Rank two globally generated vector bundles with $c_1 \leq 5$

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Abstract. We classify globally generated rank two vector bundles on $\mathbb{P}^n$, $n \geq 3$, with $c_1 \leq 5$. The classification is complete but for one case ($n = 3$, $c_1 = 5$, $c_2 = 12$).

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1. Introduction.

Vector bundles generated by global sections are basic objects in projective algebraic geometry. Globally generated line bundles correspond to morphisms to a projective space, more generally higher rank bundles correspond to morphism to (higher) Grassmann varieties. For this last point of view (that won’t be touched in this paper) see [10, 12, 13]. Also globally generated vector bundles appear in a variety of problems ([7] just to make a single, recent example).

In this paper we classify globally generated rank two vector bundles on $\mathbb{P}^n$ (projective space over $k$, $\overline{k} = k$, $ch(k) = 0$), $n \geq 3$, with $c_1 \leq 5$. The result is:

**Theorem 1.1.** Let $E$ be a rank two vector bundle on $\mathbb{P}^n$, $n \geq 3$, generated by global sections with Chern classes $c_1, c_2$, $c_1 \leq 5$.

1. If $n \geq 4$, then $E$ is the direct sum of two line bundles

2. If $n = 3$ and $E$ is indecomposable, then

   
   $$(c_1, c_2) \in S = \{((2, 2), (4, 5), (4, 6), (4, 7), (4, 8), (5, 8), (5, 10), (5, 12))\}.$$

   If $E$ exists there is an exact sequence:

   
   $$0 \rightarrow \mathcal{O} \rightarrow E \rightarrow \mathcal{I}_C(c_1) \rightarrow 0 \quad (\ast)$$

   where $C \subset \mathbb{P}^3$ is a smooth curve of degree $c_2$ with $\omega_C(4 - c_1) \simeq \mathcal{O}_C$. The curve $C$ is irreducible, except maybe if $(c_1, c_2) = (4, 8)$: in this case $C$ can be either irreducible or the disjoint union of two smooth conics.
3. For every \((c_1, c_2) \in S, (c_1, c_2) \neq (5, 12)\), there exists a rank two vector bundle on \(\mathbb{P}^3\) with Chern classes \((c_1, c_2)\) which is globally generated (and with an exact sequence as in 2.).

The classification is complete, but for one case: we are unable to say if there exist or not globally generated rank two vector bundles with Chern classes \(c_1 = 5, c_2 = 12\) on \(\mathbb{P}^3\).

2. Rank two vector bundles on \(\mathbb{P}^3\).

2.1. General facts.

For completeness let’s recall the following well known results:

**Lemma 2.1.** Let \(E\) be a rank \(r\) vector bundle on \(\mathbb{P}^n, n \geq 3\). Assume \(E\) is generated by global sections.

1. If \(c_1(E) = 0\), then \(E \simeq r.\mathcal{O}\)

2. If \(c_1(E) = 1\), then \(E \simeq \mathcal{O}(1) \oplus (r - 1).\mathcal{O}\) or \(E \simeq T(-1) \oplus (r - n).\mathcal{O}\).

**Proof.** If \(L \subset \mathbb{P}^n\) is a line then \(E|_L \simeq \bigoplus_{i=1}^r \mathcal{O}_L(a_i)\) by a well known theorem and \(a_i \geq 0, \forall i\) since \(E\) is globally generated. It turns out that in both cases: \(E|_L \simeq \mathcal{O}_L(c_1) \oplus (r - 1).\mathcal{O}_L\) for every line \(L\), i.e. \(E\) is uniform. Then 1. follows from a result of Van de Ven ([14]), while 2. follows from IV. Prop. 2.2 of [4].

