpha+r+s\,;\,\varrho_1+2r,\,\varrho_2+2s\,;\,\beta+r+s\,;\,1,\,1].$$

Thus we have

$$(3)\quad \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(\alpha)_{r+s}\left(\frac{1}{2}\delta\right)_r\left(\frac{1}{2}+\frac{1}{2}\delta\right)_r\left(\frac{1}{2}\mu\right)_s\left(\frac{1}{2}+\frac{1}{2}\mu\right)_s(-1)^{r+s}2^{2r+2s}}{(\beta)_{r+s}(1+\delta)_r(1+\mu)_s\, r!\, s!}\cdot$$

$$\cdot F_1[\alpha+r+s;\ \varrho_1+2r, \varrho_2+2s;\ \beta+r+s;\ 1, 1]$$

$$=\frac{\Gamma(\beta)\,\Gamma(\beta+\delta+\mu-\alpha-\varrho_1-\varrho_2)}{\Gamma(\beta-\alpha)\,\Gamma(\beta+\delta+\mu-\varrho_1-\varrho_2)}.$$

Now we make use of the formula due to Appell and Kampé de Feriet [2, p. 22. equ (24)]

$$(4)\qquad F_1[\alpha;\ \beta, \gamma;\ \delta;\ 1, 1]=\frac{\Gamma(\delta)\,\Gamma(\delta-\alpha-\beta-\gamma)}{\Gamma(\delta-\alpha)\,\Gamma(\delta-\beta-\gamma)},$$

valid for $R(\delta-\alpha-\beta-\gamma)>0$.

If we use formula (4) in (3), then the condition $R(\delta-\alpha-\beta-\gamma)>0$ must be satisfied, but it is satisfied in the following cases.

(*a*) We take $\alpha=-n$ and $\rho_1+\rho_2=-n$ in (3), it gives a new summation formula for double series, on using (4)

$$(5)\qquad F\left[\begin{matrix}-n, \beta;\ \frac{1}{2}\delta, \frac{1}{2}+\frac{1}{2}\delta;\ \frac{1}{2}\mu, \frac{1}{2}+\frac{1}{2}\mu;\ 1, 1\\ \frac{1}{2}(\beta-n), \frac{1}{2}(1+\beta-n);\ 1+\delta;\ 1+\mu\end{matrix}\right]=$$

$$=\frac{(1-\beta+\delta+\mu)_n}{(1-\beta)_n}.$$

In case we put $\mu=0$ or $\delta=0$ in (5), it reduces to a known result [10, p. 65, (2.4. 2.2)].

(*b*) Next we take $\mu=-m$, $\delta=-n$ and $\rho_1+\rho_2=-m-n$ in (3) and using (4), it gives another new summation formula for double series

$$(6)\quad F\left[\begin{matrix}\alpha, \beta;\ -\frac{1}{2}n, \frac{1}{2}-\frac{1}{2}n;\ -\frac{1}{2}m, \frac{1}{2}-\frac{1}{2}m;\ 1, 1\\ \frac{1}{2}(\beta+\alpha), \frac{1}{2}(1+\beta+\alpha);\ 1-n;\ 1-m;\end{matrix}\right]=\frac{(\beta)_{m+n}}{(\beta+\alpha)_{m+n}}.$$

In case $m=0$ or $n=0$ in (6), it reduces to a known result [10, p. 65 (2.4 2.2)].

Now we prove the summation formula

$$(7)\quad F\left[\begin{matrix}-n,\ \varrho+n,\ \frac{1}{2}(\varrho-\beta-1);\ \frac{1}{4}(\varrho-\beta);\ \frac{1}{4}(\varrho-\beta);\ 1,\ 1\\ \frac{1}{2}\varrho,\ \frac{1}{2}(1+\varrho),\ \varrho-\beta;\ -;\ -;\end{matrix}\right]=$$

$$=\frac{(1+\beta)_n}{(\varrho)_n},$$

provided $\rho-\beta$ is not a negative integer.

