A CHARACTERIZATION OF COMPACT FILTERBASIS IN COMPLETE METRIC SPACES (*)

by Massimo Furi and Mario Martelli (in Firenze) (**) 

SUMMARY. - In this paper, using the number $\alpha$ of Kuratowski, we give a characterization of compact filterbasis in complete metric spaces (Theorem 1). As a consequence of this characterization we extend some known results of Cantor-Kuratowski, Kuratowski and Painlevé-Kuratowski (Theorem 2, Theorem 3 and Theorem 4 respectively).

1. Let $\mathcal{B}$ and $\mathcal{B}'$ be two filterbasis. $\mathcal{B}'$ is subordered to $\mathcal{B}$, written $\mathcal{B}' \sqsubset \mathcal{B}$, iff, for any $A \in \mathcal{B}$, there exists $A' \in \mathcal{B}'$ such that $A' \subseteq A$ [1]. A filterbasis $\mathcal{B}$ in a topological space is said to be compact iff each $\mathcal{B}'$ such that $\mathcal{B}' \sqsubset \mathcal{B}$ has cluster points.

In this paper, using the number $\alpha$ of Kuratowski [2], we give a characterization of compact filterbasis in complete metric spaces (Theorem 1). As a consequence of this characterization we extend some known results of Cantor-Kuratowski [3], Kuratowski [2], and Painlevé-Kuratowski [3] (Theorem 2, Theorem 3 and Theorem 4 respectively).

We recall first some terminology.

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(**) Indirizzo degli Autori: Istituto Matematico «Ulisse Dini» - Università - Viale Morgagni 67/A — 50134 Firenze.
Let $X$ be a metric space. Throughout the paper $\alpha(A)$ denotes the Kuratowski number of $A \subseteq X$; i.e., if $A$ is bounded, $\alpha(A)$ is the infimum of all $\varepsilon > 0$ such that $A$ admits a finite covering of sets with diameter less than $\varepsilon$; if $A$ is unbounded, then $\alpha(A) = +\infty$. About the number $\alpha$ and its properties see [4] and [5].

Let $\mathcal{B}$ be a filterbasis in a metric space. We define $\alpha(\mathcal{B}) = \inf \{ \alpha(A) : A \in \mathcal{B} \}$. Obviously, if $\mathcal{B}' \mathrel{\mathrel{\subseteq}} \mathcal{B}$ then $\alpha(\mathcal{B}') \leq \alpha(\mathcal{B})$. Moreover, denoting by $\overline{A}$ the adherence of $A$, we put $\overline{\mathcal{B}} = \{ \overline{A} : A \in \mathcal{B} \}$. We shall say that $\mathcal{B}$ is a closed filterbasis iff $\mathcal{B} \mathrel{\mathrel{\subseteq}} \overline{\mathcal{B}}$ ($\mathcal{B}$ is equivalent to $\overline{\mathcal{B}}$), i.e. $\mathcal{B} \mathrel{\mathrel{\subseteq}} \overline{\mathcal{B}}$ and $\overline{\mathcal{B}} \mathrel{\mathrel{\subseteq}} \mathcal{B}$.

2. Our main result is Theorem 1. To prove this Theorem we need the following two Lemmas.

**Lemma 1.** Let $\mathcal{S}$ be a family of sets with the finite intersection property (f.i.p.) and let $A \in \mathcal{S}$. If $\{ A_k : 1 \leq k \leq n \}$ is a finite covering of $A$, then there exists $k, 1 \leq k \leq n$, such that the family $\mathcal{S}'$, obtained from $\mathcal{S}$ replacing $A$ by $A_k$, has the f.i.p.

**Proof.** Assume the contrary. Then there exist $n$ finite subfamilies $\mathcal{S}_1, \ldots, \mathcal{S}_n$ of $\mathcal{S}$, such that

$$A_k \cap (\cap \{ B : B \in \mathcal{S}_k \}) = \emptyset, \quad k = 1, \ldots, n.$$  

This implies $A \cap (\cap \{ B : B \in \cup_k \mathcal{S}_k \}) = \emptyset$. But this contradicts the f.i.p. of $\mathcal{S}$.

