PARABOLICITY AND EXISTENCE
OF BOUNDED OR DIRICHLET FINITE
POLYHARMONIC FUNCTIONS (*)

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THEOREM 1. $O^N_G \cap \bar{O}^N_{H^{kG}} \neq \emptyset$ for $N \geq 2, k \geq 2$.

PROOF. Consider the $N$-dimensional «beam»

$$T: \{(x, y_1, \ldots, y_{N-1}) \mid |x| < \infty, |y_i| \leq \pi\},$$

with each pair of opposite faces $y_i = \pi, y_i = -\pi$ identified by a parallel translation orthogonal to the $x$-axis. Endow $T$ with the metric

$$ds^2 = e^{-2x^2} dx^2 + e^{-2x(N-1)} \sum_{i=1}^{N-1} dy_i^2.$$

Since

$$\Delta f(x) = -e^{x^2} (e^{-2x} e^{x^2} f')('),$$

$f(x)$ is harmonic if and only if $f(x) = ax + b$. Therefore, the harmonic measure vanishes, and $T \in O^N_G$.

Set

$$u_2(x) = \int_0^x e^{-2t} dt,$$

$$u_k(x) = \int_0^x \int_{u_{k-1}(t)} e^{-t} dt \ dx$$

with $k \geq 3$. Straightforward computation shows that $u_k \in H^k$. The proof that $u_k \in B$ is by induction. Clearly $u_2 \in B$. Suppose $u_m \in B$. For all sufficiently large $x$,

$$\int_x^\infty u_m(t) e^{-t} dt < 2 \int_x^\infty e^{-t} dt = e^{-x^2}$$

and $u_{m+1}(x) < u_2(x)$. Since $u_{m+1}$ is odd, $|u_{m+1}(x)| < |u_2(x)|$ for large $x$, hence $u_k \in H^k B$ for all $k \geq 2$. Moreover,

$$D(u_2) = c \int_{-\infty}^{\infty} (u_2')^2 e^{x^2} e^{-x^2} dx = c \int_{-\infty}^{\infty} e^{-2x^2} dx < \infty,$$

$$D(u_m) = c \int_{-\infty}^{\infty} \left( \int_{u_{m-1}(t)}^{\infty} e^{-t} dt \right)^2 dx < D(u_2) < \infty,$$

so that $u_k \in H^k C$ and $T \in \bar{O}^N_{H^{kG}}$. 
2. Our next problem is to find a parabolic space which excludes both bounded and Dirichlet finite polyharmonic functions.

**Theorem 2.** \( O_{G}^{N} \cap O_{H_{k}B}^{N} \cap O_{H_{k}D}^{N} \neq \emptyset \) for \( N \geq 2, k \geq 2 \).

**Proof.** We make use of the punctured \( N \)-space

\[
E_{a}^{N} = \{ 0 < r < \infty \}, \quad r = |x|, \quad x = (x_1, \ldots, x_N),
\]

\[
ds = r^{a}|dx|, \quad \alpha \text{ constant}.
\]

Parabolicity for \( N = 2 \) is invariant under conformal metrics. Since the punctured Euclidean plane \( E_{a}^{2} \) is parabolic, so is \( E_{a}^{N} \) for all \( \alpha \). A function \( \gamma (r) \) is harmonic on \( E_{a}^{N}, N > 2 \), if

\[
\gamma'' (r) + [N - 1 + (N - 2) \alpha] r^{-1} \gamma' (r) = 0,
\]

that is,

\[
\gamma (r) = \begin{cases} 
(ar^{-(N-2)(\alpha+1)} + b, & \alpha \neq -1, \\
 \log r + d, & \alpha = -1.
\end{cases}
\]

Therefore the harmonic measure of the ideal boundary vanishes if and only if \( \alpha = -1 \). We conclude that \( E_{-1}^{N} \in O_{a}^{N} \) for all \( N \geq 2 \), and \( E_{a}^{N} \in E_{a}^{N} \) for all \( \alpha \neq -1, N \geq 3 \).

3. To exclude \( H^{k}B \)- and \( H^{k}D \)-functions we use a series representation for an arbitrary \( H^{k} \)-function on \( E_{a}^{N} \). For later use we include any \( \alpha \).

