FREE EXTENSIONS
OF DISTRIBUTIVE LATTICES (*)

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SOMMARIO. - Si prova l'esistenza di $\mathcal{K}$-estensioni libere $D_\alpha(\mathcal{K})$ di un reticolo distributivo $D$ per certe classi $\mathcal{K}$ di reticoli distributivi $\alpha$-completi e si indaga quando $D_\alpha(\mathcal{K})$ è isomorfa ad un $\alpha$-anello di insiemi. Se $\mathcal{K}$ soddisfa le identità $\lor$ e $\land$ distributive $\alpha$-infinite, si assegna una condizione sufficiente affinché $D_\alpha(\mathcal{K})$ sia isomorfa ad un $\alpha$-anello di insiemi e si fa vedere che se $\alpha$ è numerabile, $D_\alpha(\mathcal{K})$ è sempre isomorfa ad un tale anello.

SUMMARY. - We prove the existence of the free $\mathcal{K}$-extensions $D_\alpha(\mathcal{K})$ of a distributive lattice $D$ for certain classes $\mathcal{K}$ of $\alpha$-complete distributive lattices and examine when $D_\alpha(\mathcal{K})$ is isomorphic to an $\alpha$-ring of sets. When $\mathcal{K}$ satisfies the join and meet $\alpha$-infinite distributive identities we give a sufficient condition for $D_\alpha(\mathcal{K})$ to be isomorphic to an $\alpha$-ring of sets, and show that if $\alpha$ is countable, then $D_\alpha(\mathcal{K})$ is always isomorphic to an $\alpha$-ring of sets.

Let $D$ be a distributive lattice and let $\mathcal{K}$ be a class of $\alpha$-complete distributive lattices. A free $\mathcal{K}$-extension of $D$ is a lattice $D_\alpha(\mathcal{K}) \in \mathcal{K}$ such that $D_\alpha(\mathcal{K})$ is $\alpha$-generated by a sublattice $D_\alpha(\mathcal{K})$ isomorphic to $D$, and every homomorphism of $D_\alpha(\mathcal{K})$ into a distributive lattice $C \in \mathcal{K}$ can be extended to an $\alpha$-homomorphism of $D_\alpha(\mathcal{K})$ into $C$. If $\mathcal{K}$ is the class of all $\alpha$-complete distributive lattices, then $D_\alpha(\mathcal{K})$ is called a free $\alpha$-extension of $D$ and is denoted by $D_\alpha$. Free $\alpha$-extensions of Boolean algebras were investigated by the author [6] and G. Day [1] (see also [5], § 36), and free $\mathcal{K}$-extensions of abstract algebras are studied in [4]. In this paper we investigate free $\mathcal{K}$-extensions of distributive lattices and their representation

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by \(\alpha\)-rings of sets, thus extending the results of [6] to distributive lattices. In § 2 we show (Theorem 2.1) that the free \(\alpha\) extension \(D_\alpha\) of a distributive lattice \(D\) always exists, and in § 3 we investigate the free \(\mathcal{K}\)-extension \(D_\alpha(\mathcal{K})\) of \(D\), where \(\mathcal{K}\) is the class of all \(\alpha\)-complete distributive lattices satisfying the join and meet \(\alpha\)-infinite distributive identities. After proving (Theorem 3.1) the existence of \(D_\alpha(\mathcal{K})\) we show (Theorem 3.3) that \(D_\alpha(\mathcal{K})\) is always isomorphic to a \(\sigma\)-ring of sets. We also give (Theorem 3.2) a sufficient condition in order that \(D_\alpha(\mathcal{K})\) be isomorphic to an \(\alpha\)-ring of sets for an arbitrary cardinal number \(\alpha\). Moreover this \(\alpha\)-ring of sets can be chosen, in a natural way, as an \(\alpha\)-ring of subsets of the Stone space of the Boolean extension of \(D\).

