FREE PRODUCTS OF COMMUTATIVE RINGS WITH AMALGAMATION (*)

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SOMMARIO. - Si dimostra un teorema che dà condizioni sufficienti sopra una classe $\mathcal{K}$ di anelli commutativi affinchè esistano in $\mathcal{K}$ prodotti liberi con amalgamazione. Questo teorema viene poi usato per mostrare l'esistenza di prodotti liberi con amalgamazione nella classe di tutti gli anelli che soddisfano all'equazione $x^n = x$. Nel caso speciale $n = 2$ si ritrova un risultato noto per gli anelli di Boole.

SUMMARY. - We prove a theorem giving sufficient conditions on a class $\mathcal{K}$ of commutative rings in order that free products with amalgamation exist in $\mathcal{K}$. This theorem is then used to show that free products with amalgamation exist in the class of all rings satisfying the equation $x^n = x$. The special case where $n = 2$ gives a known result for Boolean rings.

Let $\mathcal{K}$ be a class of commutative rings and let $\{A_t\}_{t \in T} \subseteq \mathcal{K}$. Let $B \in \mathcal{K}$ such that for every $t \in T$, there exists a monomorphism $f_t: B \to A_t$. The free product of $\{A_t\}_{t \in T}$ in $\mathcal{K}$ with amalgamated subring $B$ is a pair $(A, \{g_t\}_{t \in T})$, where $A \in \mathcal{K}$ and for every $t \in T$, $g_t: A_t \to A$ is a monomorphism and the following conditions are satisfied:

(i) For every $t_1, t_2 \in T$, $g_{t_1}f_{t_1} = g_{t_2}f_{t_2}$.

(ii) $A$ is generated by $\bigcup_{t \in T} g_t(A_t)$.

(iii) If $R \in \mathcal{K}$ and $\{h_t\}_{t \in T}$ is a set of homomorphisms such that $h_t: A_t \to R$ and for every $t_1, t_2 \in T$, $h_{t_1}f_{t_1} = h_{t_2}f_{t_2}$, then there exists a homomorphism $h: A \to R$ such that $hg_t = h_t$ for every $t \in T$.

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We say that free products with amalgamation exist in \( \mathcal{K} \) if the free product of \( \{ A_i \}_{i \in T} \) in \( \mathcal{K} \) with amalgamated subring \( B \) exists for every \( \{ A_i \}_{i \in T} \subseteq \mathcal{K} \) and \( B \in \mathcal{K} \).

The existence of free products with amalgamation in the class of all Boolean rings (with unity) was proved in [1], and free products with amalgamation in classes of universal algebras are discussed in [5]. In this note we prove a theorem (Theorem 1) giving sufficient conditions on a class \( \mathcal{K} \) of commutative rings in order that free products with amalgamation exist in \( \mathcal{K} \). We then use this theorem to show (Theorem 2) that free products with amalgamation exist in the class of all rings satisfying the equation \( x^n = x \) (for all \( x \) and a fixed integer \( n > 1 \)). The case where \( n = 2 \) gives the result for Boolean rings which we referred to earlier in [1]. Finally, we consider the existence of free products with amalgamation in the class of all rings \( A \) with the property that for every \( x \in A \) there exists an integer \( n(x) > 1 \) such that \( x^{n(x)} = x \) (See Theorem 3).

Free products with amalgamation are closely related to the following amalgamation property. A class \( \mathcal{K} \) of commutative rings has the amalgamation property if for every \( A_1, A_2, B \in \mathcal{K} \) and for every monomorphisms \( f_1 : B \rightarrow A_1 \) and \( f_2 : B \rightarrow A_2 \), there exist \( A \in \mathcal{K} \) and monomorphisms \( g_1 : A_1 \rightarrow A \) and \( g_2 : A_2 \rightarrow A \) such that \( g_1 f_1 = g_2 f_2 \).