All radial polyharmonic functions are found by solving the system of equations \( \Delta \gamma_{i} (r) = \gamma_{i-1} (r), i = 1, \ldots, k \), where \( \gamma_{0} (r) = 0 \). We seek an expansion in terms of spherical harmonics \( S_{nm} (\theta), n = 1, 2, \ldots, m = 1, 2, \ldots, m \), \( \theta = (\theta_{1}, \ldots, \theta_{N-1}) \). A simple calculation gives \( \Delta S_{nm} = n (n + N - 2) r^{-2 \alpha - 2} S_{nm} \).

This enables us to solve \( \Delta (f_{nm} (r) S_{nm}) = 0 \), and obtain the general solution \( f_{nm} (r) S_{nm} \) with \( f_{nm} (r) = ar^{p_{n}} + br^{q_{n}} \), where

\[
p_{n}, q_{n} = \frac{1}{2} \left[ - (N - 2)(\alpha + 1) \mp \sqrt{(N - 2)^{2}(\alpha + 1)^{2} + 4n(n + N - 2)} \right].
\]

Any harmonic function on the \( r \)-sphere has an eigenfunction expansion

\[
h (r, \theta) = f_{0} (r) + \sum_{n=1}^{\infty} \sum_{m=1}^{m_{n}} f_{nm} (r) S_{nm} (\theta).
\]
A proper choice of the constants $a_{nm}, b_{nm}, a, \text{ and } b$ gives

$$h = \sum_{n=1}^{\infty} \sum_{m=1}^{m_n} (a_{nm} r^{p_n} + b_{nm} r^{q_n}) S_{nm} + a \gamma_k (r) + b.$$ 

We obtain biharmonic functions by solving $\Delta u = h$ term-by-term from the above expansion. Higher order polyharmonic functions are the solutions of $\Delta^k u = h$, $k \geq 2$. Compact convergence of our series is a consequence of that for harmonic functions. A simple computation shows that an arbitrary $u \in H^k (E^N_a)$ has an expansion for $\alpha = -1$,

$$u = \sum_{j=0}^{k-1} \sum_{n=1}^{m_n} (a_{jnm} r^{p_n} + b_{jnm} r^{q_n}) (\log r)^j S_{nm} + \gamma_k (r).$$

For later reference we also give the expansion for $\alpha \neq -1$:

$$u = \sum_{j=0}^{k-1} \left( \sum_{n \neq n_j'} \sum_{m=1}^{m_n} e_{jnm} r^{p_n+(2\alpha+2)j} S_{nm} + \sum_{n \neq n_j''} \sum_{m=1}^{m_n} d_{jnm} r^{q_n+(2\alpha+2)j} S_{nm} \right)$$

$$+ \sum_{n_j'} \sum_{m=1}^{m_n} U_{nj'} \log r S_{nj'm} + \sum_{n_j''} \sum_{m=1}^{m_n} V_{nj''} \log r S_{nj''m} + \gamma_k (r),$$

where $n_j'$, $n_j''$ are the values of $n$ for which

$$p_n + \left( \frac{1}{2} N + j \right) (\alpha+1) = 0, \quad q_n + \left( \frac{1}{2} N + j \right) (\alpha+1) = 0$$

respectively, and $U_{nj'}$ and $V_{nj''}$ are polynomials in $r$.

4. We are ready to show that $E^N_{-1} \in O_{H^k B}$ for all $N, k$. For $(j, k) \neq (n, m)$, $S_{jk}$ and $S_{nm}$ are orthogonal with respect to the inner product

$$(S_{jk}, S_{nm}) = \int_{\partial B(0,1)} S_{jk} S_{nm} d\omega,$$

where $B(0, 1)$ is the unit ball about the origin and $d\omega$ the Euclidean surface element of $\partial B(0,1)$. If $u \in H^k B$, then

$$(u, S_{nm}) = \text{const} \sum_{j=0}^{k-1} (a_{jnm} r^{p_n} + b_{jnm} r^{q_n}) (\log r)^j$$
is bounded as \( r \) ranges in \([0, \infty)\). Since \( p_n \neq 0, q_n \neq 0 \), the \( a_{jmn} \) and \( b_{jmn} \) vanish for all \( j \). The radial part \( \gamma_k (r) \) of \( u \) consists of a finite linear combination of powers of \( \log r \) and must therefore vanish. We have shown that \( E_{-1}^N \in O_{H^k B}^N \).