1. Definitions and Notation.

Boolean concepts which are not defined explicitly in this paper have the same meaning as in [5]. Lattice join, meet, inclusion, and complementation are denoted respectively by \(\vee\), \(\wedge\), \(\leq\), and \(\sim\). The smallest and the largest elements of a lattice \(L\) are denoted, whenever they exist, by \(0\) and \(1\) respectively. A homomorphism \(h\) of an \(\alpha\)-complete lattice \(L\) into an \(\alpha\)-complete lattice \(L'\) is called an \textit{\(\alpha\)-complete homomorphism} (or, briefly, \(\alpha\)-homomorphism) if \(h\) preserves joins and meets of subsets of \(L\) with cardinality at most \(\alpha\). A congruence relation \(\theta\) on an \(\alpha\)-complete lattice \(L\) is called \(\alpha\)-\textit{complete} if the homomorphism corresponding to \(\theta\) is \(\alpha\)-complete. A sublattice \(S\) of an \(\alpha\)-complete lattice \(L\) is called \(\alpha\)-\textit{regular} if for every subset \(A\) of \(S\) with cardinality at most \(\alpha\), the join and meet of \(A\), whenever they exist in \(S\), coincide with the join and meet of \(A\) in \(L\). A subset \(E\) of an \(\alpha\)-complete lattice \(L\) is said to \(\alpha\)-\textit{generate} \(L\) if \(L\) has no proper, \(\alpha\)-complete, \(\alpha\)-regular sublattices containing \(E\).

An \(\alpha\)-complete distributive lattice \(D\) is called a \textit{free \(\alpha\)-complete distributive lattice on \(m\) generators} if it is \(\alpha\)-generated by a subset \(E\) with cardinality \(m\) and with the property that every mapping of \(E\) into an \(\alpha\)-complete distributive lattice \(C\) can be extended to an \(\alpha\)-homomorphism of \(D\) into \(C\). All free \(\alpha\)-complete distributive lattices on \(m\) generators are isomorphic and will be denoted here by \(\mathcal{D}_m\).

A complete distributive lattice \(D\) is said to satisfy the \textit{Join infinite distributive identity (JID)} if for every \(\alpha \in D\) and for every
subset \{ b_i \}_{i \in I} of D,

\[ a \land (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \land b_i) . \]

D is said to satisfy the meet infinite distributive identity (MID) if the dual of (\*) holds for every a \in L and every subset \{ b_i \}_{i \in I} of D. If D is \alpha-complete and (\*) holds for every subset \{ b_i \}_{i \in I} whose cardinality is at most \alpha, then D is said to satisfy the join \alpha-infinite distributive identity (\alpha-JID). The meet \alpha-infinite distributive identity (\alpha-MID) is defined dually.

2. Free \alpha-extensions.

**Definition 2.1.** Let D be a distributive lattice and let \mathcal{K} be a class of \alpha-complete distributive lattices. An \alpha-complete distributive lattice D* is called a free \mathcal{K}-extension of D if

(i) \( D^* \in \mathcal{K} . \)

(ii) \( D^* \) is \alpha-generated by a sublattice \( D_0 \) isomorphic to D.

(iii) If \( C \in \mathcal{K} \) and \( h \) is an arbitrary homomorphism of \( D_0 \) into \( C \), then there exists an \alpha-homomorphism \( h^* \) of \( D^* \) into \( C \) such that the restriction of \( h^* \) to \( D_0 \) is \( h \).

If \( \mathcal{K} \) is the class of all \alpha-complete distributive lattices, then \( D^* \) is called a free \alpha-extension of D.

We shall show in this section that for every distributive lattice D and every cardinal number \( \alpha \), the free \alpha-extension of D exists and is unique up to isomorphisms. We denote the free \alpha-extension of D by \( D_\alpha \) and we shall consider D as a sublattice of \( D_\alpha \), thus identifying it with the sublattice \( D_0 \) of definition 2.1.

Free \alpha-extensions of distributive lattices are a natural generalization of the free \alpha-complete distributive lattices which are defined in Section 1. The relationship between these two concepts is shown by the following lemma whose proof follows immediately from the definitions.

**Lemma 2.1.** Let \( D_m \) be the free distributive lattice on m generators and let \( D_{m,\alpha} \) be the free \alpha-complete distributive lattice on m generators. Then \( D_{m,\alpha} \) is the free \alpha-extension of \( D_m \).

In the following lemma we shall consider \( D_m \) as a sublattice of \( D_{m,\alpha} \), thus identifying it with its isomorphic image in \( D_{m,\alpha} \).
Lemma 2.2. Let \( \theta \) be a congruence relation on \( D_m \) and let \( \theta^* \) be the intersection of all the \( \alpha \)-complete congruence relations on \( D_{m,\alpha} \) containing \( \theta \). Then \( \theta^* \cap (D_m \times D_m) = \theta \).

