QUASIREGULAR COLLINEATION GROUPS (*)

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SOMMARIO. - Gli autori studiano la struttura delle orbite dei gruppi di collineazioni quasiregolari dei piani grafici finiti.
Si ottengono dei limiti per l'ordine del gruppo in relazione al numero delle rette e dei punti uniti e viene discussa la struttura delle orbite per i gruppi di ordine $\geq n\sqrt{n}$ (indicando con $n$ l'ordine del piano).

SUMMARY. - In this paper, the authors study the orbit structure of quasiregular collineation groups of finite projective planes.
Bounds are obtained for the order of the group in terms of the number of fixed points and lines, and the orbit structure of such groups of order $\geq n\sqrt{n}$ (where $n$ is the order of the plane) is discussed.

§ 1. Introduction.

Quasiregular collineation groups of finite projective planes have been studied in [2], [3] and [5].

Any collineation group $\Gamma$ of a finite projective plane of order $n$ must act as a faithful permutation group on at least one orbit. If $\Gamma$ is quasiregular then, since $\Gamma$ then acts as a sharply transitive group on each faithful orbit, $|\Gamma| \leq n^2 + n + 1$. In this paper we find stronger bounds for $|\Gamma|$ in terms of the number of fixed points and lines.

In [2] the orbit structure of all quasiregular collineation groups $\Gamma$ with $|\Gamma| > \frac{1}{2}(n^2 + n + 1)$ was determined. Our results suggest that a similar result is possible for $|\Gamma| > n\sqrt{n}$.

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Finally, in Theorems 2 and 3 we show that if \( \Gamma \) is quasi-regular and has no faithful point orbit then \( \Gamma \) is a \( p \)-group and \( n \) is a power of the same prime \( p \). In this case the orbit structure of \( \Gamma \) is also determined.

\[\text{§ 2. Preliminary discussion.}\]

For the basic definitions and theory of finite projective planes see Dembowski [1] or Pickert [4].

A permutation group \( \Gamma \) on a set \( S \) is called quasi-regular if, for any \( \gamma \in \Gamma \) and any \( s \in S \), \( s\gamma = s \) implies \( t\gamma = t \) for all points \( t \) in the orbit \( s\Gamma \) of \( s \), i.e. \( \Gamma \) induces a regular permutation group on each of its orbits.

Throughout this paper \( \Gamma \) will denote a quasi-regular collineation group of a finite projective plane \( \pi \) of order \( n \). For any \( \gamma \in \Gamma \) we will let \( F(\gamma) \) denote the closed configuration of fixed points and lines of \( \gamma \), and we let \( F(\Gamma) = \bigcap_{\gamma \in \Gamma} F(\gamma) \). If \( P \) is any point of \( \pi \) we will denote the orbit of \( P \) by the equivalent Gothic letter \( \mathbb{P} \). For any subset \( \pi' \) of points and lines of \( \pi \) we let \( \Gamma_{\pi'} \) denote the subgroup of \( \Gamma \) fixing every element of \( \pi' \).

A point orbit is called trivial if it consists of a single point, linear if it is a set of collinear points and triangular if it is a set of 3 non-collinear points. If \( \mathbb{P} \) is a point orbit such that \( F(\Gamma_{\mathbb{P}}) \) is a proper subplane then \( \mathbb{P} \) is called a special orbit and \( F(\Gamma_{\mathbb{P}}) \) is a special subplane. We note that if \( \pi^* \) is any special subplane, then \( \Gamma \) leaves \( \pi^* \) invariant and induces on \( \pi^* \) a collineation group \( \Gamma^* \cong \Gamma/\Gamma_{\pi^*} \). If \( F(\Gamma_{\mathbb{P}}) = \pi \) i.e. \( \Gamma_{\mathbb{P}} = 1 \) then \( \mathbb{P} \) is a faithful orbit. (Dual definitions are made for line orbits).