**Lemma 2.2.** Let \(E\) be a rank two vector bundle on \(\mathbb{P}^n, n \geq 3\). If \(E\) has a nowhere vanishing section then \(E\) splits. If \(E\) is generated by global sections and doesn’t split then \(h^0(E) \geq 3\) and a general section of \(E\) vanishes along a smooth curve, \(C\), of degree \(c_2(E)\) such that \(\omega_C(4 - c_1) \simeq \mathcal{O}_C\). Moreover \(\mathcal{I}_C(c_1)\) is generated by global sections.

**Lemma 2.3.** Let \(E\) be a non split rank two vector bundle on \(\mathbb{P}^3\) with \(c_1 = 2\). If \(E\) is generated by global sections then \(E\) is a null-correlation bundle.

**Proof.** We have an exact sequence: \(0 \rightarrow \mathcal{O} \rightarrow E \rightarrow \mathcal{I}_C(2) \rightarrow 0\), where \(C\) is a smooth curve with \(\omega_C(2) \simeq \mathcal{O}_C\). It follows that \(C\) is a disjoint union of lines. Since \(h^0(\mathcal{I}_C(2)) \geq 2, d(C) \leq 2\). Finally \(d(C) = 2\) because \(E\) doesn’t split.

This settles the classification of rank two globally generated vector bundles with \(c_1(E) \leq 2\) on \(\mathbb{P}^3\).
2.2. Globally generated rank two vector bundles with $c_1 = 3$.

The following result has been proved in [10] (with a different and longer proof).

**Proposition 2.4.** Let $E$ be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 3$ then $E$ splits.

**Proof.** Assume a general section vanishes in codimension two, then it vanishes along a smooth curve $C$ such that $\mathcal{O}_C \simeq \mathcal{O}_C(-1)$. Moreover $\mathcal{I}_C(3)$ is generated by global sections. We have $C = \bigcup_{i=1}^r C_i$ (disjoint union) where each $C_i$ is smooth irreducible with $\omega_{C_i} \simeq \mathcal{O}_{C_i}(-1)$. It follows that each $C_i$ is a smooth conic. If $r \geq 2$ let $L = \langle C_1 \rangle \cap \langle C_2 \rangle$ ($\langle C_i \rangle$ is the plane spanned by $C_i$). Every cubic containing $C$ contains $L$ (because it contains the four points $C_1 \cap L$, $C_2 \cap L$). This contradicts the fact that $\mathcal{I}_C(3)$ is globally generated. Hence $r = 1$ and $E = \mathcal{O}(1) \oplus \mathcal{O}(2)$. □

2.3. Globally generated rank two vector bundles with $c_1 = 4$.

Let’s start with a general result:

**Lemma 2.5.** Let $E$ be a non split rank two vector bundle on $\mathbb{P}^3$ with Chern classes $c_1, c_2$. If $E$ is globally generated and if $c_1 \geq 4$ then:

$$c_2 \leq \frac{2c_1^3 - 4c_1^2 + 2}{3c_1 - 4}.$$

**Proof.** By our assumptions a general section of $E$ vanishes along a smooth curve, $C$, such that $\mathcal{I}_C(c_1)$ is generated by global sections. Let $U$ be the complete intersections of two general surfaces containing $C$. Then $U$ links $C$ to a smooth curve, $Y$. We have $Y \neq \emptyset$ since $E$ doesn’t split. The exact sequence of liaison: $0 \to \mathcal{I}_U(c_1) \to \mathcal{I}_C(c_1) \to \omega_Y(4 - c_1) \to 0$ shows that $\omega_Y(4 - c_1)$ is generated by global sections. Hence $\deg(\omega_Y(4 - c_1)) \geq 0$. We have $\deg(\omega_Y(4 - c_1)) = 2g' - 2 + d'(4 - c_1)$ ($g' = p_a(Y)$, $d' = \deg(Y)$). So $g' \geq \frac{d'(c_1 - 4) + 2}{2} \geq 0$ (because $c_1 \geq 4$). On the other hand, always by liaison, we have: $g' - g = \frac{1}{2}(d' - d)(2c_1 - 4)$ ($g = p_a(C)$, $d = \deg(C)$). Since $d' = c_1 - d$ and $g = \frac{4(c_1 - 4)}{2} + 1$ (because $\omega_C(4 - c_1) \simeq \mathcal{O}_C$), we get: $g' = 1 + \frac{d(c_1 - 4)}{2} + \frac{1}{2}(c_1^2 - 2d)(2c_1 - 4) \geq 0$ and the result follows. □
Proposition 2.6. Let $E$ be a rank two globally generated vector bundle on $\mathbb{P}^3$. If $c_1(E) = 4$ and if $E$ doesn’t split, then $5 \leq c_2 \leq 8$ and there is an exact sequence: $0 \to \mathcal{O} \to E \to \mathcal{I}_C(4) \to 0$, where $C$ is a smooth irreducible elliptic curve of degree $c_2$ or, if $c_2 = 8$, $C$ is the disjoint union of two smooth elliptic quartic curves.