**Proof.** Imbed \( D_m/\theta \) in an \( \alpha \)-complete distributive lattice \( D^* \). Then, by Lemma 2.1, the natural homomorphism \( h \) of \( D_m \) onto \( D_m/\theta \) can be extended to an \( \alpha \)-homomorphism \( h^* \) of \( D_{m,\alpha} \) into \( D^* \). Let \( \Gamma \) be the kernel of \( h^* \); that is, let \( \Gamma \) be the congruence relation on \( D_{m,\alpha} \) defined by \( (a, b) \in \Gamma \) if and only if \( h^*(a) = h^*(b) \). Then \( \Gamma \) is an \( \alpha \)-complete congruence relation, and \( \Gamma \supseteq \theta^* \). Hence \( \Gamma \supseteq \theta^* \). And since \( \Gamma \cap (D_m \times D_m) = \theta \), we have \( \theta^* \cap (D_m \times D_m) = \theta \) also. This completes the proof.

Theorem 2.1. For every distributive lattice \( D \) and every cardinal number \( \alpha \), the free \( \alpha \)-extension \( D_\alpha \) of \( D \) exists and is unique up to isomorphisms.

**Proof.** Let \( |D| = m \) and let \( D_{m,\alpha} \) be the free \( \alpha \)-complete distributive lattice on \( m \) generators. Let \( \theta \) be a congruence relation on \( D_m \) such that \( D_m/\theta \cong D \), and let \( \theta^* \) be the intersection of all \( \alpha \)-complete congruence relations on \( D_{m,\alpha} \) containing \( \theta \). We shall show that \( D_{m,\alpha}/\theta^* \) is a free \( \alpha \)-extension of \( D \).

By Lemma 2.2, \( D_m/\theta \) is a sublattice of \( D_{m,\alpha}/\theta^* \) isomorphic to \( D \). And since \( D_{m,\alpha} \) is \( \alpha \)-generated by \( D_m \) and \( \theta^* \) is an \( \alpha \)-complete congruence relation, it follows that \( D_{m,\alpha}/\theta^* \) is \( \alpha \)-generated by \( D_m/\theta \). Thus it only remains to show that homomorphism of \( D_m/\theta \) can be extended to \( D_{m,\alpha}/\theta^* \). Let \( h \) be a homomorphism of \( D_m/\theta \) into an \( \alpha \)-complete distributive lattice \( C \). Let \( f \) be the natural \( \alpha \)-homomorphism of \( D_{m,\alpha} \) onto \( D_{m,\alpha}/\theta^* \) and denote the restriction of \( f \) to \( D_m \) by \( f' \). Then the homomorphism \( g = hf' \) has an extension \( g^* \) which is an \( \alpha \)-homomorphism of \( D_{m,\alpha} \) into \( C \). We define \( h^*: D_{m,\alpha}/\theta^* \to C \) by

\[
h^*(f(x)) = g^*(x), \quad x \in D_{m,\alpha}.
\]

To show that \( h^* \) is well defined, we let \( \Gamma \) be the kernel of \( g^* \). Thus \( \Gamma \) is the \( \alpha \)-complete congruence relation on \( D_{m,\alpha} \) defined by \( (x, y) \in \Gamma \) if and only if \( g^*(x) = g^*(y) \). Since \( \Gamma \supseteq \theta \), it follows from the definition of \( \theta^* \) that \( \Gamma \supseteq \theta^* \). Now let \( f(x) = f(y) \). Then \( (x, y) \in \epsilon \theta^* \subseteq \Gamma \). Hence \( g^*(x) = g^*(y) \). Thus \( h^* \) is well defined. Moreover, it can be easily verified that \( h^* \) is the desired extension of \( h \).

To show the uniqueness of the free \( \alpha \)-extension of \( D \), we let \( D_1 \) and \( D_2 \) be two free \( \alpha \)-extensions of \( D \). Let \( i \) be an isomorphism
of the sublattice $D$ of $D_1$ onto the sublattice $D$ of $D_2$. Then the isomorphism $i$ can be extended to an $\alpha$-homomorphism $i_1$ of $D_1$ onto $D_2$, and the isomorphism $i^{-1}$ can be extended to an $\alpha$-homomorphism $i_2$ of $D_2$ onto $D_1$. Let $D^* = \{ x \in D_1 \mid i_2 i_1(x) = x' \}$. Then $D^*$ is an $\alpha$-complete, $\alpha$-regular sublattice of $D_1$ containing $D$. Since $D_1$ is $\alpha$-generated by $D$, $D^* = D_1$. Hence $i_1$ is an isomorphism of $D_1$ onto $D_2$. This completes the proof of the theorem.