A special subplane is called maximal (minimal) if it is not properly contained in (does not properly contain) another special subplane. If a collineation group \( \Gamma \) has any special orbits then it must have a maximal special subplane, \( \pi_1 \) say. If \( \Gamma_1 \) is the collineation group induced on \( \pi_1 \) then a similar statement is true for \( \Gamma_1 \) on \( \pi_1 \). In this way we can construct a chain of subplanes \( \pi = \pi_0 \supset \pi_1 \supset \ldots \supset \pi_r \) with induced collineation groups \( \Gamma_1, \ldots, \Gamma_r \) such that each \( \pi_i \) is a maximal special subplane of \( \Gamma_{i-1} \) on \( \pi_{i-1} \) and \( \pi_r \) is also a minimal special subplane of \( \Gamma_{r-1} \) on \( \pi_{r-1} \). We call this chain a special chain of length \( r \).

A number of properties relating to special chains and the orbit structure of \( \Gamma \) can be found in [5]. We draw attention to
Result 1. \( \Gamma \) is a collineation group of a finite projective plane. If \( F(\Gamma) \neq \Phi \) when any triangular orbit is either special or faithful.

Result 2. \( \Gamma \) is a collineation group of a finite projective plane \( \pi \). If \( \pi_1 \) is a maximal special subplane of \( \Gamma \) on \( \pi \) and if \( \Gamma_1 \) is the collineation group induced on \( \pi_1 \), then the number of faithful point (line) orbits of \( \Gamma \) in \( \pi \) is at least the number of faithful line (point) orbits of \( \Gamma_1 \) in \( \pi_1 \).

§ 3. Quasiregular collineation groups with fixed elements.

We begin this section by showing that if \( \Gamma \) is quasiregular then \( F(\Gamma) \) cannot be a single element.

Lemma 1. \( \Gamma \) is a quasiregular collineation group of a finite projective plane \( \pi \). If \( F(\Gamma) \neq \Phi \) then \( \Gamma \) fixes at least one point and at least one line.

Proof. Clearly if \( \Gamma \) fixes two or more points \( \Gamma \) must also fix at least one line. We shall assume that \( F(\Gamma) \) is a single point, \( T \) say, of \( \pi \) and show that this leads to a contradiction.

Let \( \pi^* \) be a minimal special subplane of \( \pi \) and let \( \Gamma^* \) be the collineation group induced on \( \pi^* \) by \( \Gamma \). (Possibly \( \pi = \pi^* \) and \( \Gamma = \Gamma^* \)). Then \( F(\Gamma^*) = F(\Gamma) \neq \Phi \).

Since \( \Gamma^* \) does not fix a line of \( \pi^* \), \( \Gamma^* \) has no linear point orbits and thus, by Result 1, all points of \( \pi^* \setminus T \) are in faithful orbits of \( \Gamma^* \). Hence if \( \alpha \in \Gamma^* \), \( \alpha \neq 1 \), the only point of \( \pi^* \) fixed by \( \alpha \) is \( T \). But \( \alpha \) must fix an equal number of points and lines of \( \pi^* \). Hence \( \alpha \) fixes exactly one line, \( l \) say, of \( \pi^* \). However, since \( \Gamma^* \) is quasiregular, \( \alpha \) must fix the entire \( \Gamma^* \)-orbit of \( l \). Thus \( l \in F(\Gamma^*) \).

This contradiction proves the Lemma.

Since any faithful orbit of a quasiregular collineation group \( \Gamma \) has length \( |\Gamma| \), we know that \( |\Gamma| \leq n^2 + n + 1 \). If \( |\Gamma| = n^2 + n + 1 \), then \( \Gamma \) is transitive on the points of \( \pi \) and \( F(\Gamma) = \Phi \). If \( F(\Gamma) = \Phi \) then, clearly, \( |\Gamma| < n^2 + n + 1 \). We now show how the size of \( F(\Gamma) \) affects the upper bound for \( |\Gamma| \).

