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RH-regular transformation of unbounded double sequences

Richard F. Patterson

Abstract. *At the Ithaca meeting in 1946 it was conjectured that it is possible to construct a two-dimensional regular summability matrix* $A = \{a_{n,k}\}$ *with the property that, for every real sequence* $\{s_k\}$*, the transformed sequence*

$$t_n = \sum_{k=0}^{\infty} a_{n,k} s_k$$

possesses at least one limit point in the finite plane. It was also counter-conjectured that, for every regular summability matrix A*, there exists a single sequence* $\{s_k\}$ *such that the transformed sequence* t_n *tends to infinity monotonically. In 1947 Erdos and Piranian presented answers to these conjectures. The goal of this paper is to present a multidimensional version of the above conjectures. The first conjecture is the following: A four-dimensional RH-regular summability matrix* $A = \{a_{m,n,k,l}\}$ *can be constructed with the property that every double sequence* $\{s_{k,l}\}$ *transformed into the double sequence*

$$t_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} s_{k,l}$$

possesses at least one Pringsheim limit point in the finite plane. The multidimensional counter-conjecture is the following. For every RH-regular summability matrix A *there exists a double sequence* $\{s_{k,l}\}$ *such that the four-dimensional transformed double sequence* $\{t_{m,n}\}$ *tends to infinity monotonically Pringsheim sense. This paper established that both multidimensional conjectures are false.*

Keywords: RH-regular, double sequences Pringsheim limit point, p-convergent
MS Classification 2010: 40B05, 40C05

1. Definitions, Notations, and Preliminary Results

Definition 1.1. *[Pringsheim, [4]] A double sequence $x = [x_{k,l}]$ has a* **Pringsheim limit** *L (denoted by P-$\lim x = L$) provided that, given an $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. Such an x is described more briefly as "P-convergent".*

Definition 1.2. *[Patterson, [3]] A double sequence y is a* **double subsequence** *of x provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that, if $x_j = x_{n_j,k_j}$, then y is formed by*

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{array}$$

In [5] Robison presented the following notion of conservative four-dimensional matrix transformation and a Silverman-Toeplitz type characterization of such notion.

Definition 1.3. *A four-dimensional matrix A is said to be* **RH-regular** *if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.*

This assumption of boundedness is made because a double sequence which is P-convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [2] and [5].

Theorem 1.4. *(Hamilton [2], Robison [5]) The four dimensional matrix A is RH-regular if and only if*

- RH_1: P-$\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*
- RH_2: P-$\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} = 1$;
- RH_3: P-$\lim_{m,n} \sum_{k=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each l;*
- RH_4: P-$\lim_{m,n} \sum_{l=0}^{\infty} |a_{m,n,k,l}| = 0$ *for each k;*
- RH_5: $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent;*
- RH_6: *there exist finite positive integers Δ and Γ such that* $\sum_{k,l>\Gamma} |a_{m,n,k,l}| < \Delta$.

Definition 1.5. *Let A be a four dimensional matrix with pairwise column (m,n). Then the (i,j)-reverse L-string , denoted by, $L_{i,j}^{m,n}$ is*

$$\{a_{m,n,1,j}, a_{m,n,2,j}, a_{m,n,3,i}, \cdots, a_{m,n,i,j}, a_{m,n,i,j-1}, a_{m,n,i,j-2}, \cdots, a_{m,n,i,1},\}.$$

Given a double sequence x the (i,j)-reverse L-string , denoted by, $L_{i,j}$ is

$$\{x_{1,j}, x_{2,j}, x_{3,i}, \cdots, x_{i,j}, x_{i,j-1}, x_{i,j-2}, \cdots, x_{i,1},\}.$$

2. Main Results

THEOREM 2.1. *If A is a pairwise-row finite RH-regular summability matrix then there exists a double sequence $\{s_{k,l}\}$ such that the corresponding transformed double sequence $|t_{m,n}|$ tends to infinity, in the Pringsheim with arbitrary rapidity.*

Proof. Let A be a pairwise-row finite RH-regular summability matrix. If m_0 and n_0 are sufficiently large, then each pairwise index whose indices exceed m_0 and n_0, respectively, contains a non-zero element. For fixed pairwise column index (m,n) let C-string denote the last column of the pairwise row whose sum is non-zero, and R-string denote the last row of the pairwise row whose sum is non-zero. Using the terms from C-string and R-string along with Definition 1.5 we can now construct a last reverse L-string whose sum is non-zero. Therefore a terminal reverse L-string exists. Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ and $\beta_1, \beta_2, \beta_3, \cdots$ be the indices of the pairwise-columns that contain terminal reverse L-string. Without of loss of generality we may assume that $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$ and $\beta_1 < \beta_2 < \beta_3 < \cdots$. Define the terms of $\{s_{k,l}\}$ such that

$$k \neq \alpha_1, \alpha_2, \alpha_3, \cdots$$

and

$$l \neq \beta_1, \beta_2, \beta_3, \cdots$$

be arbitrary. Since A is pairwise row finite, each pairwise-column contains at most a finite number of pairwise-terminal reverse L-string of elements, that is, for each pairwise column the pairwise-terminal reverse L-string of element are bounded away from zero. If $f(m,n)$ is any arbitrary real function the terms

$$\begin{matrix} s_{k_1,l_1} & s_{k_1,l_2} & \cdots \\ s_{k_2,l_1} & s_{k_2,l_2} & \cdots \\ \cdots & \cdots & \ddots \end{matrix}$$

can be chosen large enough so that $|t_{m,n}| > f(m,n)$; $m > m_0$ and $n > n_0$. $\square$

THEOREM 2.2. *If A is an RH-regular summability matrix then there exists a double sequence $\{s_{m,n}\}$ such that the transformed double sequence $\{t_{m,n}\}$ has no P-limit points in the finite plane.*

Proof. Let c be a constant such that $\sum_{k,l=0,0}^{\infty,\infty} |a_{m,n,k,l}| < \frac{c}{5}$ for all (m,n). Such a constant exists by RH_5 of the RH-regularity conditions of A. We can choose $m_0 = n_0$ sufficiently large such that, regularity conditions RH_3, RH_4, and RH_5 of A assure us, that there exists a pair (α_1, β_1) such that

$$\sum_{\{(k,l):k>\alpha_1 \text{ or } l>\beta_1\}} |a_{m_0,n_0,k,l}| < \frac{1}{c^2}.$$

Now choose m_1 and n_1 with $m_1 > m_0$ and $n_1 > n_0$ such that

$$\sum_{\{(k,l):0\leq k\leq\alpha_1;0\leq l\leq\beta_1\}} |a_{m,n,k,l}| < \frac{1}{5}$$

for $m > m_1$ and $n > n_1$ by RH_1. Let us construct the second stage. Conditions RH_3, RH_4, and RH_5 assure us that we can choose (α_2, β_2) with $\alpha_2 > \alpha_1$ and $\beta_2 > \beta_1$ such that

$$\sum_{\{(k,l):k>\alpha_2 \text{ or } l>\beta_2\}} |a_{m,n,k,l}| < \frac{1}{c^4}$$

whenever $m, n \leq m_1, n_1$, respectively. Using RH_1, we can now choose m_2 and n_2 with $m_2 > m_1$ and $n_2 > n_1$ such that

$$\sum_{\{(k,l):0\leq k\leq\alpha_2;0\leq l\leq\beta_2\}} |a_{m,n,k,l}| < \frac{1}{5}$$

for $m > m_2$ and $n > n_2$. Using the RH-regularity conditions of A the general stage is constructed as follows. Let (α_r, β_s)be such that $\alpha_r > \alpha_{r-1}$ and $\beta_s > \beta_{s-1}$ with

$$\sum_{\{(k,l):k>\alpha_r \text{ or } l>\beta_s\}} |a_{m,n,k,l}| < \frac{1}{c^{r+s}}$$

where $m, n \leq m_{r-1}, n_{s-1}$, respectively. Now we choose m_r and n_s with $m_r > m_{r-1}$ and $n_s > n_{s-1}$ such that

$$\sum_{\{(k,l):0\leq k\leq\alpha_r;0\leq l\leq\beta_s\}} |a_{m,n,k,l}| < \frac{1}{5}$$

for $m > m_r$ and $n > n_s$, where $r, s = 1, 2, 3, \ldots$. Let us now consider following double sequence

$$s_{k,l} = \begin{cases} \left(1+\frac{1}{c}\right)^{r+s} & \text{if} \quad \alpha_{r-1} < k \leq \alpha_r \ \text{ and/or } \beta_{s-1} < l \leq \beta_s \\ 0 & \text{if} \quad \text{otherwise} \\ r, s = 1, 2, 3, \ldots & \end{cases}.$$

Let us now partition the A transformation of $\{s_{k,l}\}$ into three parts with

$$m_{r-1} < m \leq m_r \ \text{ and/or } n_{s-1} < n \leq n_s.$$

The first partition satisfy the following inequality

$$\sum_{k,l=0,0}^{\alpha_{r-1},\beta_{s-1}} |a_{m,n,k,l}| < \frac{1}{5}\left(1+\frac{1}{c}\right)^{r+s-2} \quad \text{with } r, s = 2, 3, 4, \ldots \tag{1}$$

and the second satisfies the inequality

$$
\begin{aligned}
& \sum_{k,l=\alpha_{r+1}+1,\beta_{s+1}+1}^{\infty,\infty} |a_{m,n,k,l}| \qquad (2)\\
< \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4} + \frac{1}{c^{r+s+4}}\left(1+\frac{1}{c}\right)^{r+s+6} + \cdots \\
= \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\left[1+\frac{1}{c^2}\left(1+\frac{1}{c}\right)^{2}+\frac{1}{c^4}\left(1+\frac{1}{c}\right)^{4}+\cdots\right] \\
= \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\sum_{i=0}^{\infty}\frac{1}{c^{2i}}\left(1+\frac{1}{c}\right)^{2i} \\
= \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\frac{1}{1-\frac{1}{c^2}\left(1+\frac{1}{c}\right)^2} \\
\leq \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\left(\frac{1}{1-\frac{4}{25}}\right) \\
\leq \quad & \frac{1}{c^{r+s+2}}\left(1+\frac{1}{c}\right)^{r+s+4}\frac{25}{21} \\
\leq \quad & \frac{1}{21c^{r+s}}\left(1+\frac{1}{c}\right)^{r+s+4} \\
\text{with} \quad & r,s=0,1,2,\ldots.
\end{aligned}
$$

The final partition satisfies the equality

$$
\begin{aligned}
\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l} \quad = \quad & \sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l}s_{k,l} \\
& + \sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l} \\
= \quad & \sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l}\left(1+\frac{1}{c}\right)^{r+s} \\
& + \sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}\left(1+\frac{1}{c}\right)^{r+s+2} \qquad (3)
\end{aligned}
$$

In addition,the final partition also satisfies the following inequality

$$\begin{aligned}
\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l} &= \left(1+\frac{1}{c}\right)^{r+s}\left[\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l}\right. \\
&\quad \left. + \left(1+\frac{1}{c}\right)^{2}\sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}\right] \\
&> \left(1+\frac{1}{c}\right)^{r+s}\left[\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l}\right. \\
&\quad + \sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l} \\
&\quad \left. + \frac{1}{c}\sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}\right].
\end{aligned}$$

Observe that, if m and n are sufficiently large the following is true by the RH-regularity of A:

$$\frac{1}{c}\sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l} < \frac{1}{5} \tag{4}$$

and

$$\left|\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_r,\beta_s} a_{m,n,k,l} + \sum_{k,l=\alpha_r+1,\beta_s+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}\right| > \frac{3}{5}. \tag{5}$$

Therefore, for m and n sufficiently large, inequalities (1) through (5) imply the following

$$\begin{aligned}
\left|\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}s_{k,l}\right| &= \left|\sum_{k,l=0,0}^{\alpha_{r-1},\beta_{s-1}} a_{m,n,k,l}s_{k,l}\right. \\
&\quad \left. + \sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l} + \sum_{k,l=\alpha_{r+1}+1,\beta_{s+1}+1}^{\infty,\infty} a_{m,n,k,l}s_{k,l}\right| \\
&\geq \left|\sum_{k,l=\alpha_{r-1}+1,\beta_{s-1}+1}^{\alpha_{r+1},\beta_{s+1}} a_{m,n,k,l}s_{k,l}\right| \\
&\quad - \sum_{k,l=0,0}^{\alpha_{r-1},\beta_{s-1}} |a_{m,n,k,l}|s_{k,l} \\
&\quad - \sum_{k,l=\alpha_{r+1}+1,\beta_{s+1}+1}^{\infty,\infty} |a_{m,n,k,l}|s_{k,l} \\
&> \left(1+\frac{1}{c}\right)^{r+s}\left[\frac{2}{5} - \frac{1}{5(1+\frac{1}{c})} - \frac{1}{21c^{r+s}}\left(1\frac{1}{c}\right)^4\right] \\
&> \frac{1}{100}\left(1+\frac{1}{c}\right)^{r+s}.
\end{aligned}$$

□

THEOREM 2.3. *If A is a four-dimensional RH-regular summability matrix then there exists a double sequence $\{s_{k,l}\}$ such that $t_{m,n} = \rho_{m,n}e^{i\theta_{m,n}}$, with*

$$P-\lim_{m,n}\rho_{m,n} = \infty \text{ and } P-\lim_{m,n}\theta_{m,n} = 0.$$

If the matrix A is also real then the double sequence $s_{m,n}$ can be chosen so that the double sequence $t_{m,n}$ is real and positive.

In the proof of Theorem 2.2, replace $\frac{1}{5}$ with a Pringsheim null double sequence and replace $\{s_{k,l}\}$ with the following sequence, or a sequence similar to the following, with respect to order.

$$s'_{k,l} = \begin{cases} \left(1+\frac{1}{c}\right)^{\sqrt{r+s}} & \text{if} \quad \alpha_{r-1} < k \leq \alpha_r \text{ and/or } \beta_{s-1} < l \leq \beta_s \\ 0 & \text{if} \quad \text{otherwise} \\ r,s = 1,2,3,\ldots \end{cases}.$$

The result then follows from RH_1, RH_3, RH_4, and RH_5 of the RH-regularity conditions of A.

THEOREM 2.4. *If the double real valued function $f(m,n)$ is such that*

$$P - \lim_{m,n} f(m,n) = \infty$$

then there exists an RH-regular summability matrix A such that, for every double sequence $\{s_{m,n}\}$ to which transformation A is applicable, the inequality

$$|t_{m,n}| < f(m,n) \tag{6}$$

is satisfied for infinitely many ordered pairs (m,n).

Proof. This asserts that there exists an RH-regular transformation that transforms every double sequence to which it is summable either into a double sequence with at least one finite Pringsheim limit point or else into a double sequence whose terms tend to infinity at an arbitrary slow rate, independent of the double sequence. The following four-dimensional summability matrix satisfies the conditions of the theorem.

$$a_{m,n,k,l} = \begin{cases} 1 & \text{if both } m \text{ and } n \text{ are even with } k = \frac{m}{2} \text{ and } l = \frac{n}{2} \\ 0 & \text{if both } m \text{ and } n \text{ are even with } k \neq \frac{m}{2} \text{ and } l \neq \frac{n}{2} \\ 1 & \text{if both } m \text{ and } n \text{ are odd with } k = \frac{m-1}{2} \text{ and } l = \frac{n-1}{2} \\ 0 & \text{if both } m \text{ and } n \text{ are odd with } k < \frac{m-1}{2} \text{ and } l < \frac{n-1}{2} \\ 0 & \text{if both } m \text{ and } n \text{ are odd with } k > \frac{m-1}{2} \text{ and } l > \frac{n-1}{2} \\ & \text{except when } k = k_1, k_2, k_3, \dots \text{ and } l = l_1, l_2, l_3, \dots \\ 2^{-(r+s)} & \text{if both } m \text{ and } n \text{ are odd with } k < \frac{m-1}{2} \text{ and } l < \frac{n-1}{2} \\ & k = k_1, k_2, k_3, \dots \text{ and } l = l_1, l_2, l_3, \dots \\ r, s = 1, 2, 3, \dots. & \end{cases}.$$

Suppose that the double sequence $\{s_{m,n}\}$ is such that inequality (6) does not hold infinitely often in the Pringsheim sense. Choose index sequences $\{k_r\}$, $\{l_s\}$ such that $f(k_r, l_s) > 2^{r+s}$; and if each element of (m,n) is odd and $k_r > \frac{m-1}{2}$ and $l_s > \frac{n-1}{2}$, $a_{m,n,k_r,l_s} = \frac{1}{2^{r+s}}$.