**Lemma 2.** Let $\mathcal{S}$ be a family with the f.i.p.. Let, for each $A \in \mathcal{S}$, $\Phi(A)$ be a finite covering of $A$. Then for each $A \in \mathcal{S}$ we can select $U_A \in \Phi(A)$ such that the family $\{ U_A : A \in \mathcal{S} \}$ has the f.i.p..

**Proof.** Let $\Omega$ be the set of all mappings $\psi$ with the following properties

\( a) \) $\mathcal{D}(\psi) \subseteq \mathcal{S}$, where $\mathcal{D}(\psi)$ denotes the domain of $\psi$;

\( b) \) $\psi(A) \in \Phi (A), \quad \forall \ A \in \mathcal{D}(\psi)$;

\( c) \) the family obtained from $\mathcal{S}$, replacing each $A \in \mathcal{D}(\psi)$ by $\psi(A)$ has the f.i.p.

We have only to prove that there exists $\varphi \in \Omega$ such that $\mathcal{D}(\varphi) = \mathcal{S}$. Obviously the set $\Omega$ is nonempty by Lemma 1. Moreover $\Omega$ can be partially ordered as follows: $\psi_1 < \psi_2$ if and only if $\mathcal{D}(\psi_1) \subseteq \mathcal{D}(\psi_2)$ and $\psi_1(A) = \psi_2(A)$ for all $A \in \mathcal{D}(\psi_1)$. By Zorn's
Lemma $\Omega$ has a maximal element $\varphi$. Therefore, by Lemma 1, $D(\varphi) = \emptyset$.

We now prove

**Theorem 1.** Let $\mathcal{B}$ a filterbasis in a complete metric space $X$. Then $\mathcal{B}$ is compact if and only if $\alpha(\mathcal{B}) = 0$.

**Proof.** Suppose $\alpha(\mathcal{B}) > 0$. Let us prove that there exists a filterbasis $\mathcal{B}' \subset \mathcal{B}$ with no cluster point. Put $\mathcal{B}' = \{ U' \subset X : \exists U \in \mathcal{B} \text{ such that } \alpha(U \setminus U') < \alpha(\mathcal{B}) \}$. Obviously each $U' \in \mathcal{B}'$ is nonempty. Let $U'_1, U'_2 \in \mathcal{B}'$. There exist $U_1, U_2 \in \mathcal{B}$ such that $\alpha(U_1 \setminus U'_1)$, $\alpha(U_2 \setminus U'_2) < \alpha(\mathcal{B})$ and $W \in \mathcal{B}$, $W \subset U_1 \cap U_2$. Since

$$\alpha(W \setminus (U'_1 \cap U'_2)) = \alpha((W \setminus U'_1) \cup (W \setminus U'_2)) =$$

$$\max \{ \alpha(W \setminus U'_1), \alpha(W \setminus U'_2) \} \leq \max \{ \alpha(U_1 \setminus U'_1), \alpha(U_2 \setminus U'_2) \} < \alpha(\mathcal{B})$$

then $U'_1 \cap U'_2$ belongs to $\mathcal{B}'$. Moreover if $V' \supset U'$, with $U' \in \mathcal{B}'$, then $V' \in \mathcal{B}'$. Hence $\mathcal{B}'$ is a filter. Clearly $\mathcal{B}' \supset \mathcal{B}$, therefore $\mathcal{B}' \vdash \mathcal{B}$. It remains only to prove that $\mathcal{B}'$ has no cluster point.

Indeed, let $x \in X$ and consider the neighborhood of $x$, $B(x, \varepsilon) = \{ y \in X : d(x, y) < \varepsilon \}$ with $0 < 2\varepsilon < \alpha(\mathcal{B})$. Since, for any $U' \in \mathcal{B}'$, we have $U \setminus (U' \setminus B(x, \varepsilon)) \subset (U \setminus U') \cup B(x, \varepsilon)$, it turns out that

$$\alpha(U \setminus (U' \setminus B(x, \varepsilon))) \leq \alpha((U \setminus U') \cup B(x, \varepsilon)) \leq$$

$$\leq \max \{ \alpha(U \setminus U'), \alpha(B(x, \varepsilon)) \} < \alpha(\mathcal{B}).$$

Hence $U' \setminus B(x, \varepsilon) \in \mathcal{B}'$. Then $x$ is not a cluster point of $\mathcal{B}'$ since $(U' \setminus B(x, \varepsilon)) \cap B(x, \varepsilon) = \emptyset$.

