EXISTENCE OF GLOBAL SOLUTIONS
OF DELAYED DIFFERENTIAL EQUATIONS
ON COMPACT MANIFOLDS (*)

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Proposition is a consequence of several facts from algebraic topology. For these we refer to [1] and keep to notations introduced in [1].

**Proof of Proposition.** Consider singular homology modules, the scalar ring being \( \mathbb{Z}/2 \) — the ring of integers mod 2. Assume that \( f(X) \cong X \) and find \( u \in X - f(X) \). Obviously \( H_n(f) : H_n(X) \to H_n(X, X - u) \) is a zero-homomorphism and by homotopy

\[
(2) \quad H_n(id) : H_n(X) \to H_n(X, X - u)
\]

is a zero-homomorphism as well. On the other hand \( H_n(X) = \mathbb{Z}/2 \) (cf [1], (22.30)) and the desired contradiction results from the fact that (2) is an isomorphism. This follows from the commutative diagram preceding (22.24) in [1]. In this diagram we put \( A = X, B = \{u\} \). It follows that \( \Gamma A \cong \mathbb{Z}/2 \cong \Gamma B, \) \( r \) is an isomorphism and \( j_B^A = H_n(id) \). The diagram mentioned above may be given the following form

\[
\begin{array}{ccc}
H_n(X) & \xrightarrow{j_A} & \mathbb{Z}/2 \\
\downarrow \quad H_n(id) & & \downarrow \\
H_n(X, X - u) & \xrightarrow{j_B} & \\
\end{array}
\]

\( j_A, j_B \) being isomorphisms (cf. [1], (22.24), (22.1)). Hence (2) is an isomorphism and Proposition is proved.

**Note 1.** By means of index theory Proposition is proved in [2] (II, Theorem 18,2) under the additional assumption of orientability of \( X \).

**Proof of Theorem.** Choose any Riemannian form \( \omega \) on \( X \). \( \omega \) induces a metric on \( X \) and a norm on \( TX \) (on \( T_wX \) for any \( w \in X \)). For any \( v \in X, s \in R \) and \( T > 0 \) there exists a solution \( x = F(v, s) \) of (1) on \( \langle s - T, \infty \rangle \) such that \( x(t) = v \) for \( t \in \langle s - T - 1, s - T \rangle \) (\( x \) is defined on \( \langle s - T - 1, \infty \rangle \) and fulfills (1) on \( \langle s - T, \infty \rangle \)). \( F(v, s)(t) \) depends continuously on \( (v, t) \) and \( F(v, s)(s - T) = v \) for \( v \in X \). Define \( f : X \to X \) by \( f(v) = F(v, s)(s) \). By Proposition \( f(X) = X \). Fix \( u \in X, s \in R \). For any \( T > 0 \) there
exists a solution $x_{[T]}$ of (1) on $(s - T, \infty)$ such that $x_{[T]}(s) = u$. As $X$ is compact and $g$ is continuous, for any compact interval $\langle a, b \rangle \subset R, g$ is bounded on $X \times X \times \langle a, b \rangle$ and the set of all

$$x_{[T]}|_{\langle a, b \rangle}, \quad T \geq \max (0, s - a)$$

(solutions $x_{[T]}$ restricted to $\langle a, b \rangle$) is compact (equicontinuous).

Hence there exists such a sequence $T_k \to \infty, \ k = 1, 2, 3, \ldots$ and $x: R \to X$ that

$$x_{[T_k]} \to x \text{ uniformly on any compact interval } \langle a, b \rangle,$$

$k \to \infty$. Obviously $x$ is a global solution of (1) and $x(s) = u$. Proof of Theorem is complete.

**Note 2.** It is sufficient to assume that $g$ is continuous as there exist $g_j, j = 1, 2, 3 \ldots$ of class $C^{(1)}$ such that $g_j \to g$ for $j \to \infty$, the convergence being uniform on $X \times X \times J$ for any compact interval $J$. The procedure works for functional differential equations on $X$ in case that there is guaranteed local existence of solutions, uniqueness, prolongability of solutions for $t \to + \infty$, continuous dependence of solutions on initial condition and equicontinuity of solutions. Again the result may be extended by passing to a limit.

**Note 3.** In some cases there passes just a unique global solution through any $u \in X$ at any $s \in R$. This was proved in [3] (Theorem 4) in case that $X$ is a torus and $g(x, t) = g_1(x(t), t) + h(x, t), g_1$ being a vectorfield and $h$ being sufficiently small. The same holds for any compact manifold (the author intends to deduce it from Theorem in [4]).

**References**