PROOF. To prove (7), we start with the left side of (7).

$$F\left[\begin{matrix}-n,\ \varrho+n,\ \frac{1}{2}(\varrho-\beta-1);\ \frac{1}{4}(\varrho-\beta);\ \frac{1}{4}(\varrho-\beta);\ 1,\ 1\\ \frac{1}{2}\varrho,\ \frac{1}{2}(1+\varrho),\ \varrho-\beta;\ -;\ -;\end{matrix}\right]$$

$$=\frac{(1+\beta)_n}{(\varrho-\beta)_n}\sum_{r=0}^{r+s}\leq\sum_{s=0}^{n}\frac{(-n)_{r+s}(\varrho+n)_{r+s}\left(\frac{1}{2}\varrho-\frac{1}{2}\beta-\frac{1}{2}\right)_{r+s}\left(\frac{1}{4}\varrho-\frac{1}{4}\beta\right)_r}{\left(\frac{1}{2}\varrho\right)_{r+s}\left(\frac{1}{2}+\frac{1}{2}\varrho\right)_{r+s}\left(3/2+\frac{1}{2}\varrho+\beta\right)_{r+s}}\cdot$$

$$\cdot\frac{\left(\frac{1}{4}\varrho-\frac{1}{4}\beta\right)_s}{r!\ s!}\ {}_3F_2\left[\begin{matrix}-n+r+s,\ \beta+\frac{1}{2}-\frac{1}{2}\varrho,\ \varrho+n+r+s;\ 1\\ \frac{1}{2}+\frac{1}{2}\varrho+r+s,\ 1+\beta;\end{matrix}\right],$$

we have used Saalschutz's theorem [10, p. 243. equ. (III. 2)]

$$(8)\quad {}_3F_2\left[\begin{matrix}-n,\ n+a,\ b;\ 1\\ c,\ 1+a+b-c;\end{matrix}\right]=\frac{(c-b)_n(1+a-c)_n}{(c)_n(1+a+b-c)_n}$$

$$=\frac{(1+\beta)_n}{(\varrho-\beta)_n}\sum_{p=0}^{n}\frac{(-n)_p(\varrho+n)_p\left(\frac{1}{2}+\beta\right)_p}{\left(\frac{1}{2}+\frac{1}{2}\varrho\right)_p(1+\beta)_p\,p!}\cdot$$

$$\cdot F\left[\begin{matrix}-p,\ \frac{1}{2}\left(\varrho-\beta-\frac{1}{2}\right);\ \frac{1}{4}(\varrho-\beta);\ \frac{1}{4}(\varrho-\beta);\ 1,\ 1\\ \frac{1}{2}\varrho,\ \frac{1}{2}+\frac{1}{2}\varrho-\beta-p;\ -;\ -;\end{matrix}\right],$$

now we use a result due to Sharma [14]

$$(9)\quad F\begin{bmatrix} -n, \alpha;\ \gamma_1;\ \gamma_2;\ 1, 1 \\ \mu, 1+\alpha+\gamma_1+\gamma_2-\mu-n;\ -;\ -; \end{bmatrix} = \frac{(\mu-\alpha)_n(\mu-\gamma_1-\gamma_2)_n}{(\mu)_n(\mu-\alpha-\gamma_1-\gamma_2)_n},$$

$$= \frac{(1+\beta)_n}{(\varrho-\beta)_n}\, {}_4F_3\begin{bmatrix} -n, \varrho+n, \frac{1}{2}+\frac{1}{2}\beta, \frac{1}{2}\beta;\ 1 \\ \frac{1}{2}\varrho, \frac{1}{2}+\frac{1}{2}\varrho, 1+\beta; \end{bmatrix} = \frac{(1+\beta)_n}{(\varrho)_n} \quad \text{by (2).}$$

This completes the proof of (7). In case $\beta=0$ in (7), it reduces to the following form

$$(10)\quad F\begin{bmatrix} -n, \varrho+n, \frac{1}{2}(\varrho-1);\ \frac{1}{4}\varrho;\ \frac{1}{4}\varrho;\ 1, 1 \\ \frac{1}{2}\varrho, \frac{1}{2}(1+\varrho), \varrho;\ -;\ -; \end{bmatrix} = \frac{n!}{(\varrho)_n},$$

provided that ρ is not a negative integer. Next we prove the summation formula.

