THE TWO-DIMENSIONAL LAPLACE TRANSFORM FOR G-FUNCTIONS (*)

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SUMMARY. - The object of this paper is to obtain new operational relations between the original and the image for two dimensional Laplace transform that involve Meijer's $G$-function and Whittaker's confluent hypergeometric functions.

1. Introductory.

The integral equation

$$\Phi (p, q) = pq \int_0^\infty \int_0^\infty e^{-px-qq} f(x, y) \, dy \, dx, \quad R(p, q) > 0$$

(1.1)

represents the classical Laplace transform of two variables and the functions $\Phi (p, q)$ and $f(x, y)$ related by (1.1), are said to be operationally related to each other. $\Phi (p, q)$, is called the image and $f(x, y)$ the original.

Symbolically we can write

$$\Phi (p, q) \rightrightarrows f(x, y) \quad \text{or} \quad f(x, y) \rightrightarrows \Phi (p, q),$$

(1.2)

and the symbol $\rightrightarrows$ is called « operational ».

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Meijer's \( G \)-function [1] is defined by a Mellin-Barnes type integral:

\[
G_{p,q}^{m,n}(x \Big| \begin{array}{c} a_p \\ b_q \end{array}) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} s^x ds,
\]

where \( m, n, p, q \) are integers with \( q \geq 1; 0 \leq n \leq p; 0 \leq m \leq q \), the parameters \( a_j \) and \( b_j \) are such that no poles of \( \Gamma(b_j - s); j = 1, 2, \ldots, m \) coincides with any pole of \( \Gamma(1 - a_j + s); j = 1, 2, \ldots, n \). The poles of integrand must be simple and those of \( \Gamma(b_j - s); j = 1, 2, \ldots, m \) lie on one side of the contour \( L \) and those of \( \Gamma(1 - a_j + s); j = 1, 2, \ldots, n \) must lie on the other side. The integral converges if \( p + q < 2(m + n) \) and \( |\arg s| < \left( m + n - \frac{1}{2} p - \frac{1}{2} q \right)\pi \).

For sake of brevity \( (a_p, e_p) \) denotes \( (a_1, e_1), \ldots, (a_p, e_p) \).

The object of this paper is to obtain new operational relations between the original and the image in two variables that involve Meijer's \( G \) function.

2. The main result.

If

(i) \( \bar{\delta} = \alpha + \beta - \frac{1}{2} (\gamma + \delta) > 0, \quad |\arg \theta| < \bar{\delta} \pi \)

(ii) \( 0 \leq \beta \leq \gamma, \quad 0 \leq \alpha \leq \delta, \quad \delta \geq 1 \)

(iii) \( Re\left(b_j - \sigma - \frac{v}{n}\right) > -\frac{5}{2n}, \quad j = 1, 2, \ldots, \alpha \)

(iv) \( Re\left(a_j - \sigma - \frac{v}{n}\right) < 1 - \frac{7}{2n}, \quad j = 1, 2, \ldots, \beta \)

(v) \( a_j - b_h \) is not a positive integer, \( j = 1, 2, \ldots, \beta; \)
\( h = 1, 2, \ldots, \alpha \)

(vi) \( \Delta(a; n) \) represents the sequence \( \frac{a}{n}, \frac{a+1}{n}, \ldots, \frac{a+n-1}{n} \)

then
\[
\begin{align*}
(\text{2.1}) & \quad p^{-1/2} (pq)^{2-\nu-n\sigma/2} G_{\gamma+n, \delta}^{a, \beta+n} \left( \frac{\theta n^n (pq)^{n/2}}{b_\delta} \right) \\
& \quad \cdot \frac{n-1}{2} n^{5/2-n\sigma-2\nu} (\pi y)^{-1/2} (4xy)^{\nu+n\sigma/2-3/2} \\
& \quad \cdot \frac{1}{2} n^{2n} \left( \frac{\theta n^n}{(4xy)^{n/2}} \right)^{A}[n\sigma-1; n], a_\gamma, A[n\sigma+2\nu-2; n].
\end{align*}
\]

**Proof:** The Laplace transform of a $G$-function is given by :

\[
(\text{2.2}) \quad \int_0^\infty e^{-pt} t^{1-n\sigma} G_{\gamma, \delta}^{a, \beta} \left( \frac{\theta t^n}{b_\delta} \right) dt
\]

\[
\equiv (2\pi)^{1/2-1/2n} n^{5/2-n\sigma} \pi^{\nu-2} G_{\gamma+n, \delta}^{a, \beta+n} \left( \frac{\theta n^n}{p^n} \right)^{A}[2-n-n\sigma; -n], a_\gamma
\]

where $\bar{\delta} = \alpha + \beta - \frac{1}{2} \gamma - \frac{1}{2} \delta > 0$, $|\arg\theta| < \bar{\delta} \pi$, $R(b_j - n\sigma) > -2$, $j = 1, 2, \ldots, \beta$.