The amalgamation property has been investigated for a number of algebraic systems and a detailed discussion of this property with references to the literature is given in [4]. Clearly, if free products with amalgamation exist in a class \( \mathcal{K} \) of commutative rings, then \( \mathcal{K} \) has the amalgamation property. The converse, however, does not hold: The class \( \mathcal{F} \) of all fields has the amalgamation property [4] but free products with amalgamation do not exist in \( \mathcal{F} \) (not even free products exist in \( \mathcal{F} \) [5]). It is not difficult to show, however, that if \( \mathcal{K} \) is a variety (i.e. an equationally defined class), then \( \mathcal{K} \) has the amalgamation property if and only if free products with amalgamation exist in \( \mathcal{K} \) (see Lemma 2).

Throughout the following, \( \mathcal{K} \) will denote a variety of commutative rings. Moreover, for every \( \{ A_i \}_{i \in T} \subseteq \mathcal{K} \) and \( B \in \mathcal{K} \) such that for every \( i \in T \), there exists a monomorphism \( f_i : B \rightarrow A_i \), we define the ideal \( I(\{ A_i \}_{i \in T}, B) \) as follows. Let \( (E, \{ i_t \}_{t \in T}) \) be the free product of \( \{ A_i \}_{i \in T} \in \mathcal{K} \) ([5], p. 103), and for simplicity of notation identify each \( A_i \) with \( i_t(A_i) \). Then \( I(\{ A_i \}_{i \in T}, B) \) is the ideal of \( E \) generated by \( \{ f_t(x) - f_u(x) \mid t_1, t_2 \in T, x \in B \} \).
The following two lemmas follow easily from the preceding definitions.

**Lemma 1.** The free product of \( \{ A_t \}_{t \in T} \) in \( \mathcal{K} \) with amalgamated subring \( B \) exists if and only if \( I(\{ A_t \}_{t \in T}, B) \cap A_t = \{0\} \) for every \( t \in T \).

**Proof.** Let \( (I \{ A_t \}_{t \in T}, B) = I \). To show the necessity of the condition, let \( (A, \{ g_t \}_{t \in T}) \) be the free product of \( \{ A_t \}_{t \in T} \) in \( \mathcal{K} \) with amalgamated subring \( B \). Since \( E \) is the free product of \( \{ A_t \}_{t \in T} \), there exists a homomorphism \( g : E \to A \) such that for every \( t \in T \), \( g|_{A_t} = g_t \), where \( g|_{A_t} \) denotes the restriction of \( g \) to \( A_t \). Let \( J \) be the kernel of \( g \). Then for every \( x \in B \), \( g(f_t(x) - f_t(x)) = 0 \), and it follows from the definition of \( I \) that \( I \subseteq J \). But for every \( t \in T \), \( J \cap A_t = \{0\} \), hence \( I \cap A_t = \{0\} \).

Conversely, suppose that \( I \cap A_t = \{0\} \) for every \( t \in T \), and let \( g_t \) be the restriction to \( A_t \) of the natural homomorphism of \( E \) onto \( E/I \). Then it can be shown, in exactly the same way as in ([1], p. 228), that \( (E/I, \{ g_t \}_{t \in T}) \) is the free product of \( \{ A_t \}_{t \in T} \) in \( \mathcal{K} \) with amalgamated subring \( B \).

**Lemma 2.** Let \( \mathcal{K} \) be a variety of commutative rings. Then free products with amalgamation exist in \( \mathcal{K} \) if and only if \( \mathcal{K} \) has the amalgamation property.