Lemma 2. \( \Gamma \) is a quasiregular collineation group of a finite projective plane of order \( n \). \( F(\Gamma) \) contains exactly \( s + 1 \) (\( s \geq 2 \)) points \( P_0, P_1, ... P_s \) on a line \( l \) and at least one line \( m \parallel l \) such that \( P_0 \in m \). If \( F(\Gamma) \) contains \( t \) points of \( m \) distinct from \( P_0 \), then \( |\Gamma| \leq n - t \).
Remark. Before proving this lemma we note that the only possible values for $t$ are 0, 1 or $s$. In this last case $F (\Gamma')$ is then a subplane of order $s$.

Proof. Let $\mathcal{R}_i$ be a maximal orbit of points on $m$, and let $|\mathcal{R}_i| = t_i$. Then since $F (\Gamma_{\mathcal{W}_i})$ contains a quadrangle, $\mathcal{R}_i$ is either special or faithful. If $\mathcal{R}_i$ is faithful then $|\Gamma| = |\mathcal{R}_i| = t_i \leq n - t$.

Suppose $\mathcal{R}_i$ is not faithful, then $F (\Gamma_{\mathcal{W}_i})$ is a subplane $\pi_i$ of $\pi$, of order $m_i \geq t_i + t$.

Let $\mathcal{R}_2$ be a maximal orbit of points, under $\Gamma_{\mathcal{W}_i}$, on $m$, and let $|\mathcal{R}_2| = t_2 \leq t_i$. Then $\Gamma_{\mathcal{W}_i} \setminus \mathcal{R}_2$ fixes a subplane $\pi_2$, of order $m_2$, and $|\Gamma| = |\mathcal{R}_i| / |\mathcal{R}_2| / |\Gamma_{\mathcal{W}_i} \setminus \mathcal{R}_2|$. Hence $|\Gamma| \leq t_i^2 / |\Gamma_{\mathcal{W}_i} \setminus \mathcal{R}_2|$. Defining $\mathcal{R}_i$ and $t_i (i = 3, 4 \ldots r)$ in a similar manner, we obtain a chain of special subplanes $\pi_1 \subset \pi_2 \subset \ldots \subset \pi_r = \pi$.

We let $m_i$ denote the order of $\pi_i$. Then $|\Gamma| = |\mathcal{R}_i| / |\mathcal{R}_2| \ldots |\mathcal{R}_r| / |\Gamma_{\mathcal{W}_i} \setminus \mathcal{R}_r| = t_1 t_2 \ldots t_r - 1 \leq t_1^r$.

But, since $m_i \geq t + t_i$, we have $t_i \leq m_i - t$, and thus $|\Gamma| \leq (m_i - t)^r \leq m_i^r - t$. Each $\pi_i$ is a proper subplane of $\pi_{i+1}$ so that $m_i^2 \leq m_{i+1}$. Hence, by induction, $m_i^{2r-1} \leq n$. Thus $m_i^r \leq n^{r/21-r} \leq n$, and so $|\Gamma| \leq n - t$.

Corollary. If $\pi'$ is any special subplane, of order $m$, then $|\Gamma \pi'| \leq n - m$.

Lemma 3. $\Gamma$ is a quasiregular collineation group of a finite projective plane, of order $n$. If $F (\Gamma)$ consists of $k + 1$ collinear points ($k \geq 1$) and the line joining them, then $|\Gamma| \leq \max \{nk, n^{\sqrt{n}}\}$.

Proof. Let the fixed line of $\Gamma$ be $l$. If any orbit of points on $l$ is faithful then $|\Gamma| \leq n - k$. So we shall assume that every point of $l$ is in a non-faithful orbit.

Suppose $\Gamma$ has special orbits. (Note that this implies $k \leq \sqrt{n}$) Then some line $l' \parallel l$ through one of the fixed points must lie in a special subplane $\pi'$ of order $m$. Clearly $|\Gamma' \setminus l' \parallel l| \leq m \leq \sqrt{n}$. By Lemma 2, $\Gamma' \leq n$ and thus $|\Gamma| \leq n^{\sqrt{n}}$.