Suppose $\alpha(\mathcal{B}) = 0$. We must prove that any filterbasis $\mathcal{B}' \vdash \mathcal{B}$ has cluster points. Since $\alpha(\mathcal{B}') = 0$, it is sufficient to show that the condition $\alpha(\mathcal{B}) = 0$ implies the existence of cluster points of $\mathcal{B}$.

There are two possibilities:

a) there exists $A^* \in \mathcal{B}$ such that $\alpha(A^*) = 0$;

b) $\alpha(A) > 0$ for all $A \in \mathcal{B}$.

In the first case let $\mathcal{B}'$ be the filterbasis consisting of all members of $\mathcal{B}$ contained in $A^*$. Clearly for any $A \in \mathcal{B}'$ the adherence $\overline{A}$ of $A$ is compact. Since the family $\{ \overline{A} : A \in \mathcal{B}' \}$ has the f.i.p. it follows that $\cap \{ \overline{A} : A \in \mathcal{B}' \} = \emptyset$. Then $\mathcal{B}$ has cluster points since $\mathcal{B}' \vdash \mathcal{B}$.
In the second one, for any \( A \in \mathcal{B} \), let \( \Phi(A) \) be a finite covering of \( A \) consisting of subsets of \( A \) with diameter less than \( 2\alpha(A) \). By Lemma 2 we can select \( U_A \in \Phi(A) \) such that the family \( \{ U_A : A \in \mathcal{B} \} \) has the f.i.p. Clearly \( \{ U_A : A \in \mathcal{B} \} \) is a subbasis of a Cauchy filterbasis \( \mathcal{B}' \). Since \( X \) is complete, \( \mathcal{B}' \) converges to a point \( x \in X \), so \( x \) is a cluster point of \( \mathcal{B} \).

3. In this section we give some consequences of Theorem 1.

**Theorem 2.** Let \((X, d)\) be a complete metric space and let \( \mathcal{S} = \{ A_j : j \in J \} \) be a family of closed subsets of \( X \), with the f.i.p. If for any \( \varepsilon > 0 \) there exists a finite subset \( K_\varepsilon \) of \( J \) such that \( \alpha(\bigcap \{ A_j : j \in K_\varepsilon \}) < \varepsilon \), then \( \alpha(\bigcap \{ A_j : j \in J \}) \) is nonempty and compact.

**Proof.** By the assumption \( \mathcal{S} \) is a subbasis of a filterbasis \( \mathcal{B} \), such that \( \alpha(\mathcal{B}) = 0 \). Since \( A_j \) is closed for each \( j \in J \), the set \( M \) of all cluster points of \( \mathcal{B} \) is precisely \( \bigcap \{ A_j : j \in J \} \). Therefore \( M \) is closed and, by Theorem 1, nonempty. Moreover \( M \subseteq B \) for any \( B \in \mathcal{B} \), and this implies \( 0 \leq \alpha(M) \leq \alpha(\mathcal{B}) = 0 \). Then \( M \) is compact by the completeness of \( X \).

Theorem 2 contains a well known result in the case when all sets \( A_j \) are compact, and a result of Cantor-Kuratowski [3] when the family \( \mathcal{S} \) is a nonincreasing sequence \( \{ A_n \} \) of nonempty closed sets such that \( \lim_n \alpha(A_n) = 0 \).

Let \((X, d)\) be a complete metric space and let \( \mathcal{C}(X) \) be the family of all nonempty and closed subsets of \( X \). For every pair \( A, B \) of elements of \( \mathcal{C}(X) \) the Hausdorff distance \( D(A, B) \) is \( \max \{ \varrho(A, B), \varrho(B, A) \} \) where \( \varrho(A, B) = \sup \{ d(a, b) : a \in A \} \).