Proof. A general section of $E$ vanishes along $C$ where $C$ is a smooth curve with $\omega_C = \mathcal{O}_C$ and where $\mathcal{I}_C(4)$ is generated by global sections. Let $C = C_1 \cup \ldots \cup C_r$ be the decomposition into irreducible components: the union is disjoint, each $C_i$ is a smooth elliptic curve hence has degree at least three.

By Lemma 2.5 $d = \deg(C) \leq 8$. If $d \leq 4$ then $C$ is irreducible and is a complete intersection which is impossible since $E$ doesn’t split. If $d = 5$, $C$ is smooth irreducible.

Claim: If $8 \geq d \geq 6$, $C$ cannot contain a plane cubic curve.

Assume $C = P \cup X$ where $P$ is a plane cubic and where $X$ is a smooth elliptic curve of degree $d - 3$. If $d = 6$, $X$ is also a plane cubic and every quartic containing $C$ contains the line $(P) \cap (X)$. If $\deg(X) \geq 4$ then every quartic, $F$, containing $C$ contains the plane $(P)$. Indeed $F|H$ vanishes on $P$ and on the $\deg(X) \geq 4$ points of $X \cap (P)$, but these points are not on a line so $F|H = 0$. In both cases we get a contradiction with the fact that $\mathcal{I}_C(4)$ is generated by global sections. The claim is proved.

It follows that, if $8 \geq d \geq 6$, then $C$ is irreducible except if $C = X \cup Y$ is the disjoint union of two elliptic quartic curves.

Now let’s show that all possibilities of Proposition 2.6 do actually occur. For this we have to show the existence of a smooth irreducible elliptic curve of degree $d$, $5 \leq d \leq 8$ with $\mathcal{I}_C(4)$ generated by global sections (and also that the disjoint union of two elliptic quartic curves is cut off by quartics).

Lemma 2.7. There exist rank two vector bundles with $c_1 = 4, c_2 = 5$ which are globally generated. More precisely any such bundle is of the form $\mathcal{N}(2)$, where $\mathcal{N}$ is a null-correlation bundle (a stable bundle with $c_1 = 0, c_2 = 1$).

Proof. The existence is clear (if $\mathcal{N}$ is a null-correlation bundle then it is well known that $\mathcal{N}(k)$ is globally generated if $k \geq 1$). Conversely if $E$ has $c_1 = 4, c_2 = 5$ and is globally generated, then $E$ has a section vanishing along a smooth, irreducible quintic elliptic curve (cf 2.6). Since $h^0(\mathcal{I}_C(2)) = 0$, $E$ is stable, hence $E = \mathcal{N}(2)$.