Now suppose that there are no special orbits. Then every point of $\pi \setminus l$ is in a faithful orbit. If $\Gamma$ contains an element $\alpha$ fixing exactly $k + 1$ points of $l$, then $\alpha$ must also fix $k + 1$
concurrent lines. If \( l' \) is one of these lines then, by the quasiregularity of \( \Gamma' \), \( \alpha \) fixes every element of \( l' \) and, hence \( |l'| \leq k \). But, as before, by Lemma 2 \( |\Gamma_v| \leq n \) and thus \( |\Gamma| \leq n^k \).

Suppose every element of \( \Gamma \) fixes more than \( k + 1 \) points of \( l \). Let \( A \in I \), be such that \( |A| = 1 \) is minimal for all the non-fixed points of \( l \), and put \( |A| = t \). Then if \( \Gamma^* \) is the permutation group induced by \( \Gamma \) on the \( n - k \) non-fixed points of \( l \), each element of \( \Gamma^* \) fixes at least \( t \) points. Furthermore the number of orbits of this permutation group is at most \( \frac{n - k}{t} \). But if \( \chi(\alpha) \) denotes the number of fixed elements of \( \alpha \) then

\[
\sum_{\alpha \in \Gamma^*} \chi(\alpha) = |\Gamma^*| \cdot \text{number of orbits of } \Gamma^* \text{ (see [7], Exercise 3.10)}.
\]

Thus, again considering \( \Gamma \) as a permutation group on the \( n - k \) non-fixed points of \( l \), we have

\[
t \left( |\Gamma^*| - 1 \right) + n - k \leq \sum_{\alpha \in \Gamma^*} \chi(\alpha) = |\Gamma^*| \cdot \text{number of orbits of } \Gamma^* \leq \frac{n - k}{t} |\Gamma^*|.
\]

Rearranging gives \( |\Gamma^*| (t^2 - (n - k)) \leq t^2 - t(n - k) \). But, clearly, \( t \leq n - k \). Thus \( t^2 - (n - k) \leq 0 \) or \( t \leq \sqrt{n - k} \).

If \( l' \neq l \) is any line through \( A \) then \( l' \) is faithful and \( |l'| \leq \leq n^k t \). Thus \( |\Gamma| = |l'| \leq n^k \sqrt{n - k} < n^k \).

This exhausts all possibilities and proves Lemma 3.

In the proof of Lemma 3 it is shown that if \( |\Gamma| > n\sqrt{n} \) and \( F(\Gamma) \) satisfied the conditions of the lemma, then \( \Gamma \) has no special orbits. We now show that this is true for all \( \Gamma \) with \( |\Gamma| > n\sqrt{n} \) and \( F(\Gamma) = \Phi \).

**Theorem 1.** \( \Gamma \) is a quasiregular collineation group of a finite projective plane \( \pi \) of order \( n \). If \( |\Gamma| > n\sqrt{n} \) and \( F(\Gamma) = \Phi \) then \( \Gamma \) has no special orbits.

**Proof.** Let \( \pi^* \) be a maximal special subplane of maximal order \( m \), (i.e. if \( \pi' \) is any other maximal special subplane of order \( m' \), then \( m' \leq m \)), and let \( \Gamma^* \) be the collineation group induced
on $\pi^*$. Since $F(I^*) \cong F$, $|I^*| \leq m^2$ (see [2] Theorem 4) and, since $\pi^*$ is maximal, $I_{\pi^*}$ is semi-regular on the points of $\pi \setminus \pi^*$. Thus $|I_{\pi^*}| = \frac{n - m}{t}$, where $t \geq 1$.

By Lemma 1, $F(I)$ contains a point $P$. If $P$ is fixed linewise in $\pi^*$ then the dual of Lemma 3 proves the theorem. Let $l$ be any non-fixed line of $\pi^*$ such that $P \in l$. Then, clearly, $|I| \leq m + 1$. Let $T$ be any point of $l$ such that $T \notin \pi^*$, then $T$ contains a quadrangle and is either faithful or special.

