A CHARACTERIZATION OF COMPACT FILTERS
IN COMPLETE UNIFORM SPACES (*)

by GIANNI FACINI (in Trieste) (**) 

SOMMARIO. - I risultati sulla compattezza di un filtro stabiliti da Furi e Martelli per gli spazi metrici vengono qui generalizzati, mediante estensione della nozione di numero di Kuratowski, per gli spazi uniformi. Vengono dedotte alcune conseguenze.

SUMMARY. - The results on compactness of filters obtained by Furi and Martelli for metric spaces are here generalized, by means of a suitable definition of the Kuratowski number, to uniform spaces. Some consequences are drawn.

The purpose of this paper is to extend to complete uniform spaces the results obtained by M. Furi and M. Martelli for complete metric spaces [1]. Throughout this paper unless we, expressly affirm the contrary, $E$ denotes a uniform space, $\mathcal{U}$ the uniformity of $E$, $V$ a member of $\mathcal{U}$, $\mathcal{F}$ a filter in $E$, $\mathcal{U}$ an ultrafilter in $E$ finer than $\mathcal{F}$. We say that a set $\mathcal{C}$ of subsets of $E$ is $V$-small if every member of $\mathcal{C}$ is $V$-small.

$\mathcal{F}$ is a quasi-Cauchy filter if for every $V \in \mathcal{U}$ there exists $E \in \mathcal{F}$ and a finite $V$-small cover of $E$.

It is easily seen that $\mathcal{F}$ is a quasi-Cauchy filter if every $\mathcal{U}$ finer than $\mathcal{F}$ is a Cauchy filter.

The necessity of the condition is evident if we remember that an ultrafilter, in a set $X$, containing as an element the union of a finite family $\mathcal{C}$ of subsets of $X$ contains as an element one member of $\mathcal{C}$ at least.

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(**) Indirizzo dell’Autore: Istituto di Matematica dell’Università — Piazzale Europa, 1 — 34100 Trieste.
Let us prove the sufficiency of the condition. Suppose that there exists \( V \in \mathcal{U} \) such that no \( F \in \mathcal{F} \) admits finite \( V \)-small covers.

Let \( \mathcal{P}_v \) be the set of all subsets of \( E \) with finite \( V \)-small covers and put \( \mathcal{F}' = \{ F - G : F \in \mathcal{F}, G \in \mathcal{P}_v \} \). It is easily seen that \( \mathcal{F}' \) is a filter. We only prove that if \((F_1 - G_1) \in \mathcal{F}' \) and \((F_2 - G_2) \in \mathcal{F}' \) then \((F_1 - G_1) \cap (F_2 - G_2) \in \mathcal{F}' \). Indeed \((F_1 - G_1) \cap (F_2 - G_2) \supset (F_1 \cap F_2) - (G_1 \cup G_2) \). But \((F_1 \cap F_2) \in \mathcal{F} \) and \((G_1 \cup G_2) \in \mathcal{P}_v \). Let \( \mathcal{U} \) be an ultrafilter finer than \( \mathcal{F}' \) (and hence finer than \( \mathcal{F} \)). It is evident that \( \mathcal{U} \) has no \( V \)-small element and hence \( \mathcal{U} \) is not a Cauchy filter. Thus also the sufficiency of the condition is proved. Since a convergent filter in \( E \) is a Cauchy filter, we have that a necessary condition for a filter \( \mathcal{F} \) in \( E \) to be compact is that \( \mathcal{F} \) be a quasi-Cauchy filter. The condition is also sufficient if \( E \) is complete.

Thus we are now able to generalize to uniform spaces the Theorem I of Furi and Martelli concerning metric spaces.

**THEOREM 1.** — *Let \( \mathcal{F} \) be a filter in a complete uniform space \( E \). Then \( \mathcal{F} \) is compact if and only if \( \mathcal{F} \) is a quasi-Cauchy filter.*

Though it is easy to see that in a metric space the conditions for a filter \( \mathcal{F} \) to be a quasi-Cauchy filter and to satisfy the equality \( a(\mathcal{F}) = 0 \) (\(^1\)) coincide, we can draw the formal expression of our Theorem I nearer to that of Furi and Martelli by generalizing to uniform spaces the notion of Kuratowski number of a subset of a metric space.

To this purpose we first generalize to uniform spaces the notion of diameter of a set.

Let \( \mathcal{U} \) be the uniformity of the uniform space \( E \). \( \mathcal{U} \) is a filter and hence a lattice and may be incomplete. We can embed it in a complete lattice as follows. In the set \( \mathcal{P}(\mathcal{U}) \) of all subsets of \( \mathcal{U} \) we define a transitive binary relation assuming that if \( G_1 \) and \( G_2 \) are members of \( \mathcal{P}(\mathcal{U}) \) then \( G_1 \leq G_2 \) if for every \( V \in G_2 \) there is a \( V' \in G_1 \) such that \( V' \subseteq V \). Since the relation \( R \) defined by \( G_1 \leq G_2 \) is transitive, reflexive and symmetric, we can consider the quotient \( \mathcal{U}^* = \mathcal{P}(\mathcal{U})/R \).

It can be easily proved that \( \mathcal{U}^* \) in regard to the order induced by the transitive relation defined in \( \mathcal{U} \) is a complete lattice.

\(^1\) For the meaning of \( a(\mathcal{F}) \) see [1].
We are now able to pose our definitions.

The diameter of a subset $F$ of $E$ is the image through the canonical map $\varphi$ of $\mathcal{P}(\mathcal{U})$ into $\mathcal{U}^*$ of the set of all members $V$ of $\mathcal{U}$ such that $F$ is $V$-small.