It is worth pointing out that a slight modification of the proof of the above theorem yields the following result.

**Theorem 2.2.** If $\theta$ is a congruence relation on a distributive lattice $D$, then the free $\alpha$-extension of $D/\theta$ is isomorphic to $D_\alpha/\theta^*$, where $\theta^*$ is the intersection of all the $\alpha$-complete congruence relations on $D_\alpha$ containing $\theta$.

3. Representation by $\alpha$-rings of sets.

Throughout this section, $D$ will denote a distributive lattice, $\mathcal{K}$ the class of all $\alpha$-complete distributive lattices satisfying ($\alpha$-JID) and ($\alpha$-MID) (see section 1.), and $D_\alpha(\mathcal{K})$ the free $\mathcal{K}$-extension of $D$. We shall be concerned with the problem of representing $D_\alpha(\mathcal{K})$ by an $\alpha$-ring of sets. We begin by the following definition.

**Definition 3.1.** Let $D$ be a distributive lattice. By a Boolean extension of $D$ we shall understand any Boolean algebra $B$ which is generated by a sublattice $D_0$ isomorphic to $D$.

The next lemma shows that all Boolean extensions of $D$ are isomorphic. We shall denote the Boolean extension of $D$ by $B(D)$ and, again, we shall identify $D$ with its isomorphic image in $B(D)$.

**Lemma 3.1.** Let $B(D)$ be a Boolean extension of a distributive lattice $D$. Then every lattice homomorphism of $D$ into a Boolean algebra $C$ can be extended to a Boolean homomorphism of $B$ into $C$. Thus all Boolean extensions of $D$ are isomorphic.

**Proof.** Let $C$ be an arbitrary Boolean algebra and let $h$ be a lattice homomorphism of $D$ into $C$. To show that $h$ can be extended to a Boolean homomorphism it suffices to show, by Theorem 12.2 of [5], that for every $d_1, d_2 \in D$, $d_1 \wedge \overline{d_2} = 0$ implies $h(d_1) \wedge \overline{h(d_2)} = 0$. But if $d_1 \wedge \overline{d_2} = 0$, then $d_1 \leq d_2$. Hence $h(d_1) \leq h(d_2)$, and $h(d_1) \wedge \overline{h(d_2)} = 0$. Thus $h$ can be extended to a Boolean homomorphism of $B$ into $C$. 
The uniqueness of $B^*(D)$ can be proved in exactly the same way in which we showed that $D_\alpha$ is unique (see the last paragraph in the proof of Theorem 2.1).

Funayama [2] showed that every complete lattice satisfying (JID) and (MID) can be imbedded regularly in a complete Boolean algebra (an imbedding is called regular if it preserves all joins and meets whenever they exist). This fact is also proved in ([3], Theorem 10.14); moreover, the proof in [3] also shows that if $D$ is an $\alpha$-complete distributive lattice satisfying (JID) and (MID), then $D$ can be imbedded regularly in a complete (hence $\alpha$-complete) Boolean algebra. Thus we have the following lemma.

**Lemma 3.2.** Every $\alpha$-complete distributive lattice satisfying (JID) and (MID) can be imbedded regularly in an $\alpha$-complete Boolean algebra.

We are now ready to prove the existence of $D_\alpha(\mathcal{K})$, where $\mathcal{K}$ is the class of all $\alpha$-complete distributive lattices satisfying (JID) and (MID).

**Theorem 3.1.** $D_\alpha(\mathcal{K})$ exists and is unique up to isomorphisms. In fact, if $B^*(D)$ is the free $\alpha$-extension of the Boolean algebra $B(D)$, then $D_\alpha(\mathcal{K})$ is isomorphic to the $\alpha$-complete, $\alpha$-regular sublattice $D^*$ of $B^*(D)$ $\alpha$-generated by $D$.

**Proof.** We recall [6] that $B^*(D)$ is an $\alpha$-complete Boolean algebra $\alpha$-generated by $B(D)$ such that any Boolean homomorphism of $B(D)$ into an $\alpha$-complete Boolean algebra $C$ can be extended to an $\alpha$-homomorphism of $B^*(D)$ into $C$. Since $D^*$ is an $\alpha$-complete, $\alpha$-regular sublattice of the $\alpha$-complete Boolean algebra $B^*(D)$, $D^*$ satisfies (JID) and (MID). Hence $D^* \in \mathcal{K}$. Thus to show that $D^*$ is a free $\mathcal{K}$-extension of $D$, it suffices to show that condition (iii) of Definition 2.1 is satisfied. Let $h$ be a lattice homomorphism of $D$ into an $\alpha$-complete lattice $L \in \mathcal{K}$. We shall show that $h$ can be extended to an $\alpha$-complete homomorphism $h^*$ of $D^*$ into $L$.