If $T$ is faithful then $|I| = |T| \leq |I| |n - m| \leq (m + 1)(n - m)$. Thus $n \sqrt{n} \leq |I| \leq (m + 1)(n - m)$. But either $n = m^2$ or $n \geq m^2 + m + 2$ (see [6] Theorem 7). In either case simple computation shows that $(m + 1)(n - m) < n \sqrt{n}$. Hence $T$ must be special.

In this case $F(I_T)$ is a subplane $\pi'$ of order $m'$ with $m' \leq m$. But, since $T$ contains at least $\frac{n - m}{t}$ collinear points outside $\pi^*$, $m' \geq \frac{n - m}{t}$. Hence, since $|I| = |I_{\pi^*}| |I^*|$ we have

$$n \sqrt{n} \leq |I| = |I_{\pi^*}| |I^*| = \frac{n - m}{t} |I^*| \leq m \cdot m^2 = m^3 \leq n \sqrt{n}.$$  

This contradiction shows that $I$ cannot have any special orbits.

It was shown in [5] that if $I$ has no faithful point orbit then $F(I)$ must consist of a set of collinear points and the line joining them, or the dual configuration. We now improve this result.

**Theorem 2.** $I$ is a quasiregular collineation group of a finite projective plane of order $n$. If there is a special chain of length $\geq 2$ then $I$ has faithful point and line orbits.

**Proof.** Let $\pi \supset \pi_1 \supset ... \supset \pi_r$ be a chain of special subplane such that each $\pi_i$ is maximal in $\pi_{i-1}$ and $\pi_r$ is minimal, and let $I_i$ be the collineation group induced on $\pi_i$. In view of Result 2, it is sufficient to prove that $I_{i-2}$ has faithful point and line orbits on $\pi_{i-2}$.

To simplify the notation we shall write $I$ for $I_{i-2}$ and $\pi$ for $\pi_{i-2}$ (i.e. we shall assume $r = 2$).

Suppose $I$ has no faithful line orbit. Then, by Result 2, $I_i$ has no faithful point orbit. Thus, by [5] (Lemma 3.6), we have

(a) $F(I)$ is a line $l$ of $\pi$ and the $m + 1$ points of $l$ in $\pi_2$, where $m$ is the order of $\pi_2$.  


(b) the order of \( \pi_4 \) is \( m^2 \).
(c) \( \Gamma_2 \) is a group of elations with \( |\Gamma_2| > m \) and \( \Gamma_4 \) is a \( p \)-group where \( p | m \).

Hence by a result of Dembowskii (see [1] p. 188) \( m \) is a prime power. Put \( m = p^s \) then, by (b) the order of \( \pi_4 \) is \( p^{2s} \).

Since \( \pi_2 \) is maximal in \( \pi_1, |\Gamma_{1,\pi_2}| \mid p^{2s} - p^s \) and thus, since by (c) \( \Gamma_1 \) is a \( p \)-group, \( |\Gamma_{1,\pi_2}| \mid p^s \). But \( |\Gamma_1| = |\Gamma_2| \cdot |\Gamma_{1,\pi_2}| \) and thus \( |\Gamma_1| \leq p^{3s} \).

Since every line orbit is non-faithful, each point of \( l \) must also be in a non-faithful orbit and thus in a special subplane. For any non-fixed point \( A \in l \) there are \( n \) non-fixed lines through \( A \). Since any maximal special subplane has order \( \leq p^{2s} \) there are at least \( \frac{n}{p^{2s}} \) special subplanes containing \( A \). Every special subplane must contain the \( p^s + 1 \) fixed points of \( \Gamma \). There are \( n - p^s \) non-fixed points of \( l \) and no more than \( p^{2s} - p^s \) are in the same maximal special subplane; thus, if \( t \) is the number of maximal special subplanes, we have

\[
t \geq \frac{n}{p^{2s}} \cdot \frac{n-p^s}{p^{2s} - p^s} \geq \frac{p^{4s} (p^{4s} - p^s)}{p^{2s} (p^{2s} - p^s)} = p^{2s} (p^{2s} + p^s + 1).
\]

Let \( x = \min |\Gamma_{\pi'}| \) for any maximal special subplane \( \pi' \) then, since the collineation group induced on \( \pi' \) has order \( \leq p^{3s} \) we have \( |\Gamma| \leq p^{3s} x \).