Since A is such that its pairwise row contains only one nonzero element, then $|s_{m,n}| > f(m,n)$ for all sufficiently large m and n. Therefore, for odd m and n, the series

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} s_{k,l}$$

contains infinity many terms whose absolute value is 1. Therefore the four-dimensional A transformation is not applicable to the double sequence $\{s_{m,n}\}$. □

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Manifolds over Cayley-Dickson algebras and their immersions

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Abstract. *Weakly holomorphic manifolds over Cayley-Dickson algebras are defined and their embeddings and immersions are studied.*

Keywords: manifold, noncommutative, weakly holomorphic, Cayley-Dickson algebra
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1. Introduction

Real and complex manifolds are widely used in different branches of mathematics [4, 16, 17, 18, 20, 21, 22, 34]. On the other hand, Cayley-Dickson algebras $\mathcal{A}_r$, particularly, the quaternion skew field $\mathbf{H} = \mathcal{A}_2$ and the octonion algebra $\mathbf{O} = \mathcal{A}_3$, have found many-sided applications not only in mathematics, but also in theoretical physics (see [2], [7] - [14], [16, 22, 36, 35] and references therein). Theory of functions of quaternion and octonion variables is presented in these works and cited below. Various classes of such functions and different variants of their super-differentiability were investigated and described depending on needs of mathematics and theoretical physics over quaternions, octonions and some other alternative algebras.

This paper continues previous works of the author, where different theory from the cited above publications was developed. Functions of Cayley-Dickson variables were studied earlier [23, 24, 29, 33]. Their super-differentiability was defined in terms of representing them words and phrases as a differentiation which is real-linear, additive and satisfying Leibniz' rule on an algebra of phrases over $\mathcal{A}_r$ (see in details Chapter 1 §§2.1 and 2.2 in the book [28] or in the articles [23, 29]). That is a weak version of a super-differentiability used in super-analysis. Though such weak super-differentiability over $\mathcal{A}_r$ of a function f on an open domain implies that f is locally analytic in an $\mathcal{A}_r$-variable with $\mathcal{A}_r$-coefficients in power series with definite order of the multiplication in each additive. A super-differentiable function on a domain U in $\mathcal{A}_r^n$ or $l_2(\mathcal{A}_r)$ of $\mathcal{A}_r$-variables is also called $\mathcal{A}_r$-differentiable (or weakly $\mathcal{A}_r$-holomorphic). For $r \geq 4$ the Cayley-Dickson algebras are non-associative and non-alternative. This approach appeared to be effective for investigations of problems of analysis, partial differential equations, operator theory, noncommutative geometry [25], [26] - [32].

This article is devoted to investigations of $\mathcal{A}_r$-differentiable manifolds (weakly holomorphic manifolds). Their embeddings and immersions are studied. Results and notations of previous papers [23, 24, 29, 33] are used below.

Main results of this paper are obtained for the first time.

2. Manifolds over Cayley-Dickson algebras

Definition 2.1. *An $\mathbf{R}$ linear space X which is also left and right $\mathcal{A}_r$ module will be called an $\mathcal{A}_r$ vector space. We present X as the direct sum*

(DS) $X = X_0 i_0 \oplus ... \oplus X_m i_m \oplus ...$, *where* $X_0,...,X_m,...$ *are pairwise isomorphic real linear spaces, where $i_0,...,i_{2^r-1}$ are generators of the Cayley-Dickson algebra $\mathcal{A}_r$ such that $i_0 = 1$, $i_k^2 = -1$ and $i_k i_j = -i_j i_k$ for each $k \ge 1$ and $j \ge 1$ so that $k \ne j$, $2 \le r$.*

Let X and Y be two $\mathbf{R}$ linear normed spaces which are also left and right $\mathcal{A}_r$ modules, where $1 \le r$, such that

(1) $0 \le \|ax\|_X \le |a| \|x\|_X$ *and* $\|xa\|_X \le |a| \|x\|_X$ *and*

(2) $\|ax_j\|_X = |a| \|x_j\|_X$ *and*

(3) $\|x+y\|_X \le \|x\|_X + \|y\|_X$

for all $x, y \in X$ and $a \in \mathcal{A}_r$ and $x_j \in X_j$. Such spaces X and Y will be called $\mathcal{A}_r$ normed spaces.

Suppose that X and Y are two normed spaces over the Cayley-Dickson algebra $\mathcal{A}_v$. A continuous $\mathbf{R}$ linear mapping $\theta : X \to Y$ is called an $\mathbf{R}$ linear homomorphism. If in addition $\theta(bx) = b\theta(x)$ and $\theta(xb) = \theta(x)b$ for each $b \in \mathcal{A}_v$ and $x \in X$, then θ is called a homomorphism of $\mathcal{A}_v$ (two sided) modules X and Y.

If a homomorphism is injective, then it is called an embedding ($\mathbf{R}$ linear or for $\mathcal{A}_v$ modules correspondingly).

If a homomorphism h is bijective and from X onto Y so that its inverse mapping h^{-1} is also continuous, then it is called an isomorphism ($\mathbf{R}$ linear or of $\mathcal{A}_v$ modules respectively).

Definition 2.2. *We say that a real vector space Z is supplied with a scalar product if a bi-$\mathbf{R}$-linear bi-additive mapping $<,>: Z^2 \to \mathbf{R}$ is given satisfying the conditions:*

(1) $< x, x > \ \ge 0$, $< x, x >= 0$ *if and only if* $x = 0$;

(2) $< x, y >=< y, x >$;

(3) $< ax + by, z >= a < x, z > +b < y, z >$ *for each real numbers $a, b \in \mathbf{R}$ and vectors $x, y, z \in Z$.*

Then an $\mathcal{A}_r$ vector space X is supplied with an $\mathcal{A}_r$ valued scalar product, if a bi-$\mathbf{R}$-linear bi-$\mathcal{A}_r$-additive mapping $< *, * >: X^2 \to \mathcal{A}_r$ is given such that

(4) $< f, g >= \sum_{j,k} < f_j, g_k > i_j^* i_k$,

where $f = f_0 i_0 + ... + f_m i_m + ...$, $f, g \in X$, $f_j, g_j \in X_j$, each X_j is a real linear space with a real valued scalar product, $(X_j, < *, * >)$ is real linear isomorphic *with* $(X_k, <*,*>)$ *and* $<f_j, g_k>\in \mathbf{R}$ *for each* j, k. *The scalar product induces the norm:*

(5) $\|f\| := \sqrt{<f,f>}$.

An $\mathcal{A}_r$ *normed space or an* $\mathcal{A}_r$ *vector space with* $\mathcal{A}_r$ *scalar product complete relative to its norm will be called an* $\mathcal{A}_r$ *Banach space or an* $\mathcal{A}_r$ *Hilbert space respectively.*

A Hilbert space X *over* $\mathcal{A}_r$ *is denoted by* $l_2(\lambda, \mathcal{A}_r)$, *where* λ *is a set of the cardinality* $card(\lambda) \ge \aleph_0$ *which is the topological weight of* X_0, *i.e.* $X_0 = l_2(\lambda, \mathbf{R})$.

A mapping $f : U \to l_2(\lambda, \mathcal{A}_r)$ *can be written in the form*

$$f(z) = \sum_{j\in\lambda} f^j(z) e_j,$$

where $\{e_j : j \in \lambda\}$ *is an orthonormal basis in the Hilbert space* $l_2(\lambda, \mathcal{A}_r)$, U *is a domain in* $l_2(\psi, \mathcal{A}_r)$, $f^j(z) \in \mathcal{A}_r$ *for each* $z \in U$ *and every* $j \in \lambda$. *If* f *is Frechét differentiable over* $\mathbf{R}$ *and each function* $f^j(z)$ *is differentiable by each Cayley-Dickson variable* ${}_kz$ *on* U, *then* f *is called* $\mathcal{A}_r$*-differentiable on* U, *where*

$$z = \sum_{k\in\psi} {}_kz q_k,$$

while $\{q_k : k \in \psi\}$ *denotes the standard orthonormal basis in* $l_2(\psi, \mathcal{A}_r)$, ${}_kz \in \mathcal{A}_r$.

Definition 2.3. *Let* M *be a set such that*

(1) $M = \bigcup_j U_j$, M *is a Hausdorff topological space,*

(2) *each* U_j *is open in* M,

(3) $\phi_j : U_j \to \phi_j(U_j) \subset X$ *are homeomorphisms,* $\phi_j(U_j)$ *is open in* X *for each* j,

(4) *if* $U_i \cap U_j \ne \emptyset$, *the transition mapping* $\phi_i \circ \phi_j^{-1}$ *of charts is bijective and is* $\mathcal{A}_r$*-differentiable on its domain, while*

(5) $\phi_i : M \to X$ *with* $\phi_i \circ \phi_j^{-1}$ *being* $\mathcal{A}_r$*-differentiable on* $\phi_j(U_j)$ *for each* $i \ne j$;

where X *is either* $\mathcal{A}_r^m$ *with* $m \in \mathbf{N}$ *or a Hilbert space* $l_2(\lambda, \mathcal{A}_r)$ *over the Cayley-Dickson algebra* $\mathcal{A}_r$. *Then* M *is called an* $\mathcal{A}_r$*-differentiable manifold (or a weakly holomorphic manifold).*

Proposition 2.4. *Let* M *be an* $\mathcal{A}_r$*- differentiable manifold. Then there exists a tangent bundle* TM *which has the structure of an* $\mathcal{A}_r$*- differentiable manifold such that each fibre* T_xM *is the vector space over the Cayley-Dickson algebra* $\mathcal{A}_r$.

Proof. The Cayley-Dickson algebra $\mathcal{A}_r$ has the real shadow, which is the Euclidean space $\mathbf{R}^{\mathbf{2^r}}$, since $\mathcal{A}_r$ is the algebra over $\mathbf{R}$. Therefore, a manifold M

has also a real manifold structure. Each $\mathcal{A}_r$- differentiable mapping is infinite differentiable in accordance with Theorems 2.15 and 3.10 in [33, 23]. Then the tangent bundle TM exists, which is C^∞-manifold such that each fibre T_xM is a tangent space, where $x \in M$, T is the tangent functor. If $At(M) = \{(U_j, \phi_j) : j\}$, then $At(TM) = \{(TU_j, T\phi_j) : j\}$, $TU_j = U_j \times X$, where X is the $\mathcal{A}_r$ vector space on which M is modeled, $T(\phi_j \circ \phi_k^{-1}) = (\phi_j \circ \phi_k^{-1}, D(\phi_j \circ \phi_k^{-1}))$ for each $U_j \cap U_k \ne \emptyset$. Each transition mapping $\phi_j \circ \phi_k^{-1}$ is $\mathcal{A}_r$- differentiable on its domain, then its (strong) differential coincides with the super-differential $D(\phi_j \circ \phi_k^{-1}) = D_z(\phi_j \circ \phi_k^{-1})$, since $\tilde{\partial}(\phi_j \circ \phi_k^{-1}) = 0$. Therefore, the super-differential $D(\phi_j \circ \phi_k^{-1})$ is $\mathbf{R}$-linear and $\mathcal{A}_r$-additive, hence it is an automorphism of the $\mathcal{A}_r$ vector space X. But $D_z(\phi_j \circ \phi_k^{-1})$ is $\mathcal{A}_r$- differentiable as well, consequently, TM is the $\mathcal{A}_r$- differentiable manifold.

□

DEFINITION 2.5. *A C^1-mapping $f : M \to N$ is called an immersion, if the real rank of df is $rang(df|_x : T_xM \to T_{f(x)}N) = m_M$ for each $x \in M$, where $m_M := dim_{\mathbf{R}}M$. An immersion $f : M \to N$ is called an embedding, if f is a homeomorphism on its image.*

THEOREM 2.6. *Let M be a compact $\mathcal{A}_r$- differentiable manifold, $dim_{\mathcal{A}_r}M = m < \infty$, where $2 \le r \in \mathbf{N}$.*

(I). Then there exists an $\mathcal{A}_r$- differentiable embedding $\tau : M \hookrightarrow \mathcal{A}_r^{2m+1}$ and an $\mathcal{A}_r$- differentiable immersion $\theta : M \to \mathcal{A}_r^{2m}$ correspondingly.

(II). If M is a paracompact $\mathcal{A}_r$- differentiable manifold with countable atlas on $l_2(\lambda, \mathcal{A}_r)$, where $card(\lambda) \ge \aleph_0$, then there exists a $\mathcal{A}_r$- differentiable embedding $\tau : M \hookrightarrow l_2(\lambda, \mathcal{A}_r)$.

Proof. (I). For the proof of this theorem identities of Cayley-Dickson algebras are used below. This permits to supply the unit sphere of suitable dimension multiple of 2^r with the structure of an $\mathcal{A}_r$ differentiable manifold (see below), where $2 \le r \in \mathbf{N}$. Then charts of a suitable refined atlas with $\mathcal{A}_r$- differentiable transition mappings are used.

Let at first M be compact. Since M is compact, then it is finite dimensional over the Cayley-Dickson algebra $\mathcal{A}_r$, $dim_{\mathcal{A}_r}M = m \in \mathbf{N}$, such that $dim_{\mathbf{R}}M = 2^r m$ is its real dimension. Take an atlas $At'(M)$ refining the initial atlas $At(M)$ of M such that $({U'}_j, \phi_j)$ are charts of M, where each ${U'}_j$ is $\mathcal{A}_r$- differentiable diffeomorphic to an interior of the unit ball $Int(B(\mathcal{A}_r^m, 0, 1))$, where $B(\mathcal{A}_r^m, y, \rho) := \{z \in \mathcal{A}_r^m : |z - y| \le \rho\}$. In view of compactness of the manifold M a covering $\{{U'}_j : j\}$ has a finite subcovering, hence $At'(M)$ can be chosen finite. Denote for convenience the latter atlas as $At(M)$. Let (U_j, ϕ_j) be the chart of the atlas $At(M)$, where U_j is open in M, hence $M \setminus U_j$ is closed in M.

Consider the space $\mathcal{A}_r^m \times \mathbf{R}$ as the $\mathbf{R}$-linear space $\mathbf{R^{2^r m+1}}$, i.e. its real shadow. The unit sphere $S^{2^r m} := S(\mathbf{R}^{2^r m+1}, 0, 1) := \{z \in \mathbf{R}^{2^r m+1} : |z| = 1\}$

in $\mathcal{A}_r^m \times \mathbf{R}$ can be supplied with two charts (V_1, ϕ_1) and (V_2, ϕ_2) such that $V_1 := S^{2^r m} \setminus \{0, ..., 0, 1\}$ and $V_2 := S^{2^r m} \setminus \{0, ..., 0, -1\}$, where ϕ_1 and ϕ_2 are stereographic projections from poles $\{0, ..., 0, 1\}$ and $\{0, ..., 0, -1\}$ of V_1 and V_2 respectively onto $\mathcal{A}_r^m$. Then the transition mapping between two charts $\phi_2 \circ \phi_1^{-1} : \mathbf{E} \setminus \{0\} \to \mathbf{E} \setminus \{0\}$ is given by the formula $\phi_2 \circ \phi_1^{-1}(y) = y/|y|^2$ where $y = (y_1, ..., y_{2^r m}) \in \mathbf{E} \setminus \{0\}$, $\mathbf{E} = \mathbf{R^{2^r m}}$ (see §1.1.3 [20]). On the other hand the Euclidean space $\mathbf{E}$ is the real shadow of $\mathcal{A}_r^m$. We denote the unit sphere in $\mathcal{A}^m \times \mathbf{R}$ by $S^{2^r m}$ also.

To rewrite a function from the real variables z_j in the z-representation or vice versa the following identities are used:

$$z_j = \frac{1}{2}\left[-z i_j + i_j (2^r - 2)^{-1}\left(-z + \sum_{k=1}^{2^r - 1} i_k (z i_k^*)\right)\right] \tag{1}$$

for each $j = 1, 2, ..., 2^r - 1$,

$$z_0 = \frac{1}{2}\left[z + (2^r - 2)^{-1}\left(-z + \sum_{k=1}^{2^r - 1} i_k (z i_k^*)\right)\right] \tag{2}$$

where $2 \le r \in \mathbf{N}$, z is a Cayley-Dickson number decomposed as

$$z = z_0 i_0 + ... + z_{2^r - 1} i_{2^r - 1} \in \mathcal{A}_r \tag{3}$$

with $z_j \in \mathbf{R}$ for each j, $i_k^* = \tilde{i}_k = -i_k$ for each $k > 0$, $i_0 = 1$, since $i_j(i_j i_k) = -i_k$ and $(i_k i_j) i_j = -i_k$ for each $j > 0$, also $i_j i_k = -i_k i_j$ for each $j \ne k$ with $j > 0$ and $k > 0$, while $i_k(i_0 i_k^*) = 1$ for each k. Formulas (1)-(3) define the real-linear projection operators $\pi_j : \mathcal{A}_r \to \mathbf{R}$ so that

$$\pi_j(z) = z_j \tag{4}$$

for each Cayley-Dickson number $z \in \mathcal{A}_r$ and every $j = 0, 1, ..., 2^r - 1$.