5. To exclude \( H^k D\)-functions on \( E_{-1}^N \), we use the Dirichlet orthogonality of spherical harmonics, readily verified as follows. Let \( \Omega \) be a subregion \( \{ 0 < r_1 < r < r_1 < \infty \} \), and \( * \) the Hodge star operator. Then for \( f, g \in C^1 (E_\alpha^N) \), \( D_\alpha (f, g) = \int_\Omega df \wedge * dg \), and for \( (n, m) \neq (k, l) \), it is easily verified that \( D_\alpha (S_{nm}, S_{kl}) = 0 \).

Our expansion for an arbitrary \( u \in H^k (E_{-1}^N) \) can be written as \( u = \sum_{i=0}^{\infty} u_i \) with \( u_0 \) the radial part and \( u_n \) the sum of the terms involving the \( S_{nm} \) for a fixed \( n \). For \( v = u - u_n \) we obtain \( D_\alpha (u) = D_\alpha (u_n) + D_\alpha (v) + 2D_\alpha (u_n, v) \). Compact convergence of the series for \( u \) and its partial derivatives gives

\[
D_\alpha (u_n, v) = \lim_{j \to \infty} D_\alpha \left( u_n, \sum_{i=0}^{j} u_i \right).
\]

Our objective is to show that \( D_\alpha (u) \supseteq D_\alpha (u_n) \). It suffices to prove that

\[
D_\alpha \left( u_n, \sum_{i=0}^{j} u_i \right) = \sum_{i=0}^{j} D_\alpha (u_{n_i}, u_{i}) = 0.
\]

The functions \( u_n, u_i \) are of the form \( f (r) S_{nm} \) or \( g (r) S_{kl} \). Let \( \text{grad}_r \) be the radial component of the gradient vector, and set \( \text{grad}_\theta = \text{grad} - \text{grad}_r \). With possibly one \( S_{nm} \) or \( S_{jk} \) reducing to 1, we have

\[
D_\alpha (u_n, u_i) = \int_\Omega \text{grad} u_n \cdot \text{grad} u_i \, dV
\]

\[
= \int_\Omega (\text{grad}_r u_n \cdot \text{grad}_r u_i + \text{grad}_\theta u_n \cdot \text{grad}_\theta u_i) \, dV
\]

\[
= \int_\Omega (f' (r) g' (r) r^{-2a} S_{nm} + f (r) g (r) \text{grad} S_{nm} \cdot \text{grad} S_{kl}) \, dV
\]

\[
= \sigma \int_\Omega \text{grad} S_{nm} \cdot \text{grad} S_{kl} \, dV = 0.
\]
An exhaustion $\Omega \to E^N_\alpha$ gives $D(u) \geq D(u_n)$. A simple computation shows that $D(u_n) = \infty$ for every nonconstant $u_n$. We have $E^N_{-1} \in O^N_{H^{k_D}}$, and the proof of Theorem 2 is complete.

6. The Euclidean $N$-ball is a hyperbolic manifold which carries bounded Dirichlet finite polyharmonic functions. It remains to find a hyperbolic $N$-manifold without $H^k B$- or $H^k D$-functions.

**Theorem 3.** $\tilde{O}^N_\alpha \cap O^N_{H^{k_B}} \cap O^N_{H^{k_D}} \neq \emptyset$ for $N \geq 2, k \geq 2$.

**Proof.** We first consider $N \geq 3$. In the proof of Theorem 2 we showed that $E^N_{\alpha} \in \tilde{O}^N_\alpha$ for $N \geq 3, \alpha \neq -1$. For $u \in H^k B (E^N_{\alpha})$, $n \neq n', n''$,

$$(u, S_{nm}) = \text{const} \sum_{j=0}^{k-1} (c_{jnm} r^{p_n+(2\alpha+3)j} + d_{jnm} r^{q_n+(2\alpha+2)j}).$$