But no non-identity element of \( \Gamma \) can fix 2 distinct maximal special subplanes pointwise. Thus there are at least \( t (x - 1) \) distinct elements in \( \Gamma \) which fix a maximal special subplane pointwise. Hence \( xp^{3s} \geq |\Gamma| > p^{2s} (p^{2s} + p^s + 1) (x - 1) \). Simple computation shows that this is impossible, and thus \( \Gamma \) must have faithful point and line orbits.

As an immediate corollary to Theorem 2, and Lemma 3.6 in [5], we have

**Theorem 3.** \( \Gamma \) is a quasiregular collineation group of a finite projective plane \( \pi \) of order \( n \). If \( \Gamma \) has no faithful point orbits, then, either (i) \( \Gamma \) is a group of elations with a common centre, such that \( |\Gamma| > n \), or (ii) \( n = m^2 \), and \( \Gamma \) fixes a line \( l \) of a Baer subplane \( \pi' \) of \( \pi \), pointwise and induces on \( \pi' \) a group of elations with axis \( l \) and order \( > m \), and every element of \( \Gamma \) is either an elation of \( \pi \) or fixes a Baer subplane pointwise. In either case (i) or (ii), \( \Gamma \) is a \( p \)-group and \( n \) is a power of \( p \).
§ 4. Quasiregular collineation groups with no fixed elements.

We now consider the situation where $\Gamma$ has no fixed points or lines. If $\Gamma$ has no special orbits then the following lemma is trivial.

**Lemma 4.** $\Gamma$ is a quasiregular collineation group of a finite projective plane $\pi$ of order $n$. If $F(\Gamma) = \Phi$ and $\Gamma$ has no special orbits then either

(i) all orbits are faithful and $|\Gamma| \geq n^2 + n + 1$

(ii) there is a unique triangular orbit and $|\Gamma| \geq 3(n - 1)$

(iii) there is more than one triangular orbit and $|\Gamma| = 9$.

If, however, $\Gamma$ has special orbits then the situation becomes more complex. But for $|\Gamma|$ sufficiently large it is still possible to give information about the orbit structure of $\Gamma$. Once again, as the following lemma shows, the bound $|\Gamma| \geq n\sqrt{n}$ occurs in a natural way.

**Lemma 5.** $\Gamma$ is a quasiregular collineation group of a finite projective plane $\pi$ of order $n$. If $F(\Gamma) = \Phi$ and $|\Gamma| \geq n\sqrt{n}$ then, for any maximal special subplane $\pi^*$ of maximal order, all points of $\pi \setminus \pi^*$ incident with lines of $\pi^*$ are in faithful orbits of $\Gamma$.

**Proof.** The proof is very similar to the proof of Theorem 1. Let the order of $\pi^*$ be $m$ then, since $\pi^*$ is maximal, $|\Gamma_{\pi^*}| = \frac{n - m}{t}$ for some integer $t$. Further, if $\Gamma^*$ is the collineation group induced on $\pi^*$ then, (by [2], Theorem 4) either $|\Gamma^*| = m^2 + m + 1$ or $|\Gamma^*| \leq m^2 - \sqrt{m}$.

If $|\Gamma^*| = m^2 + m + 1$ then $\Gamma^*$ is transitive on the lines of $\pi^*$. If $A$ is any point of $\pi \setminus \pi^*$ on a line $l$ of $\pi^*$ then $\Gamma_A$ fixes $l$ and, thus, is the identity on $\pi^*$. Hence since $\Gamma_A$ also fixes $A \notin \pi^*$, $\Gamma_A = 1$ and $\pi^*$ is faithful.