The conjugation is given by the formula:

$$z^* = -(2^r - 2)^{-1} \sum_{p=0}^{2^r - 1} (i_p z) i_p \tag{5}$$

in $\mathcal{A}_r^m$ due to formulas (1)-(3), which provides z^* in the z-representation, where $i_0, ..., i_{2^r - 1}$ are the standard generators of the Cayley-Dickson algebra $\mathcal{A}_r$. Therefore the transition mapping $\phi_2 \circ \phi_1^{-1} : \mathcal{A}_r^m \setminus \{0\} \to \mathcal{A}_r^m \setminus \{0\}$ has the form in the z-representation:

$$\phi_2 \circ \phi_1^{-1}(z) = -\frac{(2^r - 2) z}{\sum_{k=1}^m [{}_k z \sum_{p=0}^{2^r - 1} (i_p \; {}_k z) i_p]}, \tag{6}$$

where $z = ({}_1z, ..., {}_mz)$ with ${}_jz \in \mathcal{A}_r$ for each $j = 1, ..., m$, $z \in \mathcal{A}_r^m \setminus \{0\}$. The transition mapping is presented as the fraction of two polynomials on the domain on which the denominator is non-zero. The fraction of two $\mathcal{A}_r$-differentiable functions is $\mathcal{A}_r$-differentiable on a domain where the denominator is non-zero [24, 29]. Therefore, $\phi_2 \circ \phi_1^{-1}(z)$ is the $\mathcal{A}_r$-differentiable diffeomorphism in $\mathcal{A}_r^m \setminus \{0\}$, i.e. the (weak) super-differential $D_z(\phi_2 \circ \phi_1^{-1})$ exists. Thus in the $\mathcal{A}_r$ realization $\phi_l(V_l) = \mathcal{A}_r^m \setminus \{0\}$ for $l = 1$ and $l = 2$ the unit sphere $S^{2^r m}$ is supplied with the structure of the $\mathcal{A}_r$-differentiable manifold.

If $g : M \to \mathcal{A}_r^N$ is a continuous mapping, then $g(M)$ is compact, since M is compact (see Theorem 3.1.10 [6]). Therefore, $g(M)$ is bounded and closed in $\mathcal{A}_r^N$ (see Theorems 3.1.8 and 3.1.23 [6]). Thus there exists a shift $h(z) = z + q$ on $\mathcal{A}_r^N$ such that $h(g(M))$ does not contain zero and hence $\inf\{|z| : z \in h(g(M))\} > 0$.

We consider $[Int(B(\mathcal{A}_r^m, 0, 1)) + q] \subset \mathcal{A}_r^m \setminus \{0\}$ with $q \in \mathcal{A}_r^m$ such that $|q| > 1$ and $At'(M)$ as above. The finite union of such balls $[Int(B(\mathcal{A}_r^m, 0, 1)) + q]$ is bounded in $\mathcal{A}_r^m \setminus \{0\}$. The shift mapping $z \mapsto z + q$ is $\mathcal{A}_r$-differentiable on $\mathcal{A}_r^m$. On the other hand, the manifold M is compact and each its atlas has a finite subatlas, where an atlas of M satisfies Conditions $3(1-5)$ above.

Simplifying the notation we can choose an atlas $\{(E_j, \xi_j) : j = 1, ..., n\}$ of M with mappings ξ_j satisfying the following properties: each $\xi_j : E_j \to \xi_j(E_j)$ is the $\mathcal{A}_r$-differentiable diffeomorphism onto the subset $\xi_j(E_j)$ in the ball $B(\mathcal{A}_r^m, q, b)$ with $|q| > 4b > 0$, whilst $\xi_j : M \to \mathcal{A}_r^m$ is $\mathcal{A}_r$-differentiable, $cl_M(E_j) \subset H_j$, $E_j \subset H_j$, H_j is open in M for each j, the restriction $\xi_j|_{H_j}$ is bijective, $\xi_j(M) \subset B(\mathcal{A}_r^m, q, 2b)$,

$\inf\{|x - y| : x \in \partial\xi_j(E_j), \ y \in \partial\xi_j(H_j)\} > b/2$,
where $\bigcup_j E_j = M$, $cl_M(E)$ denotes the closure of E in M, $\partial V := cl_{\mathcal{A}_r^m}(V) \setminus Int_{\mathcal{A}_r^m}(V)$ for a subset V in $\mathcal{A}_r^m$.

The function of the form

$$f_j(z) = \exp\left(\sum_{k=1}^{m} b_{k,j}\left[({}_kz - {}_kw_j)\sum_{p=0}^{2^r-1}\left(i_p({}_kz - {}_kw_j)\right)i_p\right]\right) \tag{7}$$

with positive constants $b_{k,j}$ and a marked point $w_j \in \mathcal{A}_r^m$ is positive $\mathcal{A}_r$-differentiable bounded on $\mathcal{A}_r^m$ and tending to zero when $|z|$ tends to the infinity, see (5). For each bounded canonical closed subset W in $\mathcal{A}_r^m$ and its open covering $\mathcal{W}$ it is possible to choose a finite open covering $\{W_j : j = 1, ..., l\}$ of W which refines $\mathcal{W}$, since W is compact. We take W_j being intersections of open balls in $\mathcal{A}_r^m$ with W. There exist constants $c_j > 0$ and $b_{k,j} > 0$ and $w_j \in \mathcal{A}_r^m$ such that

$$g_j(z) = \frac{c_j f_j(z)}{\sum_{j=1}^{l} c_j f_j(z)} \tag{8}$$

is positive and $\mathcal{A}_r$-differentiable on W and $g_j(z) < g_j(y)$ for each $z \in W \setminus W_j$

and $y \in W_j$. We can choose constants so that

$$c_1 g_j(z) > c_2 g_j(y) \tag{9}$$

for each $z \in \xi_j(E_j)$ and $y \in \xi_j(M \setminus H_j)$, where $c_1 = \inf\{|x| : \ x \in \xi_j(E_j)\}$ and $c_2 = \sup\{|x| : \ x \in \xi_j(M \setminus H_j)\}$.

Evidently, $g(z) = \sum_{j=1}^{l} g_j(z)$ is identically unit on $\mathcal{A}_r^m$. The product of $\mathcal{A}_r$-differentiable functions is $\mathcal{A}_r$-differentiable.

Using charts (E_j, ξ_j) and of the atlas of M, the open covering $\{H_j : j\}$ of M as above and such functions g_j one can choose $\mathcal{A}_r$-differentiable mappings ψ_j for each j so that $\psi_j(M) \subset V_k^m$, where either $k=1$ or $k=2$, U_j and A_j are open subsets in M with $U_j \subset A_j$ for each $j = 1, ..., n$, $\bigcup_{j=1}^n U_j = M$, $\psi_j|_{A_j}$ is bijective for each j, and

$$|\bar{\phi}_k \circ \psi_j(y)| < |\bar{\phi}_k \circ \psi_j(z)| \tag{10}$$

for each $z \in U_j$ and $y \in M \setminus A_j$, where $\bar{\phi}_k = (\phi_k, ..., \phi_k) : V_k^m \to \mathcal{A}_r^m$, while $\phi_k : V_k \to \mathcal{A}_r^m$ is given above.

The family of such component mappings ψ_j induces an $\mathcal{A}_r$-differentiable diffeomorphism: $\psi : M \to (S^{2^r m})^n$ with n equal to the number of charts, where $\psi(z) := (\psi_1(z), ..., \psi_n(z))$ for each $z \in M$.

Then the mapping $\psi(z)$ is the embedding into $(S^{2^r m})^n$ and hence into ${\mathcal{A}_r}^{n(m+1)}$, since the rank is $rank[d_z\psi(z)] = 2^r m$ at each point $z \in M$. Indeed, the rank is $rank[d_z\psi_j(z)] = 2^r m$ for each $z \in U_j$ and the dimension is bounded from above $dim_{\mathcal{A}_r}\psi(U_j) \le dim_{\mathcal{A}_r} M = m$. If y and z are two distinct points in M, then there exists j so that $z \in U_j$. If $y \in A_j$, then $\psi_j(z) \ne \psi(y)$, since $\psi_j|_{A_j}$ is bijective. If $y \in M \setminus A_j$, then from inequality (10) it follows, that $\psi_j(z) \ne \psi_j(y)$. Therefore, $\psi(z) \ne \psi(y)$ for each two distinct points z and y in M, since a natural number j exists so that $\psi_j(z) \ne \psi_j(y)$.

Let $M \hookrightarrow \mathcal{A}_r^N$ be the $\mathcal{A}_r$- differentiable embedding as above. There is also the $\mathcal{A}_r$- differentiable embedding of M into $(S^{2^r m})^n$ as it is shown above, where $(S^{2^r m})^n$ is the $\mathcal{A}_r$- differentiable manifold as the product of $\mathcal{A}_r$- differentiable manifolds.

Let $P\mathbf{R}^n$ denote the real projective space formed from the Euclidean space $\mathbf{R}^{n+1}$, denote by $\phi : \mathbf{R}^{n+1} \setminus \{0\} \to P\mathbf{R}^n$ the corresponding projective mapping. Geometrically $P\mathbf{R}^n$ is considered as S^n/τ, where $S^n := \{y \in \mathbf{R}^{n+1} : \ \|y\| = 1\}$ is the unit sphere in $\mathbf{R}^{n+1}$, while τ is the equivalence relation making identical two spherically symmetric points, i.e. points belonging to the same straight line containing zero and intersecting the unit sphere.

We consider $\mathcal{A}_r^n$ as the algebra of all $n \times n$ diagonal matrices

$$A = diag(a_1, ..., a_n)$$

with entries $a_1, ..., a_n \in \mathcal{A}_r$. It naturally has the structure of the left- and right- $\mathcal{A}_r$-module. Then $\mathcal{A}_r^n$ is isomorphic with the tensor product of algebras

$\mathcal{A}_r^n = \mathcal{A}_r \otimes_{\mathbf{R}} \mathbf{R}^n$ over the real field, where $\mathbf{R}^n$ is considered as the algebra of all diagonal $n \times n$ matrices $C = diag(b_1, .., b_n)$ with entries $b_1, ..., b_n \in \mathbf{R}$. Using this realization of $\mathcal{A}_r^n$ we get an extension of ϕ from $\mathbf{R}^{n+1}$ onto $\mathcal{A}_r \otimes_{\mathbf{R}} \mathbf{R}^{n+1}$ by the formulas:

$$\phi(ax) = a\phi(x) \tag{11}$$

and

$$\phi(xa) = \phi(x)a \tag{12}$$

for each $a \in \mathcal{A}_r$ with $|a| = 1$ and every $x \in \mathbf{R}^{n+1} \setminus \{0\}$, also

$$\phi(x_0 i_0 + ... + x_{2^r-1} i_{2^r-1}) = \phi(x_0) i_0 \alpha_0 + ... + \phi(x_{2^r-1}) i_{2^r-1} \alpha_{2^r-1} \tag{13}$$

for each non-zero vector $x = x_0 i_0 + ... + x_{2^r-1} i_{2^r-1} \in \mathcal{A}_r^{n+1}$, where $\alpha_j := \|x_j\|/\|x\|$, $x_j \in \mathbf{R}^{n+1}$ for each j, the norm is given by the usual formula

$$\|x\|^2 = \|x_0\|^2 + ... + \|x_{2^r-1}\|^2 . \tag{14}$$

Then we put by our definition $P\mathcal{A}_r^n = \phi([\mathcal{A}_r \otimes_{\mathbf{R}} \mathbf{R}^{n+1}] \setminus \{0\})$ to be the Cayley-Dickson projective space.

If $z \in P\mathcal{A}_r^n$, then by our definition $\phi^{-1}(z)$ is the $\mathcal{A}_r$ straight line in $\mathcal{A}_r^{n+1}$. To each element $x \in \mathcal{A}_r^{n+1}$ we pose an $\mathcal{A}_r$ straight line $< \mathcal{A}_r, x\} := \phi^{-1}(\phi(x))$. That is the bundle of all $\mathcal{A}_r$ straight lines $< \mathcal{A}_r, x\}$ in $\mathcal{A}_r^{n+1}$ is considered, where $x \in \mathcal{A}_r^{n+1}$, $x \neq 0$. Then $< \mathcal{A}_r, x\}$ is the $\mathcal{A}_r$ vector space of dimension 1 over $\mathcal{A}_r$ due to formulas (11)-(14) above. Therefore, $< \mathcal{A}_r, x\}$ has the real shadow isomorphic with $\mathbf{R}^{2^r}$, since the standard generators $i_0, i_1, ..., i_{2^r-1}$ are linearly independent over the real field $\mathbf{R}$.

Fix the standard orthonormal base $\{e_1, ..., e_N\}$ in $\mathcal{A}_r^N$ and projections on $\mathcal{A}_r$-vector subspaces relative to this base

$$P^L(x) := \sum_{e_j \in L} x_j e_j \tag{15}$$

for the $\mathcal{A}_r$ vector span $L = span_{\mathcal{A}_r}\{e_i : i \in \Lambda_L\}$, $\Lambda_L \subset \{1, ..., N\}$, where

$$x = \sum_{j=1}^N x_j e_j , \tag{16}$$

$x_j \in \mathcal{A}_r$ for each j, $e_j = (0, ..., 0, 1, 0, ..., 0)$ with 1 at j-th place. This means in particular that the projective space $P\mathcal{A}_r^n$ has the dimension $n-1$ over the Cayley-Dickson algebra $\mathcal{A}_r$. In this base consider the $\mathcal{A}_r$-Hermitian scalar product

$$< x, y > := \sum_{j=1}^N x_j^* y_j . \tag{17}$$

Let $l \in P\mathcal{A}_r^{N-1}$, take an $\mathcal{A}_r$-hyperplane denoted by $(\mathcal{A}_r^{N-1})_l$ and given by the condition:

$$< x, y >= 0 \text{ for each } x \in (\mathcal{A}_r^{N-1})_l \text{ and } y \in l . \tag{18}$$

We take a vector $0 \neq [l] \in \mathcal{A}_r^N$ as a representative characterizes the equivalence class $l =< \mathcal{A}_r, [l]\}$ of unit norm $\|[l]\| = 1$. Then the orthonormal base $\{q_1, ..., q_{N-1}\}$ in $(\mathcal{A}_r^{N-1})_l$ and the vector $[l] =: q_N$ compose the orthonormal base $\{q_1, ..., q_N\}$ in $\mathcal{A}_r^N$. This provides the $\mathcal{A}_r$- differentiable projection $\pi_l : \mathcal{A}_r^N \to (\mathcal{A}_r^{N-1})_l$ relative to the orthonormal base $\{q_1, ..., q_N\}$. Indeed, the operator π_l is $\mathcal{A}_r$ left $\pi_l(bx_0) = b\pi_l(x_0)$ and also right $\pi_l(x_0b) = \pi_l(x_0)b$ linear for each $x_0 \in X_0$ and $b \in \mathcal{A}_r$, but certainly non-linear relative to $\mathcal{A}_r$. Therefore the mapping π_l is $\mathcal{A}_r$- differentiable.

To construct an immersion it is sufficient, that each projection $\pi_l : T_xM \to (\mathcal{A}_r^{N-1})_l$ has $ker[d(\pi_l(x))] = \{0\}$ for each $x \in M$. The set of all points $x \in M$ for which $ker[d(\pi_l(x))] \neq \{0\}$ is called the set of forbidden directions of the first kind. Forbidden are those and only those directions $l \in P\mathcal{A}_r^{N-1}$ for which there exists a point $x \in M$ such that $l' \subset T_xM$, where $l' = [l] + z$, $z \in \mathcal{A}_r^N$. The set of all forbidden directions of the first kind forms the $\mathcal{A}_r$- differentiable manifold Q due to formulas (11)-(18) and (1)-(3).

This manifold Q consists of points (x, l) with $x \in M$ and $l \in P\mathcal{A}_r^{N-1}$ so that $[l] \in T_xM$. The manifold M is m-dimensional over the Cayley-Dickson algebra $\mathcal{A}_r$. The tangent bundle TM has the structure of an $\mathcal{A}_r$- differentiable manifold of dimension $2m$ over the Cayley-Dickson algebra $\mathcal{A}_r$ in accordance with Proposition 4 above. Each point x in the manifold M has an open neighborhood locally homeomorphic with an open neighborhood of zero in T_xM. Then $dim_{\mathcal{A}_r}T_xM = m$ and hence $P(T_xM)^2$ is isomorphic with $P\mathcal{A}_r^{2m-1}$. On the other hand, the dimension of the projective space $P\mathcal{A}_r^{2m-1}$ over the Cayley-Dickson algebra $\mathcal{A}_r$ is $2m-1$ (see also Formulas $(1-4)$). Therefore, the manifold Q has the $\mathcal{A}_r$ dimension $(2m-1)$. Take the mapping $g : Q \to P\mathcal{A}_r^{N-1}$ given by $g(x, l) := l$. Then this mapping g is $\mathcal{A}_r$- differentiable in view of Proposition 2.4 and formulas (11)-(18) and (1)-(3).