Lemma 2.8. There exist smooth, irreducible elliptic curves, $C$, of degree 6 with $\mathcal{I}_C(4)$ generated by global sections.
Proof. Let $X$ be the union of three skew lines. The curve $X$ lies on a smooth quadric surface, $Q$, and has $\mathcal{I}_X(3)$ globally generated (indeed the exact sequence $0 \to \mathcal{I}_Q \to \mathcal{I}_X \to \mathcal{I}_{X,Q} \to 0$ twisted by $\mathcal{O}(3)$ reads like: $0 \to \mathcal{O}(1) \to \mathcal{I}_C(3) \to \mathcal{O}_Q(3,0) \to 0$). The complete intersection, $U$, of two general cubes containing $X$ links $X$ to a smooth curve, $C$, of degree 6 and arithmetic genus 1. Since, by liaison, $h^1(\mathcal{I}_C) = h^1(\mathcal{I}_X(-2)) = 0$, $C$ is irreducible. The exact sequence of liaison: $0 \to \mathcal{I}_U(4) \to \mathcal{I}_C(4) \to \omega_X(2) \to 0$ shows that $\mathcal{I}_C(4)$ is globally generated.

In order to prove the existence of smooth, irreducible elliptic curves, $C$, of degree $d = 7, 8$, with $\mathcal{I}_C(4)$ globally generated, we have to recall some results due to Mori ([11]).

According to [11] Remark 4, Prop. 6, there exists a smooth quartic surface $S \subset \mathbb{P}^3$ such that $\text{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}X$ where $X$ is a smooth elliptic curve of degree $d$ ($7 \leq d \leq 8$). The intersection pairing is given by: $H^2 = 4, X^2 = 0, H.X = d$. Such a surface doesn’t contain any smooth rational curve ([11, p. 130]). In particular: (*) every integral curve, $Z$, on $S$ has degree $\geq 4$ with equality if and only if $Z$ is a planar quartic curve or an elliptic quartic curve.

**Lemma 2.9.** With notations as above, $h^0(\mathcal{I}_X(3)) = 0$.

**Proof.** A curve $Z \in |3H - X|$ has invariants $(d_Z, g_Z) = (5, -2)$ (if $d = 7$) or $(4, -5)$ (if $d = 8$), so $Z$ is not integral. It follows that $Z$ must contain an integral curve of degree $< 4$, but this is impossible.

**Lemma 2.10.** With notations as above $|4H - X|$ is base point free, hence there exist smooth, irreducible elliptic curves, $X$, of degree $d$, $7 \leq d \leq 8$, such that $\mathcal{I}_X(4)$ is globally generated.

**Proof.** Let’s first prove the following: Claim: Every curve in $|4H - X|$ is integral.

If $Y \in |4H - X|$ is not integral then $Y = Y_1 + Y_2$ where $Y_1$ is integral with deg($Y_1$) = 4 (observe that deg($Y$) = 9 or 8).

If $Y_1$ is planar then $Y_1 \sim H$, so $4H - X \sim H + Y_2$ and it follows that $3H \sim X + Y_2$, in contradiction with $h^0(\mathcal{I}_X(3)) = 0$ (cf 2.9).

So we may assume that $Y_1$ is a quartic elliptic curve, i.e. (i) $Y_1^2 = 0$ and (ii) $Y_1, H = 4$. Setting $Y_1 = aH + bX$, we get from (i): $2a(2a + bd) = 0$. Hence (a) $a = 0$, or (b) $2a + bd = 0$.

(a) In this case $Y_1 = bX$, hence (for degree reasons and since $S$ doesn’t contain curves of degree $< 4$), $Y_2 = \emptyset$ and $Y = X$, which is integral.

(b) Since $Y_1, H = 4$, we get $2a + (2a + bd) = 2a = 4$, hence $a = 2$ and $bd = -4$ which is impossible ($d = 7$ or 8 and $b \in \mathbb{Z}$).

This concludes the proof of the claim.