Suppose $|\Gamma^*| \leq m^2 - \sqrt{m}$ and that the orbit $\pi^*$ is a special. Then $F(\Gamma_A)$ is a subplane $\pi'$ of order $m' \leq m$. But $l$ contains at least $\frac{n - m}{t}$ points of $\pi^*$, and $\Gamma_A$ fixes $l$ so that $\pi'$ contains at least one point of $l$ which is in $\pi^*$. Thus $\frac{n - m}{t} \leq m' \leq m$.

Hence $|\Gamma| = |\Gamma^*| |\Gamma_{\pi^*}| \leq (m^2 - \sqrt{m}) \frac{n - m}{t} \leq (m^2 - \sqrt{m}) m < \frac{n - m}{t} \leq n\sqrt{n}$. This proves the lemma.
If \( \pi^* \) is any special subplane of order \( m \) with induced collineation group \( \Gamma^* \) then, since \( |\Gamma| = |\Gamma^*| \cdot |\Gamma_{\pi^*}| \), any numerical restriction of \( |\Gamma| \) automatically places restrictions on the possible values for \( m \) and \( |\Gamma^*| \). In [2] the orbit structure for all \( \Gamma \) with \( |\Gamma| > \frac{1}{2}(n^2 + n + 1) \) was determined. It often happens that some weaker restriction on \( |\Gamma| \) will ensure \( |\Gamma^*| > \frac{1}{2}(m^2 + m + 1) \) and enable us to use the results of [2] on \( \pi^* \) to determine the orbit structure of \( \Gamma \). Again the bound \( |\Gamma| > n \sqrt[n]{n} \) occurs naturally.

**Theorem 4.** \( \Gamma \) is a quasiregular collineation group of a finite projective plane \( \pi \) of order \( n \). \( \pi^* \) is a maximal special subplane of order \( m \) and \( \Gamma^* \) is the collineation group induced on \( \pi^* \). If \( F(\Gamma) = \Phi, |\Gamma| \geq n \sqrt[n]{n} \) and \( \Gamma^* \) has a non-faithful orbit, then \( m \) is a square and \( \pi^* \) has a Baer subplane \( \pi' \) on which \( \Gamma^* \) acts transitively.

**Proof.** Let \( I \) be any line orbit of \( \Gamma^* \). Then, by Lemma 5, if \( A \in \pi \setminus \pi^* \) is incident with a line of \( I \) the orbit \( \mathfrak{A} \) is faithful. Hence \( |\Gamma| = |\mathfrak{A}| \leq |I| (n - m) \).

Clearly \( I \) cannot be triangular since then \( n \sqrt[n]{n} \leq |\Gamma| \leq 3(n - m) \) which is impossible. Suppose \( I \) is a special orbit, then there is a special subplane \( \pi' \) of \( \pi^* \) such that \( \pi' = F(\Gamma_{I^*}) \). If \( \pi' \) has order \( m_1 \) then \( m_1 \leq \sqrt{m} \).

From [2] either \( |I| = m_1^2 + m_1 + 1 \) or \( |I| \leq m_2 - n \sqrt[m]{m} \). But if \( |I| \leq m_1^2 - n \sqrt[m]{m} \) then \( n \sqrt[n]{n} \leq |\Gamma| \leq (m_1^2 - n \sqrt[m]{m}) (n - m) < nm \leq n \sqrt[n]{n} \). This contradiction shows \( |I| = m_1^2 + m_1 + 1 \), and hence, \( n \sqrt[n]{n} \leq |\Gamma| \leq (m_1^2 + m_1 + 1) (n - m) \). If \( m_1^2 = m \), then \( m_2^2 + m_1 + 1 < m \) (see [6] Theorem 7) and then \( |\Gamma| < mn \leq n \sqrt[n]{n} \). Thus \( m_1^2 = m \) and \( \pi' \) is a Baer subplane of \( \pi^* \).