Each paracompact manifold A modeled on $\mathcal{A}_r^p$ can be supplied with the Riemann manifold structure also. Therefore, on a manifold A there exists a Riemann volume element. In view of the Morse theorem $\mu(g(Q)) = 0$, if $N - 1 > 2m - 1$, that is, $2m < N$, where μ is the Riemann volume element in $P\mathcal{A}_r^{N-1}$. In particular, $g(Q)$ is not equal to the whole $P\mathcal{A}_r^{N-1}$ and there exists $l_0 \notin g(Q)$, consequently, there exists $\pi_{l_0} : M \to (\mathcal{A}_r^{N-1})_{l_0}$. This procedure can be prolonged, when $2m < N - k$, where k is the number of the step of projection. Hence M can be immersed into $\mathcal{A}_r^{2m}$.

Consider now the forbidden directions of the second type: $l \in P\mathcal{A}_r^{N-1}$, for which there exist $x \neq y \in M$ simultaneously belonging to l after suitable parallel translation $[l] \mapsto [l] + z$, $z \in \mathcal{A}_r^N$. The set of the forbidden directions of the second type forms the manifold $\Phi := M^2 \setminus \Delta$, where $\Delta := \{(x, x) :$

$x \in M\}$. Consider $\psi : \Phi \to P\mathcal{A}_r^{N-1}$, where $\psi(x,y)$ is the straight $\mathcal{A}_r$-line with the direction vector $[x,y]$ in the orthonormal base. Then $\mu(\psi(\Phi)) = 0$ in $P\mathcal{A}_r^{N-1}$, if $2m+1 < N$. Then the closure $cl(\psi(\Phi))$ coincides with $\psi(\Phi) \cup g(Q)$ in $P\mathcal{A}_r^{N-1}$. Hence there exists $l_0 \notin cl(\psi(\Phi))$. Then consider $\pi_{l_0} : M \to (\mathcal{A}_r)_{l_0}^{N-1}$. This procedure can be prolonged, when $2m+1 < N-k$, where k is the number of the step of projection. Hence M can be embedded into $\mathcal{A}_r^{2m+1}$.

(II). Let now M be a paracompact $\mathcal{A}_r$- differentiable manifold with countable atlas on $l_2(\lambda, \mathbf{K})$. Spaces $l_2(\lambda, \mathcal{A}_r) \oplus \mathcal{A}_r^m$ and $l_2(\lambda, \mathcal{A}_r) \oplus l_2(\lambda, \mathcal{A}_r)$ are isomorphic as $\mathcal{A}_r$ Hilbert spaces with $l_2(\lambda, \mathcal{A}_r)$, since $card(\lambda) \ge \aleph_0$. Take an additional variable $z \in \mathcal{A}_r$, when $z = j \in \mathbf{N}$. Then it gives a number of a chart. Each TU_j is $\mathcal{A}_r$- differentiably diffeomorphic with $U_j \times l_2(\lambda, \mathcal{A}_r)$. Consider $\mathcal{A}_r$- differentiable functions ψ on domains in $l_2(\lambda, \mathcal{A}_r) \oplus l_2(\lambda, \mathcal{A}_r) \oplus \mathcal{A}_r$. Then there exists an $\mathcal{A}_r$- differentiable mapping $\psi_j : M \to l_2(\lambda, \mathcal{A}_r)$ such that $\psi_j : U_j \to \psi_j(U_j) \subset l_2(\lambda, \mathcal{A}_r)$ is an $\mathcal{A}_r$- differentiable diffeomorphism. Then the mapping $(\psi_1, \psi_2, ...)$ provides the $\mathcal{A}_r$- differentiable embedding of M into $l_2(\lambda, \mathcal{A}_r)$. $\square$

REMARK 2.7. *Theorem 2.6 is the extension of the immersion and embedding Whitney theorems to $\mathcal{A}_r$-differentiable manifolds (see also Theorems 1, 2 and Footnote 4 in [37]; or Theorems 1.3.4, 1.3.5 and Proposition 2.1.0 in [18]; or Theorem in §11 Chapter II.2 [4]).*

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A Lewy-Stampacchia estimate for variational inequalities in the Heisenberg group

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Abstract. *We consider an obstacle problem in the Heisenberg group framework, and we prove that the operator on the obstacle bounds pointwise the operator on the solution. More explicitly, if $\bar{u}$ minimizes the functional*

$$\int_{\Omega} |\nabla_{\mathbb{H}^n} u|^2$$

among the functions with prescribed Dirichlet boundary condition that stay below a smooth obstacle ψ, then

$$0 \leq \Delta_{\mathbb{H}^n} \bar{u} \leq \Big(\Delta_{\mathbb{H}^n} \psi \Big)^+.$$

Moreover, we discuss how it could be possible to generalize the previous bound to a quasilinear setting once some regularity issues for the equation

$$\mathrm{div}_{\mathbb{H}^n} \Big(|\nabla_{\mathbb{H}^n} u|^{p-2} \nabla_{\mathbb{H}^n} u \Big) = f$$

are satisfied.

Keywords: obstacle problem, dual estimates, Heisenberg group
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1. Introduction

In this paper, we extend the so called Dual Estimate of [11] to the obstacle problem for the Kohn-Laplacian operator in the Heisenberg group.

The notation we use is the standard one: for $n \geq 1$, we consider $\mathbb{R}^{2n+1}$

endowed with the group law

$$\begin{aligned}&\big(x^{(1)},y^{(1)},t^{(1)}\big)\circ\big(x^{(2)},y^{(2)},t^{(2)}\big)\\&\quad:=\Big(x^{(1)}+x^{(2)},y^{(1)}+y^{(2)},t^{(1)}+t^{(2)}+2(x^{(2)}\cdot y^{(1)}-x^{(1)}\cdot y^{(2)})\Big),\end{aligned}$$

for any $(x^{(1)},y^{(1)},t^{(1)})$, $(x^{(2)},y^{(2)},t^{(2)})\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, where the "$\cdot$" is the standard Euclidean scalar product.

Then, we denote by $\mathbb{H}^n$ the n-dimensional Heisenberg group, i.e., $\mathbb{R}^{2n+1}$ endowed with this group law.

The coordinates are usually written as $(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, and, as customary, we introduce the left invariant vector fields (X,Y) induced by the group law

$$X_j:=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t}\ \text{and}\ Y_j:=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t},$$

for $j=1,\dots,n$, and the horizontal gradient $\nabla_{\mathbb{H}^n}:=(X,Y)$. The main issue of the Heisenberg group is that X and Y do not commute, that is

$$[X,Y]=-4\frac{\partial}{\partial t}\neq 0.$$

We are interested in studying the obstacle problem in this framework. For this, we consider a smooth function $\psi:\mathbb{H}^n\to\mathbb{R}$, which will be our obstacle (more precisely, ψ is supposed to have continuous derivatives of second order in X and Y).

Fixed a bounded open set Ω with smooth boundary, and $p\in(1,+\infty)$, we consider the space $W^{1,p}_{\mathbb{H}^n}(\Omega)$ to be the set of all functions u in $L^p(\Omega)$ whose distributional horizontal derivatives X_ju and Y_ju belong to $L^p(\Omega)$, for $j=1,\dots,n$.

Such space is naturally endowed with the norm

$$\|u\|_{W^{1,p}_{\mathbb{H}^n}(\Omega)}:=\|u\|_{L^p(\Omega)}+\sum_{j=1}^n\Big(\|X_ju\|_{L^p(\Omega)}+\|Y_ju\|_{L^p(\Omega)}\Big).$$

We call $W^{1,p}_{\mathbb{H}^n,0}(\Omega)$ the closure of $C^\infty_0(\Omega)$ with respect to this norm.

We fix a smooth domain $\Omega_\star\Supset\Omega$, $u_\star\in W^{1,2}_{\mathbb{H}^n}(\Omega_\star)\cap L^\infty(\Omega_\star)$ and we introduce the space

$$\mathcal{K}:=\big\{u\in W^{1,2}_{\mathbb{H}^n}(\Omega)\ \text{s.t.}\ u\le\psi,\ \text{and}\ u-u_\star\in W^{1,2}_{\mathbb{H}^n,0}(\Omega)\big\}.\tag{1}$$

Loosely speaking, $\mathcal{K}$ is the space of all the functions having prescribed Dirichlet boundary datum equal to $u_\star$ along $\partial\Omega$ and that stay below the obstacle ψ.

We deal with the variational problem

$$\inf_{u\in\mathcal{K}}\mathcal{F}(u;\Omega),\ \text{where}\ \mathcal{F}(u;\Omega):=\int_\Omega|\nabla_{\mathbb{H}^n}u|^2.\tag{2}$$

By direct methods, it is seen that such infimum is attained (see, e.g., the compactness result in [18, 5] or references therein) and so we consider a minimizer $\bar{u}$. It is worth pointing out that such minimizer may be written in terms of a variational inequality, namely

$$\int_\Omega \nabla_{\mathbb{H}^n}\bar{u}\cdot\nabla_{\mathbb{H}^n}(v-\bar{u}) \,\geq\, 0, \tag{3}$$

for any $v \in W^{1,2}_{\mathbb{H}^n}(\Omega)$ with $v \leq \psi$, and $v-\bar{u} \in W^{1,2}_{\mathbb{H}^n,0}(\Omega)$. These kind of variational inequalities[1] are now receiving a considerable attention (see, e.g., [6] and references therein).

Our main result is:

Theorem 1.1. *Let $\bar{u}$ and ψ as above then*

$$0 \leq \Delta_{\mathbb{H}^n}\bar{u} \leq \Big(\Delta_{\mathbb{H}^n}\psi\Big)^+ \tag{4}$$

in the sense of distributions. As usual, the superscript "+" denotes the positive part of a function, i.e. $f^+(x) := \max\{f(x), 0\}$.

The result in Theorem 1.1 is quite intuitive: when $\bar{u}$ does not touch the obstacle, it is free to make the operator vanish. When it touches and sticks to it, the operator computed in $\bar{u}$ is driven by the positive part of the same operator computed in the obstacle – and on these touching points the obstacle has to bend in a somewhat convex fashion, which justifies the first inequality in (4) and superscript "+" in the right hand side of (4).

Figure 1, in which the thick curve represents the touching between $\bar{u}$ and the obstacle, tries to describe this phenomena. On the other hand, the set in which $\bar{u}$ touches the obstacle may be very wild, so the actual proof of Theorem 1.1 needs to be more technical than this.

In fact, the first inequality of (4) is quite obvious since it follows, for instance, by taking $v := \bar{u} - \varphi$ in (3), with an arbitrary $\varphi \in C^\infty_0(\Omega, [0, +\infty)))$, so the core of (4) lies on the second inequality: nevertheless, we think it is useful to write (4) in this way to emphasize a control from both the sides of the operator applied to the solution.

We remark that the right hand side of (4) is always finite (due to the regularity of the obstacle). Hence, (4) is an L^∞-bound and may be seen as a regularity result for the solution of the obstacle problem.

In the Euclidean setting, the analogue of (4) was first obtained in [11] for the Laplacian case, and it is therefore often referred to with the name of Lewy-Stampacchia Estimate. It is also called Dual Estimate, for it is, in a sense, obtained by the duality expressed by the variational inequality (3).

[1] The proof of (3) is standard. See however the footnote on page 27 for a general argument.

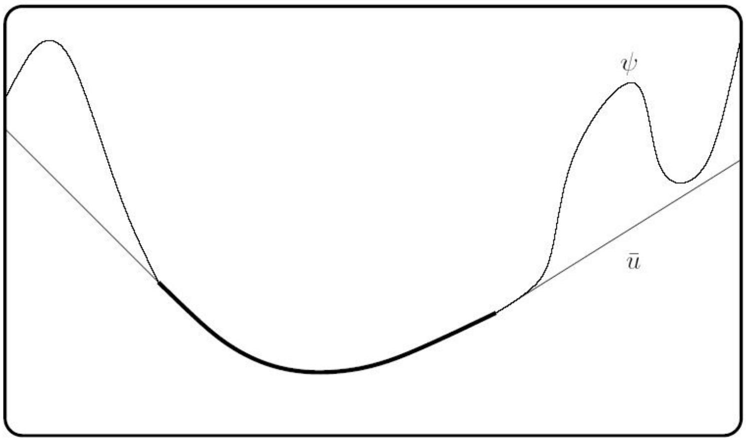

Figure 1: *Touching the obstacle*

After [11], estimates of these type became very popular and underwent many important extensions and strengthenings: see, among the others, [15, 9, 8, 1, 14].

The paper is organized as follows. First, in § 2, we discuss some possible extensions of Theorem 1.1 to the quasilinear case, once a more comprehensive regularity theory will become available. This will lead to a somewhat more general form of Theorem 1.1, namely Theorem 2.2 below (which will introduce an auxiliary parameter $\varepsilon \geq 0$). Then, in § 3, we prove Theorem 2.2 when $\varepsilon > 0$. The proof when $\varepsilon = 0$ is contained in § 4–5 and it is based on a limit argument, i.e., we consider the problem with $\varepsilon > 0$, we use Theorem 2.2 in such a case, and then we pass $\varepsilon \searrow 0$. The paper ends with an Appendix that collects some ancillary results needed in § 4.

2. Possible extension to the quasilinear case (waiting for a more exhaustive regularity theory)

Now we try to give some ideas of how Theorem 1.1 could be generalized to the quasilinear setting. In particular, we prove that for a suitable set of exponents $\mathcal{P}(\psi, \Omega)$ (see Definition 2.1 and Theorem 2.2) an analogue of Theorem 1.1 holds for the Heisenberg group version of the p-Laplace operator[2].

[2]We inform the reader that our result in Theorem 2.2 is far from being exhaustive in the quasilinear case, since, in principle, we are only able to prove explicitly that $2 \in \mathcal{P}(\psi, \Omega)$. The primary source of difficulties to decide whether $p \in \mathcal{P}(\psi, \Omega)$ is the absence of a satisfactory Hölder regularity theory for the horizontal gradient for solutions of quasilinear equations in

The notation we use is the following. Given $p \in (1, +\infty)$, a smooth domain $\Omega_\star \Supset \Omega$, $u_\star \in W^{1,p}_{\mathbb{H}^n}(\Omega_\star) \cap L^\infty(\Omega_\star)$ and $\varepsilon \geq 0$, we consider the minimization problem

$$\inf_{u \in \mathcal{K}_p} \mathcal{F}_\varepsilon(u;\Omega), \text{ where } \mathcal{F}_\varepsilon(u;\Omega) := \int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{p/2}, \tag{5}$$

and

$$\mathcal{K}_p := \big\{ u \in W^{1,p}_{\mathbb{H}^n}(\Omega) \text{ s.t. } u \leq \psi, \text{ and } u - u_\star \in W^{1,p}_{\mathbb{H}^n,0}(\Omega) \big\}. \tag{6}$$

By comparing (1) and (6), we observe that $\mathcal{K}_p$ reduces to $\mathcal{K}$ when $p = 2$. Hence, the minimization problem in (5) reduces to the one in (2) when $p = 2$ and $\varepsilon = 0$.

We notice that $\bar{u}_\varepsilon$ is a solution of the variational inequality[3]

$$\int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p-2)/2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot \nabla_{\mathbb{H}^n}(v - \bar{u}_\varepsilon) \ \geq \ 0, \tag{7}$$

for any $v \in W^{1,p}_{\mathbb{H}^n}(\Omega)$ with $v \leq \psi$, and $v - \bar{u}_\varepsilon \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$. Now, we introduce the set of p's for which our main result holds. The definition we give is slightly technical, but, roughly speaking, consists in taking the set of all the p's for

---

the Heisenberg group. Namely, if one knew that for a given p bounded solutions of $\mathrm{div}_{\mathbb{H}^n}\big((\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u\big) = f$, with f bounded, have Hölder continuous horizontal gradient, with interior estimates (this would be the Heisenberg counterpart of classical regularity results for the Euclidean case, see, e.g., Theorem 1 in [17]) then $p \in \mathcal{P}(\psi, \Omega)$. As far as we know, such a theory has not been developed yet, not even for minimal solutions (see, however, [3, 12, 13, 19] where good $C^{1,\alpha}$ estimates are proved for the case of homogeneous equations). On the other hand, we think it is worth pointing out how Theorem 1.1 could be generalized in the generality allowed by the set $\mathcal{P}(\psi, \Omega)$, since once the regularity theory becomes available, our result would be valid in general – and also because the setting we use is somewhat more general and weaker than the regularity theory itself.

We stress that the quasilinear case in the Heisenberg group is more problematic than expected at a first glance, and many basic fundamental questions are still open (see, e.g., [7], [12], [13] and [19]).

[3]Formula (7) may be easily obtained this way. Fixed $v \in W^{1,p}_{\mathbb{H}^n}(\Omega)$ with $v \leq \psi$, and $v - \bar{u}_\varepsilon \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$, for any $t \geq 0$, let $u^{(t)} := \bar{u}_\varepsilon + t(v - \bar{u}_\varepsilon)$. Notice that

$$u^{(t)} := (1-t)\bar{u}_\varepsilon + tv \leq (1-t)\psi + t\psi \leq \psi,$$

hence $u^{(t)} \in \mathcal{K}_p$. So, by the minimality of $\bar{u}_\varepsilon$, we have $\mathcal{F}_\varepsilon(u^{(0)};\Omega) = \mathcal{F}_\varepsilon(\bar{u}_\varepsilon;\Omega) \leq \mathcal{F}_\varepsilon(u^{(t)};\Omega)$ for any $t \geq 0$. Consequently,

$$\begin{aligned} 0 \quad &\leq \quad \lim_{t \searrow 0} \frac{\mathcal{F}_\varepsilon(u^{(t)};\Omega) - \mathcal{F}_\varepsilon(u^{(0)};\Omega)}{t} \\ &= \quad \int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p-2)/2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot \nabla_{\mathbb{H}^n}(v - \bar{u}_\varepsilon), \end{aligned}$$

that is (7). Once again, (7) reduces to (3) when $p = 2$ (and in this case ε does not play any role).

which a pointwise bound for the operator of a sequence of minimal solutions is stable under uniform limits.