**Proof.** Suppose first that the amalgamation property holds in \( \mathcal{K} \). Let \( \{ A_t \}_{t \in T} \subseteq \mathcal{K} \) and \( B \in \mathcal{K} \) such that for every \( t \in T \), there exists a monomorphism \( f_t : B \to A_t \). Let \( I = I(\{ A_t \}_{t \in T}, B) \). We shall show that \( I \cap A_t = \{0\} \) for every \( t \in T \). Suppose \( I \cap A_{a_0} \neq \{0\} \) for some \( t_0 \in T \), and let \( a \in I \cap A_{a_0} \), \( a \neq 0 \). Clearly \( I \) is also generated by \( \{ f_{t_0}(x) - f_t(x) \} \) \( t \in T \), \( x \in B \). Hence

\[
a = \sum_{i=1}^{n} r_i(f_{t_0}(x_i) - f_t(x_i)) + n_i(f_{t_0}(x_i) - f_t(x_i)) \quad (*)
\]

where \( r_i \in E \), \( x_i \in B \), and \( n_i \) is an integer. Since the amalgamation property holds in \( \mathcal{K} \), there exist \( C \in \mathcal{K} \) and monomorphisms \( g_i : A_{t_i} \to C \), such that \( g_if_{t_i} = g_jf_{t_j} \) for all \( i, j \), \( 0 \leq i, j \leq n \). For every \( t \in T \) such that \( t \neq t_i \), \( 0 \leq i \leq n \), let \( g_t : A_t \to C \) be the zero homomorphism. Since \( E \) is the free product of \( \{ A_t \}_{t \in T} \), there exists a homomorphism \( g : E \to C \) such that \( g|_{A_t} = g_t \) for every \( t \in T \). Then
from equation (*),

\[ g(a) = \sum_{i=1}^{n} g(r_i) (g f_{t_0} (x_i) - g f_{t_i} (x_i)) + n_i (g f_{t_0} (x_i) - g f_{t_i} (x_i)) \]

\[ = \sum_{i=1}^{n} g(r_i) (g f_{t_0} (x_i) - g f_{t_i} (x_i)) + n_i (g f_{t_0} (x_i) - g f_{t_i} (x_i)) \]

\[ = 0, \text{ since } g f_{t_i} = g f_{t_j}. \]

On the other hand, since \( a \in A_{t_0} \) and \( g_{t_0} \) is a monomorphism, \( g(a) = g_{t_0}(a) \neq 0 \). This contradiction shows that \( I \cap A_t = (0) \) for all \( t \in T \). Hence, by Lemma 1, free products with amalgamation exist in \( \mathcal{K} \). The converse is obvious.

We now prove the main theorem.

**Theorem 1.** Let \( \mathcal{K} \) be a variety of commutative rings satisfying the following two conditions:

1. For every \( A \in \mathcal{K} \), \( A \) is semisimple (i.e. the Jacobson radical of \( A \) is \( (0) \)).
2. For every \( A \in \mathcal{K} \) and every subring \( B \) of \( A \), a proper ideal \( M \) of \( B \) is maximal if and only if \( M = B \cap M' \) for some maximal ideal \( M' \) of \( A \).

Then free products with amalgamation exist in \( \mathcal{K} \).

**Proof.** By Lemma 2, it suffices to show that the amalgamation property holds in \( \mathcal{K} \). Thus let \( A_1, A_2, B \in \mathcal{K} \) and suppose that there are monomorphisms \( f_1 : B \to A_1 \) and \( f_2 : B \to A_2 \). Let \( (E; \{ t_1, t_2 \}) \) be the free product of \( A_1 \) and \( A_2 \) in \( \mathcal{K} \) and for simplicity of notation identify \( A_i \) with \( t_i(A_i) \), \( i = 1, 2 \). Let \( I \) be the ideal of \( E \) generated by \( \{ f_1(x) - f_2(x) \mid x \in B \} \). We shall show that \( I \cap A_i = (0) \), \( i = 1, 2 \). Suppose that \( I \cap A_i = (0) \), and let \( a \in I \cap A_i \), \( a \neq 0 \). Since \( A_i \) is semisimple, there exists a maximal ideal \( M_i \) of \( A_i \) such that \( a \notin M_i \). Let \( N_i = M_i \cap f_i(B) \). Then by condition (2), \( N_i = f_i(B) \) or \( N_i \) is a maximal ideal of \( f_i(B) \). Suppose that \( N_i = f_i(B) \). Let \( h_1 : A_1 \to A_1/M_1 \) be the natural homomorphism, and let \( h_2 : A_2 \to A_1/M_1 \) be the zero homomorphism. Since \( E \) is the free product of \( A_1 \) and \( A_2 \), there is a homomorphism \( h : E \to A_1/M_1 \) such that \( h | A_i = h_i \), \( i = 1, 2 \). Now since \( a \in I \),