Since $p_n \to \infty$ and $q_n \to -\infty$ as $n \to \infty$ it is easy to find an $\alpha \neq -1$ such that $p_n + (2\alpha + 2) j \neq 0$, $q_n + (2\alpha + 2) j \neq 0$ for $j = k - 1$ and all $n$. The boundedness of $u$ forces $c_{jnm}$ and $d_{jnm}$ as well as the radial part $\gamma_k (r)$ to vanish. For $n = n'$ and $n = n''$, the terms $U_{n'}$ and $V_{n''}$ are functions of powers of $r$ and log $r$ and must therefore vanish. A fortiori $E^N_{\alpha} \in O^N_{H^{k_B}}$. Computation again shows that for any choice of $\alpha, D(u_n) = \infty$, hence $D(u) = \infty$ for any $u \in H^k (E^N_{\alpha})$, and $E^N_{\alpha} \in O^N_{H^{k_D}}$.

7. For the proof of Theorem 3 in the case $N = 2$ consider the unit disk $D_\alpha = \{r < 1\}$ with the metric $ds = (1 - r^2)^\alpha |dx|$. Since $D_\alpha$ is hyperbolic, so is $D_\alpha$ for every $\alpha$. Moreover $\Delta$ and the Dirichlet integral of a function $u \in C^k (D_\alpha)$ can be taken in the Euclidean metric,

$$\Delta_{D_\alpha} u = (1 - r^2)^{-2\alpha} \Delta_{D_0} u, \quad D_{D_\alpha} (u) = D_{D_0} (u).$$

We give a simple test for the nonexistence of $H^k B$-functions. For any $\varphi \in C^{\infty}_c (D_\alpha)$ and $u \in H^k$, the self-adjointness of $\Delta$ entails $(\Delta^{k-1} u, \varphi) = \langle u, \Delta^{k-1} \varphi \rangle$. Let $u \in H^k B$. Then $\Delta^{k-1} u = h \in H$ and

$$|(\Delta^{k-1} u, \varphi)| = |(h, \varphi)| \leq K (1, |\Delta^{k-1} \varphi|),$$

with $K = \sup |u|$. 


On $D_a$ every nonzero harmonic function has Fourier series

$$h(re^{i\theta}) = \sum_{n=0}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta)$$

with $a_n^2 + b_n^2 \neq 0$ for some $n$. For an arbitrary $t \in \left(\frac{1}{2}, 1\right]$ set

$$\varphi_t(r) = \begin{cases} 
  \left(r - \frac{1}{2}\right)^{2k-1} (t-r)^{2k-1}, & r \in \left[\frac{1}{2}, t\right] \\
  0, & r \in [0, 1) - \left[\frac{1}{2}, t\right].
\end{cases}$$

The function $\varphi_t(re^{i\theta}) = \rho_t(r) (a_n \cos n\theta + b_n \sin n\theta)$ is in $C^{2k-2}(D_a)$. To use our test we first compute

$$\Delta^{k-1} \varphi_t(re^{i\theta}) = \Delta^{k-2} \left\{ - (1-r^2)^{-2a} r^{-1} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial \varphi_t}{\partial r} \right) + r^{-1} \frac{\partial^2 \varphi_t}{\partial \theta^2} \right] \right\}.$$ 

We are concerned with the term that will dominate in the integral over $D_a$ as $t \to 1$. Since the support of $\varphi_t$ is in $\left[\frac{1}{2}, t\right]$, we may disregard all powers of $r$. Each successive application of $\Delta$ gives another factor of $(1-r^2)^{-2a}$ and lowers the power of $(t-r)^{2k-1}$ in the dominating term by two. The result is that

$$\int_{D_a} |\Delta^{k-1} \varphi_t(re^{i\theta}) (1-r^2)^{2a} r| \, dr \, d\theta$$

grows at most at the rate of

$$\int_{\frac{1}{2}}^{t} (t-r) (1-r^2)^{-2a} (k-2) \, dr.$$ 

If $\alpha < 0$, $(1, \ |\Delta^{k-1} \varphi_t|) = O(1)$. It remains to estimate the growth of

$$\int_{D_a} h(re^{i\theta}) \varphi_t(re^{i\theta}) (1-r^2)^{2a} r \, dr \, d\theta$$
\[ = \text{const} \left| \int_{t_0}^{t} q_t(r) (1+r)^{2\alpha} (1-r)^{2\alpha} \gamma^{n+1} \, dr \right|. \]