DEFINITION 2.1. *Let $p \in (1, +\infty)$. We say that $p \in \mathcal{P}(\psi, \Omega)$ if the following property holds true: For any $\varepsilon > 0$, any $v \in W^{1,p}_{\mathbb{H}^n}(\Omega)$, any $M > 0$, any sequence $F_k = F_k(r, \xi) \in C([-M, M] \times \Omega)$, with $F_k(\cdot, \xi) \in C^1([-M, M])$ and*

$$0 \le \partial_r F_k \le \Big(\mathrm{div}_{\mathbb{H}^n} \ \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\Big)\Big)^+, \tag{8}$$

if $u_k : \Omega \to [-M, M]$ is a sequence of minimizers of the functional

$$\int_\Omega \frac{1}{p}(\varepsilon + |\nabla_{\mathbb{H}^n} u(\xi)|^2)^{p/2} + F_k(u(\xi), \xi)\, d\xi \tag{9}$$

over the functions $u \in W^{1,p}_{\mathbb{H}^n}(\Omega)$, $u - v \in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$, with the property that u_k converges to some u_∞ uniformly in Ω, we have that

$$\begin{aligned} 0 &\le \mathrm{div}_{\mathbb{H}^n} \ \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\infty|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} u_\infty\Big) \\ &\le \Big(\mathrm{div}_{\mathbb{H}^n} \ \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\Big)\Big)^+ \end{aligned} \tag{10}$$

in the sense of distributions.

As remarked in Lemma 5.7 at the end of this paper, we always have that

$$2 \in \mathcal{P}(\psi, \Omega). \tag{11}$$

We think that it is an interesting open problem to decide whether or not other values of p belong to $\mathcal{P}(\psi, \Omega)$, in general, or at least when the right hand side of (10) is close to zero (e.g., when the obstacle is almost flat).

With this notation, the following result holds true:

THEOREM 2.2. *If $p \in \mathcal{P}(\psi, \Omega)$ then*

$$\begin{aligned} 0 &\le \mathrm{div}_{\mathbb{H}^n} \ \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon\Big) \\ &\le \Big(\mathrm{div}_{\mathbb{H}^n} \ \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\Big)\Big)^+ \end{aligned} \tag{12}$$

in the sense of distributions.

Notice that Theorem 1.1 is a particular case of Theorem 2.2 when $p = 2$, thanks to (11). Therefore, in the sequel, we will prove Theorem 2.2 and so Theorem 1.1 will follow as a consequence.

We point out that the arguments that we present are step-free, i.e. they do not directly depend on the stratification step of $\mathbb{H}^n$ apart from the definition of the homogeneous dimension. More precisely, since our arguments are based only on the intrinsic gradient concept $\nabla_{\mathbb{H}^n}$ and the homogeneous dimension of $\mathbb{H}^n$, we can restate Theorems 1.1 and 2.2 in any nilpotent stratified Lie groups of any step $\mathbb{G}$ simply changing $\mathrm{div}_{\mathbb{H}^n}$ and $\nabla_{\mathbb{H}^n}$ with $\mathrm{div}_{\mathbb{G}}$ and $\nabla_{\mathbb{G}}$. Here $\mathrm{div}_{\mathbb{G}}$ and $\nabla_{\mathbb{G}}$ are respectively the intrinsic divergence and the intrinsic gradient in $\mathbb{G}$ (see [18]). So here we work in $\mathbb{H}^n$ only for the sake of notational simplicity.

3. Proof of Theorem 2.2 when $\varepsilon > 0$

We prove (12) in the simpler case $\varepsilon > 0$ (the case $\varepsilon = 0$ will be dealt with in § 5). The technique used in this proof is a variation of a classical penalized test function method (see, e.g., [15, 9, 8, 1, 14] and references therein), and several steps of this proof are inspired by some estimates obtained by [4] in the Euclidean case.

First of all, we set

$$\mu := -1 + \min\left\{ \inf_{\overline{\Omega}} \psi,\ \inf_{\overline{\Omega}} u_\star \right\} \in \mathbb{R}$$

and we observe that

$$\bar{u}_\varepsilon \geq \mu \tag{13}$$

a.e. in Ω. Indeed, let $w := \max\{\bar{u}_\varepsilon, \mu\}$. Since ψ and $u_\star$ are below μ in Ω, we have that $w \in \mathcal{K}$, thus

$$0 \leq \mathcal{F}_\varepsilon(w;\Omega) - \mathcal{F}_\varepsilon(\bar{u}_\varepsilon;\Omega) = -\int_{\Omega\cap\{\bar{u}_\varepsilon<\mu\}} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar{u}_\varepsilon|^2)^{p/2} \leq 0,$$

and, from this, (13) plainly follows.

Now, let $\eta \in (0,1)$, to be taken arbitrarily small in the sequel. Let also

$$h := \Big(\mathrm{div}_{\mathbb{H}^n}\ \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\Big)\Big)^+. \tag{14}$$

Notice that

$$\|h\|_{L^\infty(\Omega)} < +\infty, \tag{15}$$

because $\varepsilon > 0$. For any $t \in \mathbb{R}$, we consider the truncation function

$$H_\eta(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ t/\eta & \text{if } 0 < t < \eta, \\ 1 & \text{if } t \geq \eta. \end{cases}$$

Now, we take u_η to be a weak solution of

$$\begin{cases} \mathrm{div}_{\mathbb{H}^n}\ \Big((\varepsilon + |\nabla_{\mathbb{H}^n}u_\eta|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}u_\eta\Big) = h\cdot\big(1 - H_\eta(\psi - u_\eta)\big) & \text{in } \Omega, \\ u_\eta = \bar{u}_\varepsilon & \text{on } \partial\Omega. \end{cases} \tag{16}$$

where, as usual, the boundary datum is attained in the trace sense: such a function u_η may be obtained by the direct method in the calculus of variations, by minimizing the functional

$$\int_\Omega \frac{1}{p}(\varepsilon+|\nabla_{\mathbb{H}^n}u(\xi)|^2)^{p/2}+F_\eta(u(\xi),\xi)\,d\xi$$

over $u\in W^{1,p}_{\mathbb{H}^n}(\Omega)$, $u-\bar u_\varepsilon\in W^{1,p}_{\mathbb{H}^n,0}(\Omega)$, where

$$F_\eta(r,\xi):=\int_0^r h(\xi)\cdot\big(1-H_\eta(\psi(\xi)-\sigma)\big)\,d\sigma.$$

Now, we claim that

$$u_\eta\le\psi \text{ a.e. in } \Omega. \tag{17}$$

To establish this, we use the test function $(u_\eta-\psi)^+$ in (16). Since, on $\partial\Omega$, we have $(u_\eta-\psi)^+=(\bar u_\varepsilon-\psi)^+=0$, we obtain that

$$\begin{aligned}&-\int_\Omega\Big((\varepsilon+|\nabla_{\mathbb{H}^n}u_\eta|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}u_\eta\Big)\cdot\nabla_{\mathbb{H}^n}(u_\eta-\psi)^+\\ =\;&\int_\Omega h\cdot\big(1-H_\eta(\psi-u_\eta)\big)(u_\eta-\psi)^+=\int_\Omega h\cdot(u_\eta-\psi)^+.\end{aligned}$$

Consequently, by (14),

$$\begin{aligned}&\int_\Omega\Big[\Big((\varepsilon+|\nabla_{\mathbb{H}^n}u_\eta|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}u_\eta\Big)\\ &\qquad-\Big((\varepsilon+|\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\Big)\Big]\cdot\nabla_{\mathbb{H}^n}(u_\eta-\psi)^+\\ =\;&\int_\Omega\Big[\mathrm{div}_{\mathbb{H}^n}\,\Big((\varepsilon+|\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\Big)-h\Big]\cdot(u_\eta-\psi)^+\\ \le\;&0.\end{aligned}$$

By the strict monotonicity of the operator (i.e., by the strict convexity of the function $\mathbb{R}^{2n}\ni\zeta\mapsto(\varepsilon+|\zeta|^2)^{p/2}$), it follows that $(u_\eta-\psi)^+$ vanishes almost everywhere in Ω, proving (17).

Now, we claim that

$$\bar u_\varepsilon\ge u_\eta \text{ a.e. in } \Omega. \tag{18}$$

To verify this, we consider the test function

$$\tau:=\bar u_\varepsilon+(u_\eta-\bar u_\varepsilon)^+.$$

We notice that

$$\tau=\begin{cases}\bar u_\varepsilon & \text{in } \{u_\eta\le\bar u_\varepsilon\},\\ u_\eta & \text{in } \{u_\eta>\bar u_\varepsilon\},\end{cases}$$

hence $\tau \le \psi$, due to (17). Furthermore, on $\partial\Omega$, we have that $\tau = \bar u_\varepsilon$, due to the boundary datum in (16). Therefore the obstacle problem variational inequality (3) gives that

$$\begin{aligned} 0 &\le \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar u_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (\tau - \bar u_\varepsilon) \\ &= \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar u_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (u_\eta - \bar u_\varepsilon)^+. \end{aligned} \tag{19}$$

On the other hand, from (16),

$$\begin{aligned} &\int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \Big) \cdot \nabla_{\mathbb{H}^n} (u_\eta - \bar u_\varepsilon)^+ \\ &\qquad = - \int_\Omega h \cdot \big(1 - H_\eta(\psi - u_\eta) \big) \cdot (u_\eta - \bar u_\varepsilon)^+ \le 0. \end{aligned} \tag{20}$$

By (19) and (20), we obtain that

$$\begin{aligned} &\int_\Omega \Big[\Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \Big) \\ &\qquad - \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar u_\varepsilon \Big) \Big] \cdot \nabla_{\mathbb{H}^n} (u_\eta - \bar u_\varepsilon)^+ \le 0. \end{aligned}$$

This and the strict monotonicity of the operator implies that $(u_\eta - \bar u_\varepsilon)^+$ vanishes almost everywhere in Ω, hence proving (18).

Now, we claim that

$$\bar u_\varepsilon \le u_\eta + \eta \text{ in } \Omega. \tag{21}$$

To do this, we set

$$\theta := \bar u_\varepsilon - (\bar u_\varepsilon - u_\eta - \eta)^+.$$

Notice that $\theta \le \bar u_\varepsilon \le \psi$, and also that, on $\partial\Omega$, $\theta = \bar u_\varepsilon$. As a consequence, (3) gives that

$$\begin{aligned} 0 &\le \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar u_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (\theta - \bar u_\varepsilon) \\ &= - \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar u_\varepsilon \Big) \cdot \nabla_{\mathbb{H}^n} (\bar u_\varepsilon - u_\eta - \eta)^+. \end{aligned} \tag{22}$$

On the other hand, $(\bar u_\varepsilon - u_\eta - \eta)^+ = 0$ on $\partial\Omega$, and

$$\begin{aligned} \{\bar u_\varepsilon - u_\eta - \eta > 0\} &\subseteq \{\psi - u_\eta > \eta\} \\ &\subseteq \big\{ 1 - H_\eta(\psi - u_\eta) = 0 \big\}, \end{aligned}$$

and therefore, by (16),

$$\begin{aligned}
&\int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n}(u_\eta+\eta)|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n}(u_\eta+\eta) \Big) \cdot \nabla_{\mathbb{H}^n}(\bar u_\varepsilon - u_\eta - \eta)^+ \\
&= \int_\Omega \Big((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \Big) \cdot \nabla_{\mathbb{H}^n}(\bar u_\varepsilon - u_\eta - \eta)^+ \qquad (23)\\
&= - \int_\Omega h \cdot \big(1 - H_\eta(\psi - u_\eta)\big) \cdot (\bar u_\varepsilon - u_\eta - \eta)^+ = 0.
\end{aligned}$$

Then, (22) and (23) yield that

$$\begin{aligned}
&\int_\Omega \Big[\Big((\varepsilon + |\nabla_{\mathbb{H}^n} \bar u_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar u_\varepsilon \Big) \\
&\qquad - \Big((\varepsilon + |\nabla_{\mathbb{H}^n}(u_\eta+\eta)|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n}(u_\eta+\eta) \Big) \Big] \cdot \nabla_{\mathbb{H}^n}(\bar u_\varepsilon - u_\eta - \eta)^+ \\
&\qquad \le 0.
\end{aligned}$$

Thus, in this case, the strict monotonicity of the operator implies that $(\bar u_\varepsilon - u_\eta - \eta)^+$ vanishes almost everywhere in Ω, and so (21) is established.

In particular, by (17), (21) and (13),

$$\|u_\eta\|_{L^\infty(\Omega)} \le 2 + \|\psi\|_{L^\infty(\Omega)} + \|u_\star\|_{L^\infty(\Omega)}. \qquad (24)$$

Moreover, by (18) and (21), we have that

$$u_\eta \text{ converges uniformly in } \Omega \text{ to } \bar u_\varepsilon \qquad (25)$$

as $\eta \searrow 0$.

Furthermore

$$0 \le \partial_r F_\eta(r,\xi) \le h(\xi) = \Big(\mathrm{div}_{\mathbb{H}^n} \ \Big((\varepsilon + |\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n}\psi \Big) \Big)^+$$

hence (12) follows[4] from (25) and the fact that $p \in \mathcal{P}(\psi,\Omega)$ (recall (10) in Definition 2.1). □

4. Estimating the L^p-distance between $\nabla_{\mathbb{H}^n}\bar u_0$ and $\nabla_{\mathbb{H}^n}\bar u_\varepsilon$

The purpose of this section is to consider the solution $\bar u_\varepsilon$ of the ε-problem and the solution $\bar u_0$ of the problem with $\varepsilon = 0$, and to bound the L^p-norm of $|\nabla_{\mathbb{H}^n}\bar u_0 - \nabla_{\mathbb{H}^n}\bar u_\varepsilon|$. Such estimate is quite technical and it is different according to the cases $p \in (1,2]$ and $p \in [2,+\infty)$: see the forthcoming Propositions 4.1 and 4.2.

[4]It is worth pointing out that this is the only place in the paper where we use the condition that $p \in \mathcal{P}(\psi,\Omega)$. In particular, all the estimates in § 4 are valid for all $p \in (1,+\infty)$.

As a matter of fact, we think that the estimates proved in Propositions 4.1 and 4.2 are of independent interest, since they also allow to get around the more difficult (and in general not available in the Heisenberg group) Hölder-type estimates.

We recall the standard notation of balls in the Heisenberg group (in fact, we deal with the so called Folland-Korány balls, but the Carnot-Carathéodory balls would be good for our purposes too). For all $\xi := (z,t) \in \mathbb{R}^{2n} \times \mathbb{R}$, we define

$$\|\xi\|_{\mathbb{H}^n} := \sqrt[4]{|z|^4 + t^2}.$$

Then, for any $r > 0$, we set

$$\mathcal{B}_r := \left\{ \xi \in \mathbb{R}^{2n+1} \text{ s.t. } \|\xi\|_{\mathbb{H}^n} < r \right\}.$$

We denote by $\mathcal{L}$ the $(2n+1)$-dimensional Lebesgue measure, and we observe that $\mathcal{L}(\mathcal{B}_r)$ equals, up to a multiplicative constant r^Q, where $Q := 2(n+1)$ is the homogeneous dimension of $\mathbb{H}^n$. Also, for all $g \in L^1(\mathcal{B}_r)$, we define the average of g in $\mathcal{B}_r$ as

$$(g)_r := \frac{1}{\mathcal{L}(\mathcal{B}_r)} \int_{\mathcal{B}_r} g.$$

In what follows, we focus on L^p-estimates around a fixed point, say $\xi_\star$, of Ω. Without loss of generality, we take $\xi_\star$ to be the origin, and we fix $R \in (0,1)$ so small that $\mathcal{B}_R \Subset \Omega$.