Since $(4H - X)^2 \geq 0$, the claim implies that $4H - X$ is numerically effective. Now we conclude by a result of Saint-Donat (cf. [11, Theorem 5]) that $|4H - X|$
is base point free, i.e. \( I_{X,S}(4) \) is globally generated. By the exact sequence:
\[
0 \to O \to I_{X}(4) \to I_{X,S}(4) \to 0
\]
we get that \( I_{X}(4) \) is globally generated. 

**Remark 2.11.** If \( d = 8 \), a general element \( Y \in |4H - X| \) is a smooth elliptic curve of degree 8. By the way \( Y \neq X \) (see [1]). The exact sequence of liaison:
\[
0 \to I_{Y}(4) \to I_{X}(4) \to \omega_{Y} \to 0
\]
shows that \( h^{0}(I_{X}(4)) = 3 \) (i.e. \( X \) is of maximal rank). In case \( d = 8 \) Lemma 2.10 is stated in [2], however the proof there is incomplete, indeed in order to apply the enumerative formula of [8] one has to know that \( X \) is a connected component of \( \bigcap_{i=1}^{3} F_{i} \); this amounts to say that the base locus of \( |4H - X| \) on \( F_{1} \) has dimension \( \leq 0 \).

To conclude we have:

**Lemma 2.12.** Let \( X \) be the disjoint union of two smooth, irreducible quartic elliptic curves, then \( I_{X}(4) \) is generated by global sections.

**Proof.** Let \( X = C_{1} \sqcup C_{2} \). We have:
\[
0 \to O(-4) \to 2O(-2) \to I_{C_{1}} \to 0,
\]
 twisting by \( I_{C_{2}} \), since \( C_{1} \cap C_{2} = \emptyset \), we get:
\[
0 \to I_{C_{2}}(-4) \to 2I_{C_{2}}(-2) \to I_{X} \to 0
\]
and the result follows. \( \square \)

**Summarizing:**

**Proposition 2.13.** There exists an indecomposable rank two vector bundle, \( E \), on \( \mathbb{P}^{3} \), generated by global sections and with \( c_{1}(E) = 5 \) if and only if \( 5 \leq c_{2}(E) \leq 8 \) and in these cases there is an exact sequence:
\[
0 \to O \to E \to I_{C}(4) \to 0
\]
where \( C \) is a smooth irreducible elliptic curve of degree \( c_{2}(E) = 8 \), the disjoint union of two smooth elliptic quartic curves.

**2.4. Globally generated rank two vector bundles with \( c_{1} = 5 \).**

We start by listing the possible cases:

**Proposition 2.14.** If \( E \) is an indecomposable, globally generated, rank two vector bundle on \( \mathbb{P}^{3} \) with \( c_{1}(E) = 5 \), then \( c_{2}(E) \in \{8, 10, 12\} \) and there is an exact sequence:
\[
0 \to O \to E \to I_{C}(5) \to 0
\]
where \( C \) is a smooth, irreducible curve of degree \( d = c_{2}(E) \), with \( \omega_{C} \simeq O_{C}(1) \).

In any case \( E \) is stable.
Proof. A general section of $E$ vanishes along a smooth curve, $C$, of degree $d = c_2(E)$ with $\omega_C \simeq \mathcal{O}_C(1)$. Hence every irreducible component, $Y$, of $C$ is a smooth, irreducible curve with $\omega_Y \simeq \mathcal{O}_Y(1)$. In particular $\deg(Y) = 2g(Y) - 2$ is even and $\deg(Y) \geq 4$.

1. If $d = 4$, then $C$ is a planar curve and $E$ splits.

2. If $d = 6$, $C$ is necessarily irreducible (of genus 4). It is well known that any such curve is a complete intersection $(2, 3)$, hence $E$ splits.

3. If $d = 8$ and $C$ is not irreducible, then $C = P_1 \sqcup P_2$, the disjoint union of two planar quartic curves. If $L = \langle P_1 \rangle \cap \langle P_2 \rangle$, then every quintic containing $C$ contains $L$ in contradiction with the fact that $\mathcal{I}_C(5)$ is generated by global sections. Hence $C$ is irreducible.