The result (2.2) is either known or can be proved easily. To prove (2.2), we substitute the contour integral (1.3) for the $G$-function and change the order of integration which is permissible due to the absolute convergence of the integrals involved in the process; then evaluating the inner integral and using the definition (1.3) of $G$-function, we get the required result (2.2). The Laplace transform for $n = 1$ is given by Luke ([5], result 1, p. 166).

On writing $(pq)^{-1/2}$ for $p$ and multiplying both the sides of (2.2) by $p^{-1/2} (pq)^{-\nu}$ and then interpreting with the help of the known result ([3], p. 144) or ([2], p. 243), we get

\[
(\text{2.3}) \quad (\pi y)^{-1/2} (4xy)^{\nu/2} - \frac{1}{4} \int_0^{\infty} t^{3/2-n\sigma-\nu} J_{2\nu-1} [2 (4xy)^{1/4} t^{1/2}] \frac{\theta t^n}{b_\delta} dt
\]

\[
\equiv (2\pi)^{1/2-n/2} p^{-1/2} (pq)^{2-\nu-n\sigma/2} G_{\gamma+n, \delta}^{a, \beta+n} \left( \frac{\theta n^n (pq)^{n/2}}{b_\delta} \right)^{A}[2-n-n\sigma; -n], a_\gamma.
\]

Now evaluating the left-hand side integral by the process mentioned in (2.2) to obtain the desired result. Hence (2.1) is proved.
3. Particular cases.

On specializing the parameters the \( G \)-function can be reduced to MacRobert's \( E \)-function, generalized hypergeometric function and other higher transcendental functions. Therefore the result (2.1) leads to the generalization of many results (see for instance [2], [3], and [4]). Only two interesting particular cases are given below. Both the results are believed to be new.

In (2.1), putting \( \alpha = \delta = 2 \), \( \beta = \gamma = 0 \), \( n = 1 \), \( b_1 = b \), \( b_2 = c \) and using the formula ([5], p. 231)

\[
G_{1,2}^{2,1}\left(z \left| \begin{array}{c}
\frac{a}{b, c}
\end{array}\right.\right) =
\]

\[
= \Gamma(b - a + 1) \Gamma(\sigma - a + 1) \frac{1}{\sigma - a + 1} Z^{\frac{1}{2}} (b + c - 1) \frac{z}{\sigma} W_1^{\frac{1}{2}} (\sigma - b - c - 1), \frac{1}{2} (b - c) (z),
\]

we obtain

\[
(3.1) \quad p^{-1/2} (pq)^{1/4(b+c-2a+\gamma - \nu)} \exp\left(\frac{1}{2} \sqrt{pq} \right) W_1^{\frac{1}{2}} (\sigma - b - c - 3), \frac{1}{2} (b - c) (\sqrt{pq})
\]

\[
\frac{(\sigma - 1, \sigma + 2v - 2)}{\Gamma(b - \sigma + 2) \Gamma(\sigma + 2)} G_{2,2}^{2,1}\left(\frac{1}{2} \sqrt{xy} \left| \begin{array}{c}
\sigma - 1, \sigma + 2v - 2
\end{array}\right.\left| b, c\right.\right). \]

Similarly by taking \( \alpha = 3 \), \( \beta = \gamma = 0 \), \( \delta = 4 \), \( n = 1 \), \( \sigma = 5/4 + \kappa \), \( \nu = 1/2 - k \), \( b_1 = m - \frac{1}{4} \), \( b_2 = \frac{1}{4} \), \( b_3 = - \frac{1}{4} \), \( b_4 = - m - \frac{1}{4} \) and using the known results ([5], result 43, p. 234), we obtain

\[
(3.2) \quad p^{-1/2} (pq)^{\kappa} + \frac{7}{8} G_{1,4}^{3,1}\left(\sqrt{pq} \left| \begin{array}{c}
k + 1/4
m - \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, - m - \frac{1}{4}
\end{array}\right.\right)
\]

\[
= \Gamma(m - k + 1/2) [\Gamma(2m - 1)]^{-1} y^{-\frac{1}{2}} (4 \times y)^{-\frac{k}{2}} M_{-k, m} [2 \times y^{-\frac{1}{4}}] W_{k, m} [2 \times y^{-\frac{1}{4}}].
\]
REFERENCES