Then, we denote by $\bar{u}_0 : \Omega \to \mathbb{R}$ the minimizer of problem (2) with $\varepsilon = 0$. Then, for a fixed $\varepsilon > 0$, we take $\bar{u}_\varepsilon : \mathcal{B}_R \to \mathbb{R}$ to be the minimizer of $\mathcal{F}_\varepsilon(u; \mathcal{B}_R)$ among all the functions $u \in W^{1,p}_{\mathbb{H}^n}(\mathcal{B}_R)$, $u \le \psi$, and $u - \bar{u}_0 \in W^{1,p}_{\mathbb{H}^n,0}(\mathcal{B}_R)$. We can then extend $\bar{u}_\varepsilon$ also on $\Omega \setminus \mathcal{B}_R$ by setting it equal to $\bar{u}_0$ in such a set. By construction

$$\begin{aligned} \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p &= \mathcal{F}_0(\bar{u}_0; \Omega) - \int_{\Omega \setminus \mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p \\ &\le \mathcal{F}_0(\bar{u}_\varepsilon; \Omega) - \int_{\Omega \setminus \mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p = \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \end{aligned} \tag{26}$$

and

$$\begin{aligned} \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} &= \mathcal{F}_\varepsilon(\bar{u}_\varepsilon; \mathcal{B}_R) \\ &\le \mathcal{F}_\varepsilon(\bar{u}_0; \mathcal{B}_R) = \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{p/2}. \end{aligned} \tag{27}$$

PROPOSITION 4.1. *Assume that*

$$p \in (1, 2]. \tag{28}$$

Then, there exists $C>0$, only depending on n and p, such that

$$\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar u_0 - \nabla_{\mathbb{H}^n}\bar u_\varepsilon|^p \le C\big(1+(|\nabla_{\mathbb{H}^n}\bar u_0|^p)_R\big)^{1-(p/2)}\varepsilon^{(p/2)^2}R^Q. \tag{29}$$

Proof. The technique for this proof is inspired by the one of Lemma 2.3 of [16], where a similar result was obtained in the quasilinear Euclidean case (however, our proof is self-contained). We have

$$\begin{aligned} |\nabla_{\mathbb{H}^n}\bar u_\varepsilon - \nabla_{\mathbb{H}^n}\bar u_0|^2 &\le \big(|\nabla_{\mathbb{H}^n}\bar u_\varepsilon| + |\nabla_{\mathbb{H}^n}\bar u_0|\big)^2 \\ &\le C\big(|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2 + |\nabla_{\mathbb{H}^n}\bar u_0|^2\big). \end{aligned} \tag{30}$$

Here, C is a positive constant, which is free to be different from line to line. By (28), (27) and (30), we obtain

$$\begin{aligned} &\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2 + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)-1}|\nabla_{\mathbb{H}^n}\bar u_\varepsilon - \nabla_{\mathbb{H}^n}\bar u_0|^2 \\ \le{}& C\int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2 + |\nabla_{\mathbb{H}^n}\bar u_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2 + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{1-(p/2)}} \\ ={}& C\Bigg(\int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2 + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{1-(p/2)}} \\ &\quad + \int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n}\bar u_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2 + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{1-(p/2)}}\Bigg) \\ \le{}& C\Bigg(\int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{1-(p/2)}} + \int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n}\bar u_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2)^{1-(p/2)}}\Bigg) \\ \le{}& C\Bigg(\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{p/2} + \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2)^{p/2}\Bigg) \\ \le{}& C\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2)^{p/2}. \end{aligned} \tag{31}$$

Thus, (31) and Lemma 5.4, applied here with $a := \nabla_{\mathbb{H}^n}\bar u_0$ and $b := \nabla_{\mathbb{H}^n}\bar u_\varepsilon$, yield that

$$\begin{aligned} &\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2 + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{p/2} \\ &\qquad\le C\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2 + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)-1}|\nabla_{\mathbb{H}^n}\bar u_\varepsilon - \nabla_{\mathbb{H}^n}\bar u_0|^2 + \\ &\qquad\qquad + C\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2)^{(p/2)} \\ &\qquad\le C\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_0|^2)^{(p/2)}. \end{aligned} \tag{32}$$

Now, from (26),

$$\begin{aligned}\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_0|^2)^{(p/2)}-\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)}\\ \le\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_0|^2)^{(p/2)}-\int_{\mathcal{B}_R}|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^p\\ \le\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_0|^2)^{(p/2)}-\int_{\mathcal{B}_R}|\nabla_{\mathbb{H}^n}\bar u_0|^p.\end{aligned}\tag{33}$$

Moreover, using (28) and some elementary calculus, we see that

$$|(1+\tau)^{p/2}-\tau^{p/2}|\le C$$

for any $\tau\ge 0$. Therefore, taking $\tau:=\theta/\varepsilon$, we obtain that

$$|(\varepsilon+\theta)^{p/2}-\theta^{p/2}|\le C\varepsilon^{p/2}\tag{34}$$

for any $\theta\ge 0$. Thus, using (33) and (34) with $\theta:=|\nabla_{\mathbb{H}^n}\bar u_0|^2$, we conclude that

$$\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_0|^2)^{(p/2)}-\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)}\le C\varepsilon^{p/2}R^Q.\tag{35}$$

Now, we estimate the left hand side of (35) from below. For this scope, we define

$$h:=t\nabla_{\mathbb{H}^n}\bar u_0+(1-t)\nabla_{\mathbb{H}^n}\bar u_\varepsilon,$$

$$J:=p\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar u_\varepsilon\cdot(\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon)$$

and

$$\tilde J:=p\int_{\mathcal{B}_R}\Big[\int_0^1(1-t)\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1}h\cdot(\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon)\Big)\,dt\Big].$$

We observe that the variational inequality in (3) for $\bar u_\varepsilon$ gives that

$$J\ge 0.\tag{36}$$

Also, using the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned}&\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_0|^2)^{(p/2)}-\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)}\\ &=\int_{\mathcal{B}_R}\Big[\int_0^1\frac{d}{dt}(\varepsilon+|t\nabla_{\mathbb{H}^n}\bar u_0+(1-t)\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)}\,dt\Big]\\ &=p\int_{\mathcal{B}_R}\Big[\int_0^1(\varepsilon+|t\nabla_{\mathbb{H}^n}\bar u_0+(1-t)\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)-1}\\ &\qquad\times(t\nabla_{\mathbb{H}^n}\bar u_0+(1-t)\nabla_{\mathbb{H}^n}\bar u_\varepsilon)\cdot(\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon)\,dt\Big]\\ &=p\int_{\mathcal{B}_R}\Big[\int_0^1(\varepsilon+|h|^2)^{(p/2)-1}h\cdot(\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon)\,dt\Big].\end{aligned}$$

Integrating by parts the latter integral in t (by writing $dt = \frac{d}{dt}(t-1)\,dt$), and exploiting (36), we obtain

$$\begin{aligned}\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)} \\ = J + \tilde{J} \geq \tilde{J}.\end{aligned} \tag{37}$$

Making use of Lemma 5.3 – applied here with $a := \nabla_{\mathbb{H}^n} \bar{u}_0$ and $b := \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$ – we have that

$$\tilde{J} \geq \frac{1}{C} \int_{\mathcal{B}_R} \Big[\int_0^1 (1-t)(\varepsilon + |t\nabla_{\mathbb{H}^n} \bar{u}_0 + (1-t)\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 \, dt \Big].$$

From this and Lemma 5.5 – applied here with $\kappa := 1$ and $\Psi(x) := x^{1-(p/2)}$, which is nondecreasing, thanks to (28) – we deduce that

$$\tilde{J} \geq \frac{1}{C} \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2. \tag{38}$$

In view of (35), (37) and (38), we conclude that

$$\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 \leq C\varepsilon^{p/2} R^Q. \tag{39}$$

Then, (29) follows from (32), (39) and Lemma 5.6, applied here with $f := \nabla_{\mathbb{H}^n} \bar{u}_0$ and $g := \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$. □

In the degenerate case $p \in [2, +\infty)$ the estimate obtained in Proposition 4.1 for the singular case $p \in (1, 2]$ needs to be modified according to the following result:

Proposition 4.2. *Suppose that*

$$p \in [2, +\infty). \tag{40}$$

Then, there exists $C > 0$, only depending on n and p, such that

$$\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \leq C\big(1 + (|\nabla_{\mathbb{H}^n} \bar{u}_0|^p)_R\big)^{1-(1/p)} \varepsilon R^Q.$$

Proof. The variational inequalities (3) for $\bar{u}_0$ and $\bar{u}_\varepsilon$ imply that

$$\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^{p-2} \nabla_{\mathbb{H}^n} \bar{u}_0 \cdot (\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0) \geq 0$$

and

$$\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot (\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon) \geq 0.$$

Consequently,

$$\int_{\mathcal{B}_R} \Big(|\nabla_{\mathbb{H}^n}\bar u_0|^{p-2}\nabla_{\mathbb{H}^n}\bar u_0 - (\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar u_\varepsilon\Big)\cdot(\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon)\le 0.$$

Using this and (46) of Lemma 5.1, applied here with $A := \nabla_{\mathbb{H}^n}\bar u_0$ and $B := \nabla_{\mathbb{H}^n}\bar u_\varepsilon$, we obtain

$$\begin{aligned}
&\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^p\\
&\le C\int_{\mathcal{B}_R}\Big(|\nabla_{\mathbb{H}^n}\bar u_0|^{p-2}\nabla_{\mathbb{H}^n}\bar u_0 - |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^{p-2}\nabla_{\mathbb{H}^n}\bar u_\varepsilon\Big)\cdot(\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon)\\
&\le C\int_{\mathcal{B}_R}\Big((\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar u_\varepsilon - |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^{p-2}\nabla_{\mathbb{H}^n}\bar u_\varepsilon\Big)\cdot(\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon).
\end{aligned}$$

This and Corollary 5.2, applied here with $a := \nabla_{\mathbb{H}^n}\bar u_\varepsilon$, give

$$\begin{aligned}
&\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^p\\
&\le C\int_{\mathcal{B}_R}\Big((\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)-1} - |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^{p-2}\Big)\,|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|\,|\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon|\\
&\le C\varepsilon\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p-2)/2}\big(|\nabla_{\mathbb{H}^n}\bar u_0|+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|\big).
\end{aligned}$$

Therefore, recalling (40), noticing that

$$\frac{p-2}{p}+\frac{1}{p}+\frac{1}{p}=1$$

and using the Generalized Hölder Inequality with the three exponents $p/(p-2)$, p and p, we obtain

$$\begin{aligned}
&\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar u_0-\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^p\\
&\le C\varepsilon\left(\int_{\mathcal{B}_R}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{p/2}\right)^{(p-2)/p}\left(\int_{\mathcal{B}_R}\big(|\nabla_{\mathbb{H}^n}\bar u_0|^p+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^p\big)\right)^{1/p} R^{Q/p}.
\end{aligned}$$

Then, by the minimal property of $\bar u_0$ in (26),

$$\begin{aligned}
&\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar u_0 - \nabla_{\mathbb{H}^n}\bar u_\varepsilon|^p \\
&\le C\varepsilon \left(\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{p/2}\right)^{(p-2)/p} \left(\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^p\right)^{1/p} R^{Q/p} \\
&\le C\varepsilon \left(\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{p/2}\right)^{(p-1)/p} R^{Q/p} \\
&\le C\varepsilon \left(R^Q + \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^p\right)^{(p-1)/p} R^{Q/p} \\
&\le C\varepsilon \left(R^Q + \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n}\bar u_0|^p\right)^{(p-1)/p} R^{Q/p},
\end{aligned}$$

from which the desired result follows. □

Corollary 4.3. *For all $p \in (1,+\infty)$, we have that*

$$\lim_{\varepsilon\searrow 0} \|\nabla_{\mathbb{H}^n}\bar u_\varepsilon - \nabla_{\mathbb{H}^n}\bar u_0\|_{L^p(\mathcal{B}_R)} = 0. \tag{41}$$

Also, there exist a subsequence of ε's and a function $G \in L^p(\mathcal{B}_R)$ such that

$$|\nabla_{\mathbb{H}^n}\bar u_\varepsilon(x)| \le G(x) \tag{42}$$

for almost every $x \in \mathcal{B}_R$.

Furthermore, if we set

$$\Gamma_\varepsilon := (\varepsilon + |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\bar u_\varepsilon, \tag{43}$$

then there exist a subsequence of ε's and a function $G_\star \in L^1(\mathcal{B}_R)$ such that

$$|\Gamma_\varepsilon(x)| \le G_\star(x) \tag{44}$$

for almost every $x \in \mathcal{B}_R$.

Proof. We obtain (41) from Propositions 4.1 and 4.2, according to whether $p \in (1,2)$ or $p \in [2,+\infty)$.

From (41), one deduces (42) (see, e.g., Theorem 4.9(b) in [2]).

Now, we define $G_\star := 2^{(p/2)}(G + G^{p-1})$. We observe that $G_\star \in L^1(\mathcal{B}_R)$, since $G \in L^p(\mathcal{B}_R) \subseteq L^1(\mathcal{B}_R)$ and $G^{p-1} \in L^{p/(p-1)}(\mathcal{B}_R) \subseteq L^1(\mathcal{B}_R)$. So, in order to obtain the desired result, we have only to show that the inequality in (44) holds true.

For this, we notice that, if $p \in (1,2)$,

$$\begin{aligned}|\Gamma_\varepsilon| &= \frac{|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|}{(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{1-(p/2)}} = \frac{|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^{p-1}|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^{2-p}}{(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{1-(p/2)}}\\ &\le \frac{|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^{p-1}(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{(2-p)/2}}{(\varepsilon+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^2)^{1-(p/2)}} = |\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^{p-1} \le G^{p-1},\end{aligned}$$

which implies (44) in this case.

On the other hand, if $p \in [2,+\infty)$,

$$\begin{aligned}|\Gamma_\varepsilon| &\le 2^{(p/2)-1}\big(\varepsilon^{(p/2)-1}+|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|^{p-2}\big)\,|\nabla_{\mathbb{H}^n}\bar u_\varepsilon|\\ &\le 2^{(p/2)-1}(1+G^{p-2})G,\end{aligned}$$

which implies (44) in this case too. □

5. Proof of Theorem 2.2 when $\varepsilon = 0$

By Theorem 2.2 (for $\varepsilon > 0$, which has been proved in § 3), we know that, for a sequence $\varepsilon \searrow 0$,

$$0 \le \int_{\mathcal{B}_R} \Gamma_\varepsilon \cdot \nabla\varphi \le \int_{\mathcal{B}_R} \Big(\operatorname{div}_{\mathbb{H}^n}\ \Big((\varepsilon+|\nabla_{\mathbb{H}^n}\psi|^2)^{(p/2)-1}\nabla_{\mathbb{H}^n}\psi\Big)\Big)^+ \varphi, \qquad (45)$$

for any $\varphi \in C_0^\infty(\mathcal{B}_R,[0,+\infty))$, as long as $\mathcal{B}_R \subset \Omega$, where Γ_ε is as in (43).

By possibly taking subsequences, in the light of (41) and (44), we have that

$$\lim_{\varepsilon\searrow 0}\Gamma_\varepsilon = |\nabla_{\mathbb{H}^n}\bar u_0|^{p-2}\nabla_{\mathbb{H}^n}\bar u_0$$

almost everywhere in $\mathcal{B}_R$, and that Γ_ε is equidominated in $L^1(\mathcal{B}_R)$. Consequently, we can pass to the limit in (45) via the Dominated Convergence Theorem and obtain (12) for $\bar u_0$. This completes the proof of Theorem 2.2 also when $\varepsilon = 0$. □

Appendix

In this appendix, we collect some technical and well known estimates of general interest that will be used in the proofs of the main results of this paper.