If a quasiregular collineation group does not fix any line then it cannot contain any perspectivities. Thus any involutions must fix Baer subplanes. This simple observation gives

**Lemma 6.** \( \Gamma \) is a quasiregular collineation group of a finite projective plane \( \pi \) of order \( n \). If \( F(\Gamma) = \Phi \) and \( \pi' \) is a unique maximal special subplane, (i.e. every point of \( \pi \setminus \pi' \) is in a faithful orbit), which is not a Baer subplane, then the number of faithful point orbits is even.
Proof. Let $m$ be the order of $\pi'$. Since $m^2 \equiv n \mid |\Gamma| \equiv 1 \pmod{2}$, But the number of points of $\pi \setminus \pi'$ is $n(n + 1) - m(m + 1)$, which is even. Hence the Lemma is proved.

Finally we consider the situation where $\pi$ has a maximal special subplane, not necessarily unique, which is not a Baer subplane.

Theorem 5. $\Gamma$ is a quasiregular collineation group of a finite projective plane $\pi$ of order $n$. $F(\Gamma') = \Phi$, $|\Gamma| \geq n^2 \sqrt{n}$, $\pi^*$ is a maximal special subplane of maximal order $m = \sqrt{n}$ and any point of $\pi$ not incident with a line of $\pi^*$ is in a non-faithful orbit. If $X$ is any point not incident with a line of $\pi^*$ then $F(\Gamma_X)$ is a subplane of order $m_1$ where either $m = m_1 = \sqrt{n} - 1$ or $m = [\sqrt{n}]^{(1)}$ and $m_1 = n - (m^2 + m + 1)$.

Proof. By Lemma 5 every point of $\pi \setminus \pi^*$ which is incident with a line of $\pi^*$ is in a faithful orbit.

Let $X$ be any orbit of points not incident with a line of $\pi^*$. If $X$ is triangular then $|\Gamma_{\pi^*}| = 3$ and $|\Gamma| \leq 3(m^2 + m + 1)$. But since $n = m^2, m^2 + m + 1 < n$ and thus, since $|\Gamma| \geq n \sqrt{n}, X$ cannot be triangular.

Thus we may assume that all orbits of points not incident with a line of $\pi^*$ are in special orbits. Choose $X$ so that $F(\Gamma_X) = \pi_i$ is a subplane of order $m_i$ such that for every other orbit $Y$ of points not incident with a line of $\pi^*$, the order of $F(\Gamma_Y) \leq m_i$.

Let $l$ be any line of $\pi_i$. Clearly $l$ can contain no point of $\pi^*$, since any line through a point of $\pi^*$ is either a line of $\pi^*$, or else is in a faithful orbit.

The argument used to prove lemma 5 shows that every point of $l - \pi_i$ is in a faithful orbit. Thus every point of $l - \pi_i$ is incident with a line of $\pi^*$. Simple counting of the points of $l$ now gives $n = m^2 + m + m_i + 1$. If $m = m_i$, then $m = \sqrt{n} - 1$. If $m_i < m$, then $n < (m + 1)^2$, and so, since $m < \sqrt{n}$, we have $m = [\sqrt{n}]$.

Remark. The case $m = m_i$ is the situation which occurs in [2]. It is doubtful whether this situation can actually occur, but the result in [2] has been improved in an unpublished result by Hering (see [1] p. 183).

(1) $[a]$ is the greatest integer not exceeding $a$.  

3
With the results proved in this paper it is now possible to determine the orbit structure of quasiregular collineation groups with only a few faithful point orbits, (in [5], groups with exactly one faithful orbit were studied), and also of quasiregular collineation groups of order \( > n^\frac{1}{2}\) (in [2], groups with order \( > \frac{1}{2}(n^2+n+1)\) were determined). Naturally more possibilities will arise in each case, but there do not appear to be any great difficulties.

REFERENCES