By Lemma 3.2, we imbed $L$ regularly in an $\alpha$-complete Boolean algebra $B'$. By Lemma 3.1, $h$ can be extended to a Boolean homomorphism $h'$ of $B(D)$ into $B'$, and by the definition of $B^*(D)$, $h'$ can be extended to an $\alpha$-complete homomorphism $h''$ of $B^*(D)$ into $B'$. Let $h^*$ be the restriction of $h''$ to $D^*$. Then $h^*$ is an $\alpha$-complete lattice homomorphism of $D^*$ into $B'$. It remains to show that $h^*(D^*) \subseteq L$. Since both $h^*(D^*)$ and $L$ are $\alpha$-complete, $\alpha$-regular sublattices of $B'$, their intersection $h^*(D^*) \cap L$ is also an $\alpha$-complete,
\( \alpha \)-regular sublattice of \( B' \). And since \( h^* (D^*) \) is \( \alpha \)-generated by \( h(D) \) and \( h(D) \subseteq L \), it follows that \( h^* (D^*) = h^* (D^*) \cap L \). Hence \( h^* (D^*) \subseteq L \). This completes the proof that \( D^* \) is a free \( \alpha \)-extension of \( D \). The uniqueness of \( D_\alpha (\mathcal{K}) \) can be shown in exactly the same way in which we showed that \( D_\alpha \) is unique.

As a consequence to the last theorem we shall prove the following two results concerning the representation of \( D_\alpha (\mathcal{K}) \) by an \( \alpha \)-ring of sets.

A Boolean algebra \( B \) is called superatomic if every subalgebra of \( B \) is atomic. (See [5]).

**Theorem 3.2.** If the Boolean extension \( B(D) \) of \( D \) is superatomic, then \( D_\alpha (\mathcal{K}) \) is isomorphic to an \( \alpha \)-ring of sets.

**Proof.** By Theorem 3.1, \( D_\alpha (\mathcal{K}) \) is isomorphic to an \( \alpha \)-complete, \( \alpha \)-regular sublattice of the free \( \alpha \)-extension \( B^* (D) \) of the Boolean algebra \( B(D) \). Since \( B(D) \) is superatomic, \( B^* (D) \) is isomorphic to an \( \alpha \)-field of sets ([6] Theorems 3.1 and 3.3). Hence \( D_\alpha (\mathcal{K}) \) is isomorphic to an \( \alpha \)-ring of sets.

**Theorem 3.3.** Let \( \mathcal{K}' \) be the class of all \( \sigma \)-complete distributive lattices satisfying (\( \sigma \)-JID) and (\( \sigma \)-MID). Then \( D_\sigma (\mathcal{K}') \) is isomorphic to a \( \sigma \)-ring of sets.

**Proof.** By Theorem 3.1, \( D_\sigma (\mathcal{K}') \) is isomorphic to a \( \sigma \)-complete, \( \sigma \)-regular sublattice of the free \( \sigma \)-extension \( B^* (D) \) of \( B(D) \). But by ([6], Corollary 3.1), \( B^* (D) \) is isomorphic to a \( \sigma \)-field of sets. Hence \( D_\sigma (\mathcal{K}') \) is isomorphic to a \( \sigma \)-ring of sets.

We conclude this section by pointing out that whenever \( D_\alpha (\mathcal{K}) \) is isomorphic to an \( \alpha \)-ring of sets \( R \), then \( R \) can be chosen as the smallest \( \alpha \)-ring of subsets of the Stone space \( X \) of \( B(D) \) containing all the open-and-closed subsets of \( X \). This follows from the proofs of Theorems 3.2 and 3.3 and the fact [6] that whenever the free \( \alpha \)-extension \( B_\alpha \) of a Boolean algebra \( B \) is isomorphic to an \( \alpha \)-field of sets, then \( B_\alpha \) is isomorphic to the smallest \( \alpha \)-field of subsets of the Stone space \( S(B) \) of \( B \) containing all the open-and-closed subsets of \( S(B) \).
REFERENCES


