A FIXED POINT THEOREM IN STRICTLY
CONVEX BANACH SPACES (*)

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SOMMARIO. - Si dà una estensione di due teoremi di punto fisso dovuti a U. Barbuti - S. Guerra ed a S. P. Singh - M. I. Riggio.

SUMMARY. - Two fixed point theorems of U. Barbuti - S. Guerra and S. P. Singh - M. I. Riggio are improved.

It will be usefull to recall some definitions. Let \((X, \delta)\) be a metric space. The measure of noncompactness of the bounded set \(A \subseteq X\), denoted by \(\alpha(A)[6]\), is the infimum of \(\varepsilon > 0\) such that \(A\) admits a finite covering consisting of subsets with diameter less than \(\varepsilon\). We will use the following properties of \(\alpha\):

\[
\alpha(A) = 0 \iff A \text{ is precompact}
\]

\[
\alpha(A \cup B) = \max \{\alpha(A), \alpha(B)\}.
\]

A continuous mapping \(T : X \to X\) such that

\[
\alpha(TA) < \alpha(A)
\]

for any bounded subset \(A\) with \(\alpha(A) > 0\), is called densifying.

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Let \((Y, \| \cdot \|)\) be a Banach space and \(T\) a continuous mapping \(T : Y \to Y\). \(T\) is called \textit{generalized contraction} on \(Y\) if:

\[
\|Tx - Ty\| \leq a \|x - y\| + b \left[ \|x - Tx\| + \|y - Ty\| \right]
\]

for all \(x\) and \(y\) in \(Y\), where \(a\) and \(b\) are positive and \(a + 2b \leq 1\).

Let us finally recall the following theorem due to J. B. Diaz and F. T. Metcalf [2].

**Theorem A.** Let \(f\) be a continuous selfmapping of the metric space \((X, \delta)\) such that:

- (a) the set \(F(f) = \{x \in X : f(x) = x\}\) is nonempty;
- (b) for each \(y \in X\) such that \(y \notin F(f)\), and for each \(u \in F(f)\) we have

\[
\delta(fy, u) < \delta(y, u).
\]

Then one, and only one, of the following properties holds:

- (c) for each \(x \in X\) the Picard sequence \(\{f^n x\}\) contains no convergent subsequence;
- (d) for each \(x \in X\) the sequence \(\{f^n x\}\) converges to a point belonging to \(F(f)\).

Let us prove the following theorem:

**Theorem.** Let \(C\) be a bounded, closed and convex subset of a strictly convex Banach space \(X\), and \(T\) a generalized contraction on \(X\) which is also densifying. Then, for each \(x \in X\), the Picard sequence \(\{S^n x\}\), where

\[
1) \quad S = \lambda_0 I + \lambda_1 T + \lambda_2 T^2 + \ldots + \lambda_k T^k
\]

with

\[
\lambda_i \geq 0; \quad \lambda_i > 0; \quad \sum_{i=1}^{k} \lambda_i = 1 \quad (1)
\]

converges to a fixed point of \(T\).

(1) Such a transformation \(S\), with \(T\) contraction, was recently introduced by W. A. Kirk [5]. Let us observe that \(S\) is a selfmapping of \(C\), being \(C\) convex.
Proof. Let us prove that \( S \) is densifying. Let \( A \) be a bounded nonprecompact subset of \( C \). We have

\[
SA \subseteq \lambda_0 A + \lambda_1 TA + \ldots + \lambda_k T^k A
\]

and hence

\[
\alpha(SA) \leq \lambda_0 \alpha(A) + \lambda_1 \alpha(TA) + \ldots + \lambda_k \alpha(T^k A).
\]

Being \( T \) densifying,

\[
\alpha(TA) < \alpha(A)
\]

\[
\alpha(T^2 A) \leq \alpha(TA) < \alpha(A)
\]

\[
\ldots \ldots \ldots \ldots \ldots \ldots
\]

\[
\alpha(T^k A) \leq \alpha(T^{k-1} A) \leq \ldots < \alpha(A)
\]

(the equality in the \( n \)-th row holds iff \( \alpha(T^{n-1} A) = 0 \)) and therefore

\[
\alpha(SA) < (\lambda_0 + \lambda_1 + \ldots + \lambda_k) \alpha(A) = \alpha(A).
\]

By a theorem of M. Furi and A. Vignoli [4], \( S \) admits at least one fixed point, so the property a) of theorem \( A \) is proved for the transformation \( S \). The verification of the property b) is a part of the proof of the theorem of U. Barbuti and S. Guerra in [1]. In the same paper the authors proved that \( F(T) = F(S) \), hence, to attain our thesis, it will be sufficient to exclude the property c) of theorem \( A \). To this purpose, we shall use an idea introduced by Furi and Vignoli [3], and followed by S. P. Singh and M. L. Riggio in [7].

For \( x \in C \), let be

\[
A = \bigcup_{n=0}^{\infty} S^n x.
\]

We have \( SA = \bigcup_{n=1}^{\infty} S^n x \subseteq A \), and since \( A = \{x\} \cup SA \)

\[
\alpha(A) = \max \{\alpha(|x|), \alpha(SA)\}
\]

\[
= \max \{0, \alpha(SA)\} = \alpha(SA)
\]

Because \( S \) is densifying, \( \alpha(A) = 0 \) and hence \( A \) is precompact. Since \( X \) is a complete metric space, \( A \) is compact and therefore the sequence \( \{S^n x\} \) contains a convergent subsequence.
The theorem proved above improves the theorem of U. Barbuti and S. Guerra [1] in which \( TC \) is required to be compact (such a transformation is obviously densifying for it is completely continuous). On the other hand we have improved the result of S. P. Singh and M. I. Riggio in [7] where the same thesis is proved for the transformation

\[
T_{\lambda} x = \lambda T x + (1 - \lambda) x \quad 0 < \lambda < 1
\]

which is of the type 1) with \( k = 1 \).

BIBLIOGRAPHY