We ignore powers of \( r \) and \((1+r)\) and must estimate for some fixed \( t_0 \in \left(\frac{1}{2}, 1\right) \),

\[ \int_{t_0}^{t} (t-r)^{2k-1} (1-r)^{2\alpha} \, dr. \]

Integration by parts yields

\[ - (2\alpha+1)^{-1} (t-r)^{2k-1} (1-r)^{2\alpha+1} \bigg|_{t_0}^{t} \]

\[ - \int_{t_0}^{t} (2k-1) (2\alpha+1)^{-1} (t-r)^{2k-2} (1-r)^{2\alpha+1} \, dr. \]

After \( 2k-1 \) integrations, what remains is

\[ \text{const} \pm \text{const} \int_{t_0}^{t} (1-r)^{2\alpha+2k-1} \, dr. \]

For \( \alpha \leq -k \) this is unbounded as \( t \to 1 \), in violation of \( |(1, \Delta^{k-1} \varphi)| = O(1) \).

This contradiction gives \( D_a \in O^{2}_{H^k B} \).

8. To exclude \( H^k D \)-functions on \( D_a \), we develop another test. Let \( u \in H^k D \). If \( \varphi \in C_0^{2,3} (D_a) \), then

\[ (d\Delta^{k-2} \varphi, du) = \int_{\partial \Omega} \Delta^{k-2} \varphi^* du + (\Delta^{k-2} \varphi, \Delta u). \]

Since \( \varphi \) and all its derivatives vanish on \( \partial \Omega \),

\[ (d\Delta^{k-2} \varphi, du) = (\Delta^{k-2} \varphi, \Delta u). \]

This is equivalent to the mixed Dirichlet integral \( D (\Delta^{k-2} \varphi, u) = (\varphi, h) \),

where \( h = \Delta^{k-1} u \) is harmonic. By Schwarz's inequality we conclude that

\[ |(h, \varphi)| \leq K \sqrt{D (\Delta^{k-2} \varphi)}. \]

with \( K = \sqrt{D(u)} < \infty \).
Choose \( \varphi_i (r e^{i\theta}) \) as in No. 7. We estimate the growth for \( \alpha < 0 \) of

\[
D (A^{k-2} \varphi_i) = \int_{D_a} \left[ \left( \frac{\partial}{\partial r} A^{k-2} \varphi_i \right)^2 + \frac{1}{r^2} \left( \frac{\partial}{\partial \theta} A^{k-2} \varphi_i \right)^2 \right] r \, d\theta.
\]

Previously we showed that the dominating term of \( A^{k-2} \varphi_i \) is

\[
(t - r)^3 (1 - r^2)^{-2\alpha(k-2)}.
\]

Therefore, the order of growth as \( t \to 1 \) of the Dirichlet integral is determined by

\[
\int_{\frac{1}{2}}^1 \frac{\partial}{\partial r} \left[ (t - r)^3 (1 - r^2)^{-2\alpha(k-2)} \right]^2 dr.
\]

If \( \alpha < 0 \), we thus have \( D (A^{k-2} \varphi_i) \frac{1}{\alpha} = O(1) \), hence \( |(h, \varphi_i)| = O(1) \) as \( t \to 1 \). But \( |(h, \varphi_i)| \to \infty \) as \( t \to 1 \), if \( \alpha \leq -k \). The contradiction gives \( D_a \in O_{H^{kD}}, \, \alpha \leq -k \).

The proof of Theorem 3 is herewith complete.

9. We summarize our results:

**Theorem 4.** The totality \( \{R^N\} \) of Riemannian \( N \)-manifolds decomposes into four disjoint nonempty classes

\[
\{R^N\} = O_N^N \cap O_{H^{kX}}^N + \tilde{O}_N^N \cap \tilde{O}_{H^{kX}}^N + \tilde{O}_N^N \cap O_{H^{kX}}^N + \tilde{O}_N^N \cap \tilde{O}_{H^{kX}}^N
\]

for \( X = B, D, C; \, N \geq 2; \) and \( k \geq 2 \).
BIBLIOGRAPHY