4. If $d = 10$ and $C$ is not irreducible, then $C = P \sqcup X$, where $P$ is a planar curve of degree 4 and where $X$ is a degree 6 curve ($X$ is a complete intersection $(2, 3)$). Every quintic containing $C$ vanishes on $P$ and on the 8 points of $X \cap \langle P \rangle$, since these 8 points are not on a line, the quintic vanishes on the plane $\langle P \rangle$. This contradicts the fact that $\mathcal{I}_C(5)$ is globally generated.

5. If $d = 12$ and $C$ is not irreducible we have three possibilities:

   (a) $C = P_1 \sqcup P_2 \sqcup P_3$, $P_i$ planar quartic curves
   (b) $C = X_1 \sqcup X_2$, $X_i$ complete intersection curves of types $(2, 3)$
   (c) $C = Y \sqcup P$, $Y$ a canonical curve of degree 8, $P$ a planar curve of degree 4.

   (a) This case is impossible (consider the line $\langle P_1 \rangle \cap \langle P_2 \rangle$).
   (b) We have $X_i = Q_i \cap F_i$. Let $Z$ be the quartic curve $Q_1 \cap Q_2$. Then $X_i \cap Z = F_i \cap Z$, i.e. $X_i$ meets $Z$ in 12 points. It follows that every quintic containing $C$ meets $Z$ in 24 points, hence such a quintic contains $Z$. Again this contradicts the fact that $\mathcal{I}_C(5)$ is globally generated.
   (c) This case too is impossible: every quintic containing $C$ vanishes on $P$ and on the points $\langle P \rangle \cap Y$, hence on $\langle P \rangle$.

   We conclude that if $d = 12$, $C$ is irreducible.

   The normalized bundle is $E(-3)$, since in any case $h^0(\mathcal{I}_C(2)) = 0$ (every smooth irreducible subcanonical curve on a quadric surface is a complete intersection), $E$ is stable.

Now we turn to the existence part.

Lemma 2.15. There exist indecomposable rank two vector bundles on $\mathbb{P}^3$ with Chern classes $c_1 = 5$ and $c_2 \in \{8, 10\}$ which are globally generated.
Proof. Let \( R = \sqcup_{i=1}^{s} L_i \) be the union of \( s \) disjoint lines, \( 2 \leq s \leq 3 \). We may perform a liaison \((s, 3)\) and link \( R \) to \( K = \sqcup_{i=1}^{s} K_i \), the union of \( s \) disjoint conics. The exact sequence of liaison: 
\[ 0 \to I_U(4) \to I_K(4) \to \omega_R(5-s) \to 0 \]
shows that \( I_K(4) \) is globally generated (n.b. \( 5-s \geq 2 \)).

Since \( \omega_K(1) \simeq \mathcal{O}_K \) we have an exact sequence: 
\[ 0 \to \mathcal{O} \to \mathcal{E}(2) \to I_K(3) \to 0, \]
where \( \mathcal{E} \) is a rank two vector bundle with Chern classes 
\[ c_1 = -1, \quad c_2 = 2s - 2. \]
Twisting by \( \mathcal{O}(1) \) we get: 
\[ 0 \to \mathcal{O}(1) \to \mathcal{E}(3) \to I_K(4) \to 0 \quad (\ast). \]
The Chern classes of \( \mathcal{E}(3) \) are 
\[ c_1 = 5, \quad c_2 = 2s + 4 \text{ (i.e. } c_2 = 8, 10). \]
Since \( I_K(4) \) is globally generated, it follows from \((\ast)\) that \( \mathcal{E}(3) \) too, is generated by global sections.

Remark 2.16.