The Kuratowski number $a(F)$ of $F$ is the infimum of all the members $\mathcal{E}$ of $\mathcal{U}^*$ such that $F$ admits a finite covering of sets with diameter less than $\mathcal{E}$.

Let $\mathcal{F}$ be a filter on $E$. We define $a(\mathcal{F}) = \inf \{a(F) : F \in \mathcal{F}\}$. We denote by 0 the image $\varphi(\mathcal{P}(\mathcal{U}))$.

We can now reformulate Theorem 1 as follows:

**Theorem 1'.** Let $\mathcal{F}$ be a filter in a complete uniform space $E$. Then $\mathcal{F}$ is compact if and only if $a(\mathcal{F}) = 0$.

New let us generalize the other results of [1]. We give them in two forms, first making use of the notion of quasi-Cauchy filter, and then of the generalization to uniform spaces of notion of Kuratowski number of a set.

A filter $\mathcal{F}$ in $E$ is closed (connected) if there exists a base of $\mathcal{F}$ consisting of closed (connected) sets.

**Theorem 2.** In a complete uniform space $E$ the intersection $B$ of the members of a closed quasi-Cauchy filter $\mathcal{F}$ in $E$ is nonempty and compact.

**Theorem 2'.** In a complete uniform space $E$ the intersection $B$ of the members of a closed filter $\mathcal{F}$ such that $a(\mathcal{F}) = 0$ is nonempty and compact.

**Theorem 3.** Let $\mathcal{F}$ be a closed quasi-Cauchy filter in a complete uniform space $(E, \mathcal{U})$; then for every $V \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $F \subseteq V(B)$, where $B = \bigcap_{F \in \mathcal{F}} F$.

If $x$ and $y$ are two points of $E$, the distance $d(x, y)$ is the diameter of the set $\{x, y\}$. If $A$ is a subset of $E$ we put $d(x, A) = \inf \{d(x, y) : y \in A\}$. If $B$ is another subset of $E$ we put $\varrho(B, A) = \sup \{d(x, B) : x \in A\}$ and $D(A, B) = \sup \{\varrho(A, B), \varrho(B, A)\}$.

**Theorem 3'.** Let $\mathcal{F}$ be a closed filter such that $a(\mathcal{F}) = 0$ in a complete uniform space $E$. Then $\inf_{F \in \mathcal{F}} D(F, B) = 0$, where $B = \bigcap_{F \in \mathcal{F}} F$.

(2) The theorem obtained replacing the hypothesis « $\mathcal{F}$ is a filter » by « $\mathcal{F}$ is a filterbasis » would be only formally stronger. Indeed if $B$ is a basis the filter $\mathcal{F}$ the conditions « $B$ is compact » (« $a(B) = 0$ » and « $\mathcal{F}$ is compact » (« $a(\mathcal{F}) = 0$ ») coincide.
Theorem 4. Let $\mathcal{F}$ be a closed connected quasi-Cauchy filter in a complete uniform space $E$. Then $B = \bigcap_{F \in \mathcal{F}} F$ is a nonempty continuum.

Theorem 4'. Let $\mathcal{F}$ be a closed connected filter in a complete uniform space such that $\alpha(\mathcal{F}) = 0$. Then $B = \bigcap_{F \in \mathcal{F}} F$ is a nonempty continuum.

The Theorem 4 can be extended replacing the assumption «$\mathcal{F}$ is a connected closed quasi-Cauchy filter in $E$» by «$\mathcal{F}$ is a closed quasi-Cauchy filter in $E$ and for every $V \in \mathcal{U}$ there exists a $V$-chained member of $\mathcal{F}$. Similarly for Theorem 4'.

Proof of Theorem 2. Since $\mathcal{F}$ is closed, $B$ is closed and coincides with the set of all cluster points of $\mathcal{F}$. But as $\mathcal{F}$ is a quasi-Cauchy filter, Theorem 1 applies, and $B$ is nonempty. Since $B \in \mathcal{F}$ for every $F \in \mathcal{F}$, $B$ can be covered by a finite $V$-small family of subsets of $E$ for every $V \in \mathcal{U}$. Hence if $\mathcal{G}$ is an ultrafilter on $B$, $\mathcal{G}$ is also a Cauchy-ultrafilter and, since $B$ is complete, $\mathcal{G}$ converges. Thus $B$ is compact.

Proof of Theorem 3. Assume the contrary. Then there exists $V \in \mathcal{U}$ such that $F \notin V(B)$ for all $F \in \mathcal{F}$. Put $F' = F - V(B)$ and $\mathcal{F}' = \{F' : F \in \mathcal{F}\}$. It is easy to see that $\mathcal{F}'$ is finer than $\mathcal{F}$. Therefore $B' = \alpha \mathcal{F}'$ is nonempty by Theorem 2 and obviously $B' \cap B = \emptyset$. But this contradicts the fact that $\mathcal{F}'$ is finer than $\mathcal{F}$.

Proof of Theorem 4. Clearly $B$ is nonempty and compact by Theorem 2. Suppose $B$ disconnected. Then we can find a pair of nonempty, compact sets $B_1, B_2$ such that $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = B$. Therefore there exists $V \in \mathcal{U}$ such that $V(B) = V(B_1) \cup V(B_2)$ with $V(B_1) \cap V(B_2) = \emptyset$. By Theorem 3 there exists $F \in \mathcal{F}$ such that $F \in V(B)$. Put $C_k = F \cap V(B_k)$, $k = 1, 2, \ldots$. Obviously $F = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$: but this is impossible since $F$ is connected.

The Theorems 2', 3', 4' are clearly equivalent, in the order, to Theorems 2, 3, 4.

REFERENCES