\[ a = \sum_{j=1}^{n} r_j (f_1(x_j) - f_2(x_j)) + n_j (f_1(x_j) - f_2(x_j)), \ldots, (*) \]
where \( r_j \in E, \ x_j \in B, \) and \( n_j \) is an integer. Thus
\[
h(a) = \sum_{j=1}^{n} h(r_j)(h_1 f_1(x_j) - h_2 f_2(x_j)) + n_j (h_1 f_1(x_j) - h_2 f_2(x_j)) = 0,
\]
since \( h_i f_i(x) = 0 \) for all \( x \in B, \ i = 1, 2. \) On the other hand, since \( a \notin M_1, \ h(a) = h_1(a) \neq 0. \) This contradiction shows that \( N_1 \neq f_1(B). \) Thus \( N_i \) is a maximal ideal of \( f_1(B). \) Hence the ideal \( N_2 = f_2 f_1^{-1}(N_1) \) is maximal in \( f_2(B). \) Hence by condition (2), there is a maximal ideal \( M_2 \) of \( A_2 \) such that \( M_2 \cap A_2 = N_2. \) Let \( h_i : A_i \to A_i/M_i, \ i = 1, 2, \) be the natural homomorphism. Then it follows from condition (1) that for every \( i = 1, 2, A_i/M_i \) is a field and \( f_i(B)/N_i \) is a subfield of \( A_i/M_i. \) Since the amalgamation property holds in the class of all fields [4], there exists a field \( F \) and monomorphisms \( h_i' : A_i/M_i \to \to F, \ i = 1, 2. \) such that \( h''_i h'_i f_i = h''_2 h'_2 f_2. \) Moreover \( F \) can be chosen such that \( F \in \mathcal{K}. \) Since \( E \) is the free product of \( A_1 \) and \( A_2, \) there is a homomorphism \( h : E \to F \) such that \( h(A_i) = h_i', h_i', \ i = 1, 2. \) Now from (*)
\[
h(a) = \sum_{j=1}^{n} h(r_j)(h_1' h_1' f_1(x_j) - h_2' h_2' f_2(x_j)) + n_j (h_1' h_1' f_1(x_j) - h_2' h_2' f_2(x_j)) = 0.
\]
On the other hand, since \( a \notin M, \ h(a) = h_1(a) \neq 0. \) This contradiction shows that \( I \cap A_1 = (0). \) Similarly \( I \cap A_2 = (0). \)

Now let \( G = E/I, \ g : E \to E/I \) be the natural homomorphism, and \( g_i = g | A_i, \ i = 1, 2. \) Since \( I \cap A_i = (0), \) each \( g_i \) is a monomorphism. Moreover, since \( f_1(x) - f_2(x) \in I \) for every \( x \in B, \ g(f_1(x)) - f_2(x) = 0. \) Hence \( g_1 f_1 = g_2 f_2. \) This shows that the amalgamation property holds in \( \mathcal{K} \) and completes the proof of the theorem.