$$(11)\quad F\begin{bmatrix} -n, \frac{1}{2}(\varrho-\beta), \varrho+n;\ \frac{1}{4}(\varrho-\beta-1);\ \frac{1}{4}(\varrho-\beta-1);\ 1, 1 \\ \frac{1}{2}\varrho, \frac{1}{2}+\frac{1}{2}\varrho, \varrho-\beta;\ -;\ -; \end{bmatrix}$$

$$= \frac{(1+\beta)_n(\varrho-\beta)}{(\varrho-\beta)_n(\varrho)_n}.$$

PROOF. To prove (11), we start with the left side of (11),

$$F\begin{bmatrix} -n, \varrho+n, \frac{1}{2}(\varrho-\beta);\ \frac{1}{4}(\varrho-\beta-1);\ \frac{1}{4}(\varrho-\beta-1);\ 1, 1 \\ \varrho-\beta, \frac{1}{2}\varrho, \frac{1}{2}+\frac{1}{2}\varrho;\ -;\ -; \end{bmatrix}$$

$$= \frac{\left(1+\frac{1}{2}\varrho\right)_n (1+\beta)_n}{\left(\frac{1}{2}\varrho\right)_n (\varrho-\beta)_n}$$

$$\sum_{r=0}^{r+s\leq n}\sum_{s=0}\frac{(-n)_{r+s}(\varrho+n)_{r+s}\left(\frac{1}{2}\varrho-\frac{1}{2}\beta\right)_{r+s}\left(\frac{1}{4}\varrho-\frac{1}{4}\beta-\frac{1}{4}\right)_r\left(\frac{1}{4}\varrho-\frac{1}{4}\beta-\frac{1}{4}\right)_s}{\left(\frac{1}{2}+\frac{1}{2}\varrho\right)_{r+s}\left(1+\frac{1}{2}\varrho\right)_{r+s}\left(2+\beta+\frac{1}{2}\varrho\right)_{r+s}r!\,s!}.$$

$$\cdot\,{}_3F_2\left[\begin{matrix}-n+r+s,1+\beta-\frac{1}{2}\varrho,\varrho+n+r+s;1\\ 1+\frac{1}{2}\varrho+r+s,1+\beta;\end{matrix}\right]\quad\text{by (8)}$$

$$=\frac{\left(1+\frac{1}{2}\varrho\right)_n(1+\beta)_n}{\left(\frac{1}{2}\varrho\right)_n(\varrho-\beta)_n}\sum_{p=0}^{n}\frac{(-n)_p(\varrho+n)_p\left(1+\beta-\frac{1}{2}\varrho\right)_p}{\left(1+\frac{1}{2}\varrho\right)_p(1+\beta)_p\,p!}.$$

$$\cdot\,F\left[\begin{matrix}-p,\frac{1}{2}(\varrho-\beta);\frac{1}{4}(\varrho-\beta-1);\frac{1}{4}(\varrho-\beta-1);1,1\\ \frac{1}{2}(1+\varrho),\frac{1}{2}\varrho-\beta-p;-;-;\end{matrix}\right]\quad\text{by (9)}$$

$$=\frac{\left(1+\frac{1}{2}\varrho\right)_n(1+\beta)_n}{\left(\frac{1}{2}\varrho\right)_n(\varrho-\beta)_n}\,{}_4F_3\left[\begin{matrix}-n;\varrho+n,\frac{1}{2}+\frac{1}{2}\beta,1+\frac{1}{2}\beta;1\\ \frac{1}{2}+\frac{1}{2}\varrho,1+\frac{1}{2}\varrho,1+\beta;\end{matrix}\right],$$

now we use a result due to Carlitz [8]

$$\text{(12)}\qquad {}_4F_3\left[\begin{matrix}-n,\frac{1}{2}(a+1),\frac{1}{2}(a+2),b+n;1\\ a+1,\frac{1}{2}(b+2),\frac{1}{2}(b+1);\end{matrix}\right]$$

$$=\frac{b(b-a)}{(b+2n)(b)_n}=\frac{(1+\beta)_n(\varrho-\beta)}{(\varrho-\beta)_n(\varrho)_n}.$$

This completes the proof of (11). In case $\beta=0$ in (11), it yields

$$\text{(13)}\qquad F\left[\begin{matrix}-n,\varrho+n;\frac{1}{4}(\varrho-1);\frac{1}{4}(\varrho-1);1\\ \frac{1}{2}(1+\varrho),\varrho;-;-;\end{matrix}\right]=\frac{n!\,\varrho}{(\varrho)_n(\varrho)_n}.$$

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