We start with some classical estimates (see, e.g. Lemma 3 in [10] and references therein), which turns out to be quite useful to deal with nonlinear operators of degenerate type:

Lemma 5.1. *Let $M \in \mathbb{N}$, $M \geq 1$, and $p \in [2, +\infty)$. Then, there exists $C > 1$, only depending on M and p, such that, for any A, $B \in \mathbb{R}^M$,*

$$|A - B|^p \leq C\Big(|A|^{p-2}A - |B|^{p-2}B\Big) \cdot (A - B) \tag{46}$$

and

$$\Big||A|^{p-2}A - |B|^{p-2}B\Big| \leq C|A - B| \left(|A|^{p-2} + |B|^{p-2}\right). \tag{47}$$

Corollary 5.2. *Let $N \in \mathbb{N}$ and and $p \in [2, +\infty)$. Then, there exists $C > 1$, only depending on N and p, such that for any $\varepsilon > 0$ and any $a \in \mathbb{R}^N$*

$$\left((\varepsilon + |a|^2)^{(p/2)-1} - |a|^{p-2}\right)|a| \leq C\varepsilon(\varepsilon + |a|^2)^{(p-2)/2}.$$

Proof. We let $A := (a, \varepsilon)$ and $B := (a, 0) \in \mathbb{R}^{N+1}$ and we exploit (47). We obtain

$$\begin{aligned}
&2C\varepsilon(\varepsilon + |a|^2)^{(p-2)/2} \\
&\qquad \geq C\varepsilon\Big((\varepsilon + |a|^2)^{(p-2)/2} + |a|^{p-2}\Big) \\
&\qquad = C|A - B| \left(|A|^{p-2} + |B|^{p-2}\right) \\
&\qquad \geq \Big||A|^{p-2}A - |B|^{p-2}B\Big| \\
&\qquad = \Big|(\varepsilon + |a|^2)^{(p-2)/2}(a, \varepsilon) - |a|^{p-2}(a, 0)\Big| \\
&\qquad = \Big|\Big(\big((\varepsilon + |a|^2)^{(p-2)/2} - |a|^{p-2}\big)a,\ (\varepsilon + |a|^2)^{(p-2)/2}\varepsilon\Big)\Big| \\
&\qquad \geq \left((\varepsilon + |a|^2)^{(p-2)/2} - |a|^{p-2}\right)|a|,
\end{aligned}$$

as desired. □

In the subsequent Lemmata 5.3 and 5.4, we collect some simple, but interesting, estimates that are used in Proposition 4.1:

Lemma 5.3. *Let $N \in \mathbb{N}$, $N \geq 1$, $t \in [0, 1]$, $\varepsilon > 0$, and a, $b \in \mathbb{R}^N$. Let $h(t) := ta + (1 - t)b$. Then, there exists $C > 1$, only depending on N and p, such that*

$$\frac{d}{dt}\Big((\varepsilon + |h|^2)^{(p/2)-1}h \cdot (a - b)\Big) \geq \frac{1}{C}(\varepsilon + |ta + (1 - t)b|^2)^{(p/2)-1}|a - b|^2.$$

Proof. We have

$$\begin{aligned}
&\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1}h\cdot(a-b)\Big)=\frac{d}{dt}\Big((\varepsilon+|h|^2)^{(p/2)-1}h\Big)\cdot(a-b)\\
&\quad=(\varepsilon+|h|^2)^{(p/2)-2}\big(\varepsilon+(p-1)|h|^2\big)\frac{dh}{dt}\cdot(a-b)\\
&\quad\geq\frac{1}{C}(\varepsilon+|h|^2)^{(p/2)-1}|a-b|^2\\
&\quad=\frac{1}{C}(\varepsilon+|ta+(1-t)b|^2)^{(p/2)-1}|a-b|^2,
\end{aligned}$$

as desired. □

Lemma 5.4. *Let*

$$p\in(1,2]. \tag{48}$$

Let $N\in\mathbb{N}$, $N\geq 1$, $\varepsilon>0$, and a, $b\in\mathbb{R}^N$. Then, there exists $C>1$, only depending on N and p, such that

$$(\varepsilon+|a|^2+|b|^2)^{p/2}\leq C\Big[(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}|b-a|^2+(\varepsilon+|a|^2)^{(p/2)}\Big].$$

Proof. We have

$$|b|^2=|b-a+a|^2\leq\big(|b-a|+|a|\big)^2\leq C\big(|b-a|^2+|a|^2\big)$$

and so

$$\begin{aligned}
&(\varepsilon+|a|^2+|b|^2)^{p/2}\\
=\;&(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}(\varepsilon+|a|^2+|b|^2)\\
\leq\;&C(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}(\varepsilon+|a|^2+|b-a|^2)\\
=\;&C(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}|b-a|^2+C(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}(\varepsilon+|a|^2).
\end{aligned}$$

Therefore, by (48),

$$\begin{aligned}
&(\varepsilon+|a|^2+|b|^2)^{p/2}\\
&\quad\leq C(\varepsilon+|a|^2+|b|^2)^{(p/2)-1}|b-a|^2+C(\varepsilon+|a|^2)^{(p/2)},
\end{aligned}$$

that is the desired claim. □

The following result deals with some technical estimates on monotone integrands.

Lemma 5.5. *Let $N\in\mathbb{N}$, $N\geq 1$. Let $\kappa\in\{0,1\}$. Let ε, $\varepsilon'>0$. Let a, $b\in\mathbb{R}^N$. Let $\Psi:[\varepsilon,+\infty)\to[\varepsilon',+\infty)$ be a measurable and nondecreasing function. Then*

$$\int_0^1\frac{(1-t)^\kappa}{\Psi(\varepsilon+|ta+(1-t)b|^2)}\,dt\geq\frac{1}{2\Psi(\varepsilon+|a|^2+|b|^2)}. \tag{49}$$

Proof. If $|a| \le |b|$, for any $t \in [0,1]$,

$$\begin{aligned}|ta+(1-t)b|^2 &\le t^2|a|^2+(1-t)^2|b|^2+2t(1-t)|a||b| \\ &\le t^2|b|^2+(1+t^2-2t)|b|^2+2t(1-t)|b|^2=|b|^2.\end{aligned}$$

On the other hand, if $|a| \ge |b|$, for any $t \in [0,1]$,

$$\begin{aligned}|ta+(1-t)b|^2 &\le t^2|a|^2+(1-t)^2|b|^2+2t(1-t)|a||b| \\ &\le t^2|a|^2+(1+t^2-2t)|a|^2+2t(1-t)|a|^2=|a|^2.\end{aligned}$$

In any case,

$$\varepsilon+|ta+(1-t)b|^2 \le \varepsilon+|a|^2+|b|^2$$

and the claim follows from the monotonicity of Ψ. □

The next is a useful Hölder/L^p type estimate, that is exploited in Proposition 4.1.

LEMMA 5.6. *Let $N \in \mathbb{N}$, $N \ge 1$. Let f, $g \in L^p(\mathcal{B}_R, \mathbb{R}^N)$. Suppose that*

$$p \in (1,2]. \tag{50}$$

Then

$$\begin{aligned}&\int_{\mathcal{B}_R} |f-g|^p \\ &\quad \le \left(\int_{\mathcal{B}_R} (\varepsilon+|f|^2+|g|^2)^{(p/2)-1}|f-g|^2\right)^{p/2} \\ &\qquad \times \left(\int_{\mathcal{B}_R} (\varepsilon+|f|^2+|g|^2)^{p/2}\right)^{(2-p)/2}.\end{aligned}$$

Proof. We observe that

$$\begin{aligned}&|f-g|^p \\ =\ &\Big[(\varepsilon+|f|^2+|g|^2)^{(p/2)-1}|f-g|^2\Big]^{p/2}\Big[(\varepsilon+|f|^2+|g|^2)^{p/2}\Big]^{(2-p)/2},\end{aligned}$$

and so the desired result follows from the Hölder Inequality with exponents $2/p$ and $2/(2-p)$, which can be used here due to (50). □

To end this paper, we remark that Definition 2.1 is always nonvoid (independently of ψ and Ω), in the sense that

LEMMA 5.7. $2 \in \mathcal{P}(\psi, \Omega)$.

Proof. The functional in (9) when $p = 2$ boils down to

$$\int_\Omega \frac{1}{2}|\nabla_{\mathbb{H}^n} u(\xi)|^2 + F_k(u(\xi),\xi)\,d\xi, \tag{51}$$

up to an additive constant that does not play any role in the minimization. Hence, if u_k minimizes this functional, we have that

$$-\int_\Omega \nabla_{\mathbb{H}^n} u_k(\xi)\cdot\nabla_{\mathbb{H}^n}\varphi(\xi)\,d\xi = \int_\Omega \partial_r F_k(u_k(\xi),\xi)\,\varphi(\xi)\,d\xi$$

for any $\varphi \in C_0^\infty(\Omega)$.

Accordingly, if also u_k approaches some u_∞ uniformly in Ω, it follows that

$$\begin{aligned}\int_\Omega u_\infty \Delta_{\mathbb{H}^n}\varphi &= \lim_{k\to+\infty}\int_\Omega u_k \Delta_{\mathbb{H}^n}\varphi \\ &= \lim_{k\to+\infty} -\int_\Omega \nabla_{\mathbb{H}^n} u_k\cdot\nabla_{\mathbb{H}^n}\varphi = \lim_{k\to+\infty}\int_\Omega \partial_r F_k(u_k,\xi)\,\varphi\end{aligned} \tag{52}$$

for any $\varphi \in C_0^\infty(\Omega)$.

Also, from (8),

$$0 \le \partial_r F_k \le (\Delta_{\mathbb{H}^n}\psi)^+$$

and so (52) gives that

$$0 \le \int_\Omega u_\infty \Delta_{\mathbb{H}^n}\varphi \le \int_\Omega (\Delta_{\mathbb{H}^n}\psi)^+\,\varphi \tag{53}$$

for any $\varphi \in C_0^\infty(\Omega, [0,+\infty))$.

On the other hand, since u_k is a minimizer for (51), we have that

$$\sup_{k\in\mathbb{N}} \|\nabla_{\mathbb{H}^n} u_k\|_{L^2(\Omega)} < +\infty$$

and so, up to a subsequence, we may suppose that $\nabla_{\mathbb{H}^n} u_k$ converges to some $\nu \in L^2(\Omega)$ weakly in $L^2(\Omega)$. It follows from the uniform convergence of u_k that

$$\begin{aligned}-\int_\Omega \nu\cdot\nabla_{\mathbb{H}^n}\varphi &= -\lim_{k\to+\infty}\int_\Omega \nabla_{\mathbb{H}^n} u_k\cdot\nabla_{\mathbb{H}^n}\varphi \\ &= \lim_{k\to+\infty}\int_\Omega u_k\,\Delta_{\mathbb{H}^n}\varphi = \int_\Omega u_\infty\,\Delta_{\mathbb{H}^n}\varphi\end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)$. That is, $\nabla_{\mathbb{H}^n} u_\infty = \nu$ in the sense of distributions, and so as a function. In particular, $\nabla_{\mathbb{H}^n} u_\infty \in L^2(\Omega)$, and therefore (53) yields that

$$0 \le \int_\Omega \nabla_{\mathbb{H}^n} u_\infty\cdot\nabla_{\mathbb{H}^n}\varphi \le \int_\Omega (\Delta_{\mathbb{H}^n}\psi)^+\,\varphi,$$

for any $\varphi \in C_0^\infty(\Omega, [0,+\infty))$. This shows that u_∞ satisfies (10) in the distributional sense. ∎

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Classification of polarized manifolds by the second sectional Betti number, II

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Abstract. *Let X be an n-dimensional smooth projective variety defined over the field of complex numbers, let L be a very ample line bundle on X. Then we classify (X, L) with $b_2(X, L) = h^2(X, \mathbb{C}) + 2$, where $b_2(X, L)$ is the second sectional Betti number of (X, L).*

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1. Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers $\mathbb{C}$ and let L be an ample line bundle on X. Then we call this pair (X, L) a polarized manifold. In [11], for every integer i with $0 \leq i \leq n$, we defined the invariant $b_i(X, L)$ which is called the *ith sectional Betti number of* (X, L). If L is spanned, then we can prove that $b_i(X, L) \geq h^i(X, \mathbb{C})$ (see Remark 2.3 (iii.1) below). So it is interesting to classify (X, L) by the value of $b_i(X, L) - h^i(X, \mathbb{C})$.

In this paper, we consider the case of $i = 2$. In [13, Theorem 4.1] (resp. [14, Theorem 3.1]) we have classified polarized manifolds (X, L) such that L is spanned and $b_2(X, L) = h^2(X, \mathbb{C})$ (resp. $b_2(X, L) = h^2(X, \mathbb{C}) + 1$).

In this paper we will consider the next step and we will classify polarized manifolds (X, L) such that L is very ample and $b_2(X, L) = h^2(X, \mathbb{C}) + 2$.

2. Preliminaries

In this paper we will use the customary notation in algebraic geometry.

2.1. Review on sectional invariants of polarized manifolds

In this subsection, we will review the theory of sectional invariants of polarized manifolds which will be used in the main theorem (Theorem 3.1) and its proof.

Notation 2.1. (1) *Let X be a projective variety of dimension n, let L be an ample line bundle on X. Then the Euler-Poincaré characteristic $\chi(L^{\otimes t})$*

of $L^{\otimes t}$ *is a polynomial in* t *of degree* n*, and we can describe* $\chi(L^{\otimes t})$ *as follows.*

$$\chi(L^{\otimes t}) = \sum_{j=0}^{n} \chi_j(X,L)\binom{t+j-1}{j}.$$

(2) *Let* Y *be a smooth projective variety of dimension* i*, let* $\mathcal{T}_Y$ *be the tangent bundle of* Y*, and let* Ω_Y *be the dual bundle of* $\mathcal{T}_Y$*. For every integer* j *with* $0 \le j \le i$*, we put*

$$\begin{aligned} h_{i,j}(c_1(Y), \cdots, c_i(Y)) &:= \chi(\Omega_Y^j) \\ &= \int_Y ch(\Omega_Y^j)\, Td(\mathcal{T}_Y). \end{aligned}$$

(Here $ch(\Omega_Y^j)$ *(resp.* $Td(\mathcal{T}_Y)$*) denotes the Chern character of* Ω_Y^j *(resp. the Todd class of* $\mathcal{T}_Y$*). See [15, Examples 3.2.3 and 3.2.4].)*

(3) *Let* (X,L) *be a polarized manifold of dimension* n*. For every integers* i *and* j *with* $0 \le j \le i \le n$*, we put*

$$C_j^i(X,L) := \sum_{l=0}^{j} (-1)^l \binom{n-i+l-1}{l} c_{j-l}(X)L^l,$$

$$w_i^j(X,L) := h_{i,j}(C_1^i(X,L), \cdots, C_i^i(X,L))L^{n-i}.$$

(4) *Let* X *be a smooth projective variety of dimension* n*. For every integers* i *and* j *with* $0 \le j \le i \le n$*, we put*

$$H_1(i,j) := \begin{cases} \displaystyle\sum_{s=0}^{i-j-1} (-1)^s h^s(\Omega_X^j) & \text{if } j \ne i, \\ 0 & \text{if } j = i, \end{cases}$$

$$H_2(i,j) := \begin{cases} \displaystyle\sum_{t=0}^{j-1} (-1)^{i-t} h^t(\Omega_X^{i-j}) & \text{if } j \ne 0, \\ 0 & \text{if } j = 0. \end{cases}$$

Definition 2.2. *(See [10, Definition 2.1] and [11, Definition 3.1].) Let* (X,L) *be a polarized manifold of dimension* n*, and let* i *and* j *be integers with* $0 \le j \le i \le n$*.*

(1) *The* i*th sectional geometric genus* $g_i(X,L)$ *of* (X,L) *is defined as follows:*

$$g_i(X,L) := (-1)^i(\chi_{n-i}(X,L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

(2) *The* i*th sectional Euler number* $e_i(X,L)$ *of* (X,L) *is defined by the following:*

$$e_i(X,L) := C_i^i(X,L)L^{n-i}.$$

(3) *The* i*th sectional Betti number* $b_i(X,L)$ *of* (X,L) *is defined by the following:*

$$b_i(X,L) := \begin{cases} e_0(X,L) & \text{if } i=0, \\ (-1)^i \left(e_i(X,L) - \displaystyle\sum_{j=0}^{i-1} 2(-1)^j h^j(X,\mathbb{C}) \right) & \text{if } 1 \leq i \leq n. \end{cases}$$

(4) *The* i*th sectional Hodge number* $h_i^{j,i-j}(X,L)$ *of type* $(j,i-j)$ *of* (X,L) *is defined by the following:*

$$h_i^{j,i-j}(X,L) := (-1)^{i-j}\left\{ w_i^j(X,L) - H_1(i,j) - H_2(i,j) \right\}.$$

REMARK 2.3. (i) *For every integers* i *and* j *with* $0 \leq j \leq i \leq n$, $g_i(X,L)$, $e_i(X,L)$, $b_i(X,L)$ *and* $h_i^{j,i-j}(X,L)$ *are integer (see [11, Proposition 3.1]).*

(ii) *Let* (X,L) *be a polarized manifold of dimension* n. *For every integers* i *and* j *with* $0 \leq j \leq i \leq n$, *we have the following (see [11, Theorem 3.1]).*

(ii.1) $b_i(X,L) = \displaystyle\sum_{k=0}^{i} h_i^{k,i-k}(X,L)$.

(ii.2) $h_i^{j,i-j}(X,L) = h_i^{i-j,j}(X,L)$.

(ii.3) $h_i^{i,0}(X,L) = h_i^{0,i}(X,L) = g_i(X,L)$.

(iii) *Assume that* L *is ample and spanned. Then, for every integers* i *and* j *with* $0 \leq j \leq i \leq n$, *the following inequalities hold (see [10, Theorem 3.1] and [11, Proposition 3.3]).*

(iii.1) $b_i(X,L) \geq h^i(X,\mathbb{C})$.

(iii.2) $h_i^{j,i-j}(X,L) \geq h^{j,i-j}(X)$.

(iii.3) $g_i(X,L) \geq h^i(\mathcal{O}_X)$.

2.2. Adjunction theory of polarized manifolds

In this subsection, we will recall results on adjunction theory which will be used later.

DEFINITION 2.4. *Let* (X,L) *be a polarized manifold of dimension* n.