1. If \( \mathcal{E} \) is as in the proof of Lemma 2.15 a general section of \( \mathcal{E}(3) \) vanishes along a smooth, irreducible (because \( h^1(\mathcal{E}(-2)) = 0 \)) canonical curve, \( C \), of genus \( 1 + c_2/2 \) (\( g = 5, 6 \)) such that \( \mathcal{I}_C(5) \) is globally generated. By construction these curves are not of maximal rank (\( h^0(\mathcal{I}_C(3)) = 1 \) if \( g = 5 \), \( h^0(\mathcal{I}_C(4)) = 2 \) if \( g = 6 \)). As explained in [9] 4 this is a general fact: no canonical curve of genus \( 5 \leq g \leq 6 \) in \( \mathbb{P}^3 \) is of maximal rank. We don’t know if this is still true for \( g = 7 \).


3. The proof of 2.15 breaks down with four conics: \( I_K(4) \) is no longer globally generated, every quartic containing \( K \) vanishes along the lines \( L_i \), \( (5-s = 1) \). Observe also that four disjoint lines always have a quadrisection and hence are an exception to the normal generation conjecture (the homogeneous ideal is not generated in degree three as it should be).

Remark 2.17. The case \((c_1, c_2) = (5, 12)\) remains open. It can be shown that if \( E \) exists, a general section of \( E \) is linked, by a complete intersections of two quintics, to a smooth, irreducible curve, \( X \), of degree 13, genus 10 having \( \omega_X(-1) \) as a base point free \( g^5 \). One can prove the existence of curves \( X \subset \mathbb{P}^3 \), smooth, irreducible, of degree 13, genus 10, with \( \omega_X(-1) \) a base point free pencil and lying on one quintic surface. But we are unable to show the existence of such a curve with \( h^0(\mathcal{I}_X(5)) \geq 3 \) (or even with \( h^0(\mathcal{I}_X(5)) \geq 2 \)). We believe that such bundles do not exist.

3. Globally generated rank two vector bundles on \( \mathbb{P}^n \), 
\( n \geq 4 \).

For \( n \geq 4 \) and \( c_1 \leq 5 \) there is no surprise:

**Proposition 3.1.** Let \( E \) be a globally generated rank two vector bundle on \( \mathbb{P}^n \), 
\( n \geq 4 \). If \( c_1(E) \leq 5 \), then \( E \) splits.
Proof. It is enough to treat the case \( n = 4 \). A general section of \( E \) vanishes along a smooth (irreducible) subcanonical surface, \( S: 0 \to \mathcal{O} \to E \to 
abla S(c_1) \to 0 \). By [5], if \( c_1 \leq 4 \), then \( S \) is a complete intersection and \( E \) splits. Assume now \( c_1 = 5 \). Consider the restriction of \( E \) to a general hyperplane \( H \). If \( E \) doesn’t split, by 2.14 we get that the normalized Chern classes of \( E \) are: \( c_1 = -1 \), \( c_2 \in \{2, 4, 6\} \). By Schwarzenberger condition: \( c_2(c_2 + 2) \equiv 0 \pmod{12} \). The only possibilities are \( c_2 = 4 \) or \( c_2 = 6 \). If \( c_2 = 4 \), since \( E \) is stable (because \( E|H \) is, see 2.14), we have ([3]) that \( E \) is a Horrocks-Mumford bundle. But the Horrocks-Mumford bundle (with \( c_1 = 5 \)) is not globally generated.

The case \( c_2 = 6 \) is impossible: such a bundle would yield a smooth surface \( S \subseteq \mathbb{P}^4 \), of degree 12 with \( \omega_S \cong \mathcal{O}_S \), but the only smooth surface with \( \omega_S \cong \mathcal{O}_S \) in \( \mathbb{P}^4 \) is the abelian surface of degree 10 of Horrocks-Mumford.

Remark 3.2. For \( n > 4 \) the results in [6] give stronger and stronger (as \( n \) increases) conditions for the existence of indecomposable rank two vector bundles generated by global sections.

Putting everything together, the proof of Theorem 1.1 is complete.

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