We now apply Theorem 1 to the equationally defined class \( \mathcal{L} \) which is defined as follows. Let \( n > 1 \) be a fixed integer, and let \( \mathcal{L} \) be the class of all rings \( A \) satisfying the equation \( x^n = x \) for all \( x \in A. \) It is known [3] that for every \( A \in \mathcal{L}, \) \( A \) is commutative and semisimple. Moreover, it is shown in [2] that for every \( A \in \mathcal{L}, \) \( A \) has the congruence extension property; that is, for every subring \( B \) of \( A, \) if \( I \) is an ideal of \( B, \) then \( I = B \cap I^* \) for some ideal \( I^* \) of \( A. \)

**Theorem 2.** Free products with amalgamation exist in the class \( \mathcal{L}. \)

**Proof.** We show that conditions (1) and (2) of Theorem 1 hold in \( \mathcal{L}. \) As we already noted, condition (1) holds. To show that condition (2) holds, we first observe that if \( J \) is an ideal of \( R \in \mathcal{L}, \)
then the intersection of all the maximal ideals of $R/J$ is $(0)$. Hence every proper ideal of $R$ is the intersection of all the maximal ideals of $R$ containing it. Now, let $B$ be a subring of $A \in \mathcal{L}$, and suppose first that $M$ is a maximal ideal of $B$. Since $B$ has the congruence extension property, there exists an ideal $M^*$ of $A$ such that $M^* \cap B = M$. Moreover, $M^*$ is proper. Hence $M^*$ is the intersection of all the maximal ideals of $A$ containing $M^*$. Thus we can find a maximal ideal $M'$ of $A$ such that $M' \supseteq M^*$ and $M' \cap B$ is proper in $B$. By the maximality of $M$, $M' \cap B = M$.

Conversely, let $M'$ be a maximal ideal of $A$. Since $A/M' \in \mathcal{L}$, $A/M'$ is a field. Let $x \in B/M' \cap B$, $x \neq 0$. Then $x^n = x$. Hence $x^{n-1} = 1$, and the multiplicative inverse of $x$ is in $B/M' \cap B$. Hence $B/M' \cap B$ is a field and $M' \cap B$ is a maximal ideal of $B$. Thus condition (ii) holds and the proof is complete.

The following two corollaries follow immediately from Theorem 2. A ring $A$ is called a $p$-ring, where $p$ is a fixed prime, if for all $x \in A$, $x^p = x$ and $px = 0$.

**Corollary 1.** The class $\mathcal{L}$ has the amalgamation property.

**Corollary 2.** Free products with amalgamation exist in the class of all $p$-rings.

We now consider the class $\mathcal{L}^*$ consisting of all rings $A$ with the property that for every $x \in A$, there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$. Members of $\mathcal{L}^*$ have the congruence extension property [2], and for every $A \in \mathcal{L}^*$, $A$ is commutative and semisimple [3]. However, we cannot apply Theorem 1 to $\mathcal{L}^*$ since it is not a variety. On the other hand, the proof of Theorem 1 can be used to show that $\mathcal{L}^*$ has the amalgamation property (although the free product of an arbitrary subset of $\mathcal{L}^*$ need not exist in $\mathcal{L}^*$, the free product of a finite number of members of $\mathcal{L}^*$ does exist in $\mathcal{L}^*$). Moreover, the argument used in the proof of Lemma 2 can be also used to show that if $\mathcal{K}'$ has the amalgamation property and $\mathcal{K}'$ is a subclass of the variety $\mathcal{K}$, then the free product of $\{A_\ell | x \in \mathcal{L} \}$ in $\mathcal{K}$ with amalgamated subring $B$ exists for every $\{A_\ell | x \in \mathcal{L} \} \subseteq \mathcal{K}'$ and $B \in \mathcal{K}'$. Thus we have the following

**Theorem 3.** Free products with amalgamation need not exist in $\mathcal{L}^*$. However, if $\mathcal{K}$ is the class of all commutative rings, $\{A_\ell | x \in$
\( \subseteq \mathcal{L}^* \), \( B \in \mathcal{L}^* \), and for every \( t \in T \), there exists a monomorphism \( f_t : B \to A_t \), then the free product of \( \{ A_t \}_{t \in T} \) in \( \mathcal{R} \) with amalgamated subring \( B \) exists.

REFERENCES


