AN ISOMORPHISM THEOREM ON A BANACH ALGEBRA (*)

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SOMMARIO. - *Sia A_i $(i = 1, 2)$ un'algebra riflessiva di Banach. Dimostreremo che un isomorfismo dell'algebra di moltiplicatori A_1^m su A_2^m che sia un'isometria porta il gruppo G_1 dei moltiplicatori isometrici ed invertibili nella topologia forte di operatori (SOT) sul gruppo G_2 della topologia debole di operatori (WOT).*

SUMMARY. - *Let A_i $(i = 1, 2)$ be a reflexive Banach algebra. We shall show that an algebra isomorphism of the multiplier algebra A_1^m onto A_2^m which is an isometry maps the group of isometric and invertible multipliers G_1 in the strong operator topology (SOT) onto G_2 in the weak operator topology (WOT).*

1. Introduction.

In [2] Gaudry showed that if $m_p(G_i)$ denotes the set of all right multipliers of $L^p(G_i)$ $(i=1, 2;\ 1 \leq p < \infty;\ p \neq 2;\ G_i$ a locally compact Hausdorff group) and T is an isametric isomorphism of $m_p(G_1)$ onto $m_p(G_2)$, then G_i and G_2 are isomorphic topological groups. In this short note we try to investigate to what extent this result holds when $L^p(G)$ is replaced by a Banach algebra though $L^p(G)$ is not, in general, an algebra.

2. Notation.

Throughout this paper A denotes a reflexive Banach algebra which is without order (i. e. $A \equiv A^{**}$ and $\forall x \in A,\ xA = (0) \Rightarrow x = o$

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or $Ax=(0) \Rightarrow x=o$), A^* denotes the dual of A. A^m is the Banach algebra of left multipliers of A and G the group of isometric and invertible multipliers. It is easy to show that G is a topological group in the norm, strong and weak operator topologies (SOT and WOT). (see [3]). A^m (SOT) shall mean A^m with the strong operator topology. Similarly for A^m (*WOT*). L_x denotes the left multiplication operator; and we shall also assume throughout the note that A^2 is dense in A.

3. Before the main theorem is proved we shall consider the following important lemma:

3.1. LEMMA: *Let T be a norm bounded homomorphism of $A_1{}^m$ onto $A_2{}^m$; T restricted to norm bounded sets is continuous as a mapping of $A_1{}^m$ (SOT) onto $A_2{}^m$ (WOT).*

Proof: *Let* (φ_i) be a net in $A_1{}^m$ which is norm bounded and convergent in the SOT to φ (i. e. $\varphi_i(x) \xrightarrow{\text{norm}} \varphi(x)$ for $x \in A_1$ and $\|\varphi_i\| \leq M < \infty$).

Since

$$\|L_{\varphi_i x} - L_{\varphi x}\| = \sup_{\|y\| \leq 1} \|L_{\varphi_i x}\, y - L_{\varphi x}\, y\| = \sup_{\|y\| \leq 1} \|(\varphi_i x)\, y - (\varphi x)\, y\|$$

$$\leq \sup_{\|y\| \leq 1} \|\varphi_i x - \varphi x\|\, \|y\| \leq \|\varphi_i x - \varphi x\|$$

then if $\varphi_i\, x$ tends to $\varphi\, x$ in norm, $L_{\varphi_i x}$ tends to $L_{\varphi x}$ in the uniform norm topology of $A_1{}^m$. Since T is norm bounded, it is continuous in norm, hence $TL_{\varphi_i x}$ tends $TL_{\varphi x}$ in the norm of $A_2{}^m$. But uniform norm topology is stronger than weak operator topology; hence $f[(TL_{\varphi_i x})\, y^1]$ tends to $f\,[TL_{\varphi x})\, y^2]$ $\forall f \in A_2{}^*$ and $y^1 \in A_2$. Since $L_{\varphi_i x} = \varphi_i\, L_x$, then

(1) $$f\,[T\,(\varphi_i \cdot L_x)\, y^1] \text{ tends to } f\,[T\,(\varphi \cdot L_x)\, y^1]$$

since T is norm bounded, $(T\,\varphi_i)$ is norm bounded; and since A_2 is reflexive, $(T\,\varphi_i)$ is contained in a set which is compact in WOT (see [1]). Hence $(T\,\varphi_i)$ has a subset which converges to $\psi^1 \in A_2{}^m$ in the WOT.

i. e. $f\,[(T\,\varphi_i)\, y^1]$ tends to $f\,(\psi^1\, y^1)$ where

$$f \in A_2{}^* \text{ and } y^1 \in A_2.$$

(We have used the same notation for subnet as net.).

Hence

(2) $$f\,[(T\varphi_i\, T L_x)\, y^1] \text{ tends to } f\,[(\psi^1\, T L_x)\, y^1].$$

From (1) and (2)

$$f\,[(\psi^1\cdot T L_x)\, y^1] = f\,[(T\varphi\cdot T L_x)\, y^2]$$

$$\forall f\in A_2^* \text{ and } y\in A_2.$$

(3) $$\text{i. e. } \psi^1\cdot T L_x = T\varphi\cdot T L_x$$

Applying T^{-1} to (3), we have

$$T^{-1}\psi^1\cdot L_x = \varphi\cdot L_x\,.$$

Hence $(T^{-1}\psi^1-\varphi)\, L_x\, y = o \;\; \forall y\in A_1$ and so $(T^{-1}\psi^1-\varphi)\, xy = o \;\; \forall y\in A_1$ since A_1^2 is dnese in A_1, we have $T^{-1}\psi^1=\varphi$

$$\text{i. e. } T\varphi=\psi^1.$$

Therefore $T\varphi_i$ tends to $T\varphi$ in the WOT and the proof is complete.

3.2. THEOREM: *T, an isometric algebra isomorphism of A_1^m onto A_2^m maps* G_1 (SOT) onto G_2 (WOT).

PROOF. (i) *Tis onto*: *Let* $\varphi\in G_1$; then

(4) $$\|\varphi^1\| = \|T\varphi\| = \|\varphi\| = 1$$

where $T\varphi=\varphi^1$.

Since T is an algebra isomorphism onto, $I^1=T(I)=T(\varphi\varphi^{-1})= =T\varphi\, T(\varphi^{-1})=\varphi^1\, T(\varphi^{-1})$ where I and I^1 are the identities in A_1^m and A_2^m respectively.

Hence $T\varphi^{-1}=(\varphi^1)^{-1}$ and

(5) $$\|(\varphi^1)^{-1}\| = \|T\varphi^{-1}\| = \|\varphi^{-1}\| = 1$$

i. e. $\varphi^1\in G_2$.

Conversely if $\varphi^1\in G_2$, then $\|T^{-1}\varphi^1\| = \|\varphi^1\| = 1$. Therefore T is onto

(ii) *T is continuous*: Let $\{\varphi_i\}$ a net in G_i; $(T\varphi_i)$ is a norm bounded set in G_2 since T *is* an isometry. T is also continuous in norm for the same reason. Using 3.1, (ii) is proved.

(iii) T^{-1} *is continuous*: By the open mapping theorem, T^{-1} is bounded and hence continuous in norm. Let (φ_i) be a net in G_2, then $(T^{-1}\varphi_i^1)$ is a norm bounded set in G_1 and using 3.1 again, (iii) is proved and hence the theorem.

Remark: If T is assumed to be just norm decreasing, it has not been possible to prove theorem 3.2 without the strong conditions of SOT coinciding with WOT on G_i and G_i generating A_i in both SOT and WOT. One example of an algebra which makes 3.2 to hold however is $L^p(G)$ for compact group G.

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