Recall that if \( \{ A_\delta : \delta \in \Delta \} \) is a filterbasis, then we can regard \( \Delta \) as a directed set, defining \( \delta \prec \delta' \) if and only if \( A_\delta \supseteq A_{\delta'} \).

**Theorem 3.** Let \( \mathcal{B} = \{ A_\delta : \delta \in \Delta \} \) be a closed filterbasis in a complete metric space \( X \). Assume \( \alpha(\mathcal{B}) = 0 \). Then \( \lim_{\delta \to \Delta} D(A_\delta, M) = 0 \) where \( M = \bigcap \{ A_\delta : \delta \in \Delta \} \).

**Proof.** Assume the contrary. Then there exists \( \varepsilon > 0 \) such that \( A_\delta \not\subseteq B(M, \varepsilon) \) for all \( \delta \in \Delta \). Put \( A'_\delta = A_\delta \setminus B(M, \varepsilon) \) and \( \mathcal{B}' = \{ A'_\delta : \delta \in \Delta \} \). It is easily seen that \( \mathcal{B}' \noise \mathcal{B} \). Therefore \( M' = \bigcap \{ A'_\delta : \delta \in \Delta \} \) is nonempty by Theorem 1 and obviously \( M' \cap M = \emptyset \). But this contradicts the fact that \( \mathcal{B}' \noise \mathcal{B} \).

(1) \( B(M, \varepsilon) = \{ y \in X : d(y, M) < \varepsilon \} \).
Theorem 3, in the case when the filterbasis \( \mathcal{B} \) is a nonincreasing sequence of closed sets, gives a result due to C. Kuratowski [2].

**THEOREM 4.** Let \( \mathcal{B} = \{A_\delta : \delta \in \Delta\} \) be a closed filterbasis in a complete metric space \( X \). If \( A_\delta \) is connected for any \( \delta \in \Delta \) and \( \alpha(\mathcal{B}) = 0 \), then \( M = \cap \{A_\delta : \delta \in \Delta\} \) is a nonempty continuum.

**Proof.** Clearly \( M \) is nonempty and compact by Theorem 2. Suppose \( M \) disconnected. Then we can find a pair of nonempty, compact sets \( M_1, M_2 \) such that \( M_1 \cap M_2 = \Phi \) and \( M_1 \cup M_2 = M \). Therefore there exists \( \varepsilon > 0 \) such that \( B(M, \varepsilon) = B(M_1, \varepsilon) \cup B(M_2, \varepsilon) \) with \( B(M_1, \varepsilon) \cap B(M_2, \varepsilon) = \Phi \). By Theorem 3 there exists \( \delta \in \Delta \) such that \( A_\delta \subset B(M, \varepsilon) \). Put \( C_k = A_\delta \cap B(M_k, \varepsilon) \), \( k = 1, 2 \). Obviously \( A_\delta = C_1 \cup C_2 \) and \( C_1 \cap C_2 = \Phi \); but this is impossible since \( A_\delta \) is connected.

If, in Theorem 4, the filterbasis is a nonincreasing sequence of nonempty, closed and connected sets, we obtain a result of Painlevé-Kuratowski [3].

**Remark.** Recall that a set \( A \) of a metric space \( (X, d) \) is said to be \( \varepsilon \)-chained if for all \( x, y \in A \) there exists a finite subset \( \{x_1, \ldots, x_n\} \) such that \( x = x_1 \), \( y = x_n \) and \( d(x_i, x_{i+1}) < \varepsilon \), \( i = 1, \ldots, n-1 \). We define \( \eta(A) = \inf \{\varepsilon > 0 : A \text{ is } \varepsilon\text{-chained}\} \). Then Theorem 4 can be extended replacing the assumption «\( A \) is connected for any \( \delta \in \Delta \)» by «\( \inf \{\eta(A_\delta) : \delta \in \Delta\} = 0 \).»

**REFERENCES**