(1) *We say that* (X, L) *is a scroll (resp. quadric fibration, Del Pezzo fibration) over a normal projective variety* Y *of dimension* m *with* $1 \leq m < n$ *(resp.* $1 \leq m < n$, $1 \leq m < n-1$*) if there exists a surjective morphism with connected fibers* $f : X \to Y$ *such that* $K_X + (n-m+1)L = f^*A$ *(resp.* $K_X + (n-m)L = f^*A$, $K_X + (n-m-1)L = f^*A$*) for some ample line bundle* A *on* Y*.*

(2) (X, L) *is called a classical scroll over a normal variety* Y *if there exists a vector bundle* $\mathcal{E}$ *on* Y *such that* $X \cong \mathbb{P}_Y(\mathcal{E})$ *and* $L = H(\mathcal{E})$*, where* $H(\mathcal{E})$ *is the tautological line bundle.*

(3) *We say that* (X, L) *is a hyperquadric fibration over a smooth projective curve* C *if* (X, L) *is a quadric fibration over* C *such that the morphism* $f : X \to C$ *is the contraction morphism of an extremal ray. In this case,* $(F, L_F) \cong (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$ *for any general fiber* F *of* f*, every fiber of* f *is irreducible and reduced (see* [18] *or* [7, Claim (3.1)]*) and* $h^2(X, \mathbb{C}) = 2$*.*

REMARK 2.5. (1) *If* (X, L) *is a scroll over a smooth projective curve* C*, then* (X, L) *is a classical scroll over* C *(see [2, Proposition 3.2.1]).*

(2) *If* (X, L) *is a scroll over a normal projective surface* S*, then* S *is smooth and* (X, L) *is also a classical scroll over* S *(see [3, (3.2.1) Theorem] and [9, (11.8.6)]).*

(3) *Assume that* (X, L) *is a quadric fibration over a smooth curve* C *with* $\dim X = n \geq 3$*. Let* $f : X \to C$ *be its morphism. By [3, (3.2.6) Theorem] and the proof of [18, Lemma (c) in Section 1], we see that* (X, L) *is one of the following:*

(a) *A hyperquadric fibration over* C*.*

(b) *A classical scroll over a smooth surface with* $\dim X = 3$*.*

DEFINITION 2.6. (1) *Let* X *(resp.* Y*) be an* n*-dimensional projective manifold, and* L *(resp.* H*) an ample line bundle on* X *(resp.* Y*). Then* (X, L) *is called a simple blowing up of* (Y, H) *if there exists a birational morphism* $\pi : X \to Y$ *such that* π *is a blowing up at a point of* Y *and* $L = \pi^*(H) - E$*, where* E *is the* π*-exceptional effective reduced divisor.*

(2) *Let* X *(resp.* M*) be an* n*-dimensional projective manifold, and* L *(resp.* A*) an ample line bundle on* X *(resp.* M*). Then we say that* (M, A) *is a reduction of* (X, L) *if there exists a birational morphism* $\mu : X \to M$ *such that* μ *is a composition of simple blowing ups and* (M, A) *is not obtained by a simple blowing up of any other polarized manifold.*

THEOREM 2.7. *Let* (X, L) *be a polarized manifold with* $\dim X = n \geq 3$*. Then* (X, L) *is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) *A scroll over a smooth projective curve.*

(4) $K_X \sim -(n-1)L$*, that is,* (X, L) *is a Del Pezzo manifold.*

(5) *A hyperquadric fibration over a smooth projective curve.*

(6) *A classical scroll over a smooth projective surface.*

(7) *Let* (M, A) *be a reduction of* (X, L).

(7.1) $n = 4$, $(M, A) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.
(7.2) $n = 3$, $(M, A) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.
(7.3) $n = 3$, $(M, A) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.
(7.4) $n = 3$*,* M *is a* $\mathbb{P}^2$*-bundle over a smooth curve* C*, the nef value of* A *is* $\frac{3}{2}$*, and* $(F', A|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *for any fiber* F' *of it.*
(7.5) $K_M + (n-2)A$ *is nef.*

Proof. See [2, Proposition 7.2.2, Theorems 7.2.4, 7.3.2 and 7.3.4] and [9, (11.2), (11.7) and (11.8)]. □

NOTATION 2.8. (1) *Let* (X, L) *be a hyperquadric fibration over a smooth curve* C *and let* $f : X \to C$ *be its morphism. We put* $\mathcal{E} := f_*(L)$*. Then* $\mathcal{E}$ *is a locally free sheaf of rank* $n+1$ *on* C*. Let* $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ *be the projective bundle. Then* $X \in |2H(\mathcal{E}) + \pi^*(B)|$ *for some* $B \in \mathrm{Pic}(C)$ *and* $L = H(\mathcal{E})|_X$*, where* $H(\mathcal{E})$ *is the tautological line bundle of* $\mathbb{P}_C(\mathcal{E})$*. We put* $e := \deg \mathcal{E}$ *and* $b := \deg B$.

(2) *(See [9, (13.10)].) Let* (M, A) *be a* $\mathbb{P}^2$*-bundle over a smooth curve* C *and* $A|_F = \mathcal{O}_{\mathbb{P}^2}(2)$ *for any fiber* F *of it. Let* $f : M \to C$ *be the fibration and* $\mathcal{E} := f_*(K_M + 2A)$*. Then* $\mathcal{E}$ *is a locally free sheaf of rank* 3 *on* C*, and* $M \cong \mathbb{P}_C(\mathcal{E})$ *such that* $H(\mathcal{E}) = K_M + 2A$*. In this case,* $A = 2H(\mathcal{E}) + f^*(B)$ *for a line bundle* B *on* C*, and by the canonical bundle formula we have* $K_M = -3H(\mathcal{E}) + f^*(K_C + \det\mathcal{E})$*. Here we set* $e := \deg \mathcal{E}$ *and* $b := \deg B$.

2.3. A classification of very ample vector bundles $\mathcal{E}$ on surfaces with $c_2(\mathcal{E}) = 3$

Here we classify very ample vector bundles $\mathcal{E}$ on smooth projective surfaces with $c_2(\mathcal{E}) = 3$. We will use this result later.

THEOREM 2.9. *Let S be a smooth projective surface and let $\mathcal{E}$ be a very ample vector bundle on S with $c_2(\mathcal{E}) = 3$ and $\mathrm{rank}\mathcal{E} \geq 2$. Then $(S, \mathcal{E})$ is one of the following types.*

(i) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3})$.

(ii) $(\mathbb{P}^2, T_{\mathbb{P}^2})$, *where $T_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.*

(iii) $(\mathbb{P}^1 \times \mathbb{P}^1, [p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(2))] \oplus [p_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))])$, *where p_i is the ith projection.*

(iv) *S is a blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E} = (p^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E)^{\oplus 2}$, where $p : S \to \mathbb{P}^2$ is the morphism and E is the exceptional divisor of p.*

(v) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3))$.

(vi) *S is a Del Pezzo surface of degree 3 and $\mathcal{E} \cong \mathcal{O}(-K_S)^{\oplus 2}$.*

Proof. By a result of Noma [22, Corollary], we see that $(S, \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3))$ if $c_1(\mathcal{E})^2 \geq 4c_2(\mathcal{E}) + 1 = 13$. So we may assume that $c_1(\mathcal{E})^2 \leq 12$. We consider $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ and let $X := \mathbb{P}_S(\mathcal{E})$, $L := H(\mathcal{E})$ and $n := \dim X$. Then $H(\mathcal{E})$ is very ample and $H(\mathcal{E})^n = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) \leq 12 - 3 = 9$. Let $\pi : \mathbb{P}_S(\mathcal{E}) \to S$ be the projection. We use a classification of polarized manifolds by the degree (see [17], [19] and [5]). First of all, we prove the following claim.

CLAIM 2.10. *If $g(X, L) \leq 3$ and $c_2(\mathcal{E}) = 3$, then $(S, \mathcal{E})$ is one of the types* (i), (ii), (iii), (iv) *and* (v) *in Theorem* 2.9.

Proof. First we note that $\mathcal{E}$ is very ample.

If $g(X, L) = 0$, then by [8, (3.2) Theorem] or [4, (2.1) Theorem] we see that $c_2(\mathcal{E}) \neq 3$.

If $g(X, L) = 1$ (resp. 2, 3) and $c_2(\mathcal{E}) = 3$, then by [8, (3.3) Theorem] or [4, (2.2) Theorem] (resp. [8, (3.4) Theorem] or [4, (2.3) Theorem], [4, (2.11) Theorem] and [20, Corollary 4.7]) we see that (X, L) is either (i) or (ii) (resp. (iii) or (iv), (v)). $\square$

From now on, we assume that $g(X, L) \geq 4$. By the list of [17], we have $L^n \geq 6$.

(A) The case where $L^n = 6$. Then we see from the list of [17] that X is either a complete intersection of type $(2, 3)$ or a hypersurface in $\mathbb{P}^{n+1}$. But in each case we have $\mathrm{Pic}(X) \cong \mathbb{Z}$ and this is impossible.

(B) The case where $L^n = 7$. Then we see from the list of [17] and Table II of [1, Page 55] that (X, L) is one of the following types.

(B.1) $(X, L) = (\mathbb{P}_T(\mathcal{F}), H(\mathcal{F}))$ and $g(X, L) = 4$, where T is the blowing up of $\mathbb{P}^2$ at 6 points and $\mathcal{F}$ is a locally free sheaf on T.

(B.2) $n = 3$, $g(X, L) = 5$ and $\sigma_P : X \to Y$ is the blowing up of Y at a point P, where Y is a smooth complete intersection of type $(2, 2, 2)$.

(B.3) $g(X, L) = 6$ and the morphism $\phi : X \to \mathbb{P}^1$ defined by the complete linear system $|K_X + L|$ is a fibration over $\mathbb{P}^1$.

(B.4) X is a hypersurface of degree 7 in $\mathbb{P}^{n+1}$.

(B.I) First we consider the case (B.2). Then $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 2}$ and $\mathrm{Pic}(S) \cong \mathbb{Z}$. Next we prove the following.

Claim 2.11. *$\kappa(S) = -\infty$ holds.*

Proof. In this case, there exists an effective divisor E on X such that $E \cong \mathbb{P}^2$. We note that $\pi(E)$ is not a point because every fiber of π is $\mathbb{P}^1$. Therefore $\pi_E : E \to S$ is surjective because $E \cong \mathbb{P}^2$. Assume that π_E is not finite. Then there exists a fiber F_π of π such that F_π is contracted by σ_P. Hence [2, Lemma 4.1.13] there exists a morphism $\delta : S \to Y$ such that $\sigma_P = \delta \circ \pi$. But this is impossible because σ_P is surjective and $\dim S < \dim Y$. Therefore π_E is finite and we have $\kappa(S) = -\infty$ because $\kappa(E) = -\infty$. □

We see from Claim 2.11 and $\mathrm{Pic}(S) \cong \mathbb{Z}$ that $S \cong \mathbb{P}^2$. We note that $\mathrm{rank}\mathcal{E} = 2$ because $\dim X = 3$ in this case. Hence by [20, Corollary 4.7] $g(X, L) \leq 3$ holds and this case is ruled out.
(B.II) Next we consider the case (B.3). Since $h^0(K_X + L) = h^0(K_{\mathbb{P}_S(\mathcal{E})} + H(\mathcal{E})) = 0$, this case is also ruled out.
(B.III) Next we consider the case (B.4). This case is also ruled out because $\mathrm{Pic}(X) \not\cong \mathbb{Z}$.
(B.IV) Finally we consider the case (B.1). Then we have $\mathrm{Pic}(T) \cong \mathbb{Z}^{\oplus 7}$, $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 8}$ and $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 7}$. Since $c_2(\mathcal{E}) = 3$ and $L^n = 7$, we have $c_1(\mathcal{E})^2 = 10$. Hence we have $K_S c_1(\mathcal{E}) = -4$ because $g(S, c_1(\mathcal{E})) = g(X, L) = 4$. Next we prove the following.

Claim 2.12. *$\kappa(S) = -\infty$ holds.*

Proof. Let $\rho : X = \mathbb{P}_T(\mathcal{F}) \to T$ be the projection. Let $D_1, \ldots, D_{n-2}$ be general members of $|L|$ such that $X_{n-2} := D_1 \cap \cdots \cap D_{n-2}$ is a smooth projective surface. Here we note that $\rho_{X_{n-2}} : X_{n-2} \to T$ and $\pi_{X_{n-2}} : X_{n-2} \to S$ are birational because $L^{n-2}F_\rho = 1$ (resp. $L^{n-2}F_\pi = 1$) for any general fiber F_ρ (resp. F_π) of ρ (resp. π). Therefore S is birationally equivalent to T. So we get the assertion because $\kappa(T) = -\infty$. □

Since $\kappa(S)=-\infty$, $h^1(\mathcal{O}_S)=0$ and $\mathrm{Pic}(S)\cong\mathbb{Z}^{\oplus 7}$, we see that $K_S^2=3$. Hence we get

$$(K_S c_1(\mathcal{E}))^2=16<30=(K_S)^2(c_1(\mathcal{E}))^2,$$

but this contradicts the Hodge index theorem. Therefore this case is also impossible.

(C) The case where $L^n=8$. Then since we assume that $g(X,L)\geq 4$, we see from the list of [19] that (X,L) is one of the following types.

(C.1) $(X,L)=(\mathbb{P}_{\mathbb{Q}^2}(\mathcal{F}),H(\mathcal{F}))$ and $g(X,L)=4$, where $\mathcal{F}$ is a locally free sheaf of rank two on $\mathbb{Q}^2$.

(C.2) X is a smooth complete intersection of type $(2,2,2)$.

(C.3) The morphism $\phi:X\to\mathbb{P}^1$ defined by $|K_X+L|$ is a fibration over $\mathbb{P}^1$.

(C.4) X is a complete intersection of type $(2,4)$.

(C.5) X is a hypersurface of degree 8 in $\mathbb{P}^{n+1}$.

(C.I) First we consider the cases (C.2), (C.4) and (C.5). These cases are ruled out because $\mathrm{Pic}X\ncong\mathbb{Z}$.
(C.II) Next we consider the case (C.3). Since $h^0(K_X+L)=h^0(K_{\mathbb{P}_S(\mathcal{E})}+H(\mathcal{E}))=0$, this case is also ruled out.
(C.III) Finally we consider the case (C.1). Since $g(S,c_1(\mathcal{E}))=g(X,L)=4$ and $c_1(\mathcal{E})^2=11$, we have $K_S c_1(\mathcal{E})=-5$. Moreover $\mathrm{Pic}(X)\cong\mathbb{Z}^{\oplus 3}$ and $h^1(\mathcal{O}_X)=0$. Hence we have $\mathrm{Pic}(S)\cong\mathbb{Z}^{\oplus 2}$ and $h^1(\mathcal{O}_S)=0$. By the same argument as in the proof of Claim 2.12, we see that $\kappa(S)=-\infty$. So we have $K_S^2=8$, and

$$(K_S c_1(\mathcal{E}))^2=25<88=(K_S)^2(c_1(\mathcal{E}))^2.$$

But this contradicts the Hodge index theorem. Therefore this case is also impossible.

(D) The case where $L^n=9$. In this case, since we assume that $g(X,L)\geq 4$, we see from [6, Table III in page 104] (see also [5]) that (X,L) is one of the following types.

(D.1) $(X,L)=(\mathbb{P}_{\mathbb{Q}^2}(\mathcal{F}),H(\mathcal{F}))$ and $g(X,L)=4$, where $\mathcal{F}$ is a locally free sheaf of rank two on $\mathbb{Q}^2$.

(D.2) (X,L) is a hyperquadric fibration over $\mathbb{P}^1$, $g(X,L)=4$ and $n=3,4,5$.

(D.3) X is the Segre embedding of $\mathbb{P}^1\times Y$ in $\mathbb{P}^7$ and $g(X,L)=4$, where Y is a cubic surface in $\mathbb{P}^3$.

(D.4) The reduction (M, A) of (X, L) is $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$ and $g(X, L) = 5$.

(D.5) (X, L) is a scroll over $\mathbb{P}^2$ with five double points blown up, $g(X, L) = 5$ and $n = 3$.

(D.6) (X, L) is a scroll over the first Hirzebruch surface F_1, $g(X, L) = 5$ and $n = 3$.

(D.7) X is a blowing up of a Fano manifold Y at a point in $\mathbb{P}^7$, $g(X, L) = 6$ and $n = 3$.

(D.8) X is a hypercubic section of a cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$ in $\mathbb{P}^5$, $g(X, L) = 7$ and $n = 3$.

(D.9) (X, L) is a complete intersection of type $(3, 3)$ and $g(X, L) = 10$.

(D.10) $n = 3$, X is linked to a $\mathbb{P}^3$ in the complete intersection of a quadric and a quintic hypersurface, and $g(X, L) = 12$.

(D.11) $n = 3$, X is linked to a cubic scroll in the complete intersection of a cubic and a quartic hypersurface, and $g(X, L) = 9$.

(D.12) $n = 3$, X is a $\mathbb{P}^1$-bundle over a minimal K3 surface and L is the tautological line bundle with $g(X, L) = 8$.

(D.13) X is a hypersurface of degree 9 in $\mathbb{P}^{n+1}$ and $g(X, L) = 28$.

(D.I) First we consider the cases (D.9) and (D.13). These cases do not occur because $\mathrm{Pic}(X) \ncong \mathbb{Z}$.

(D.II) Next we consider the case (D.1). In this case we have $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 3}$. Hence $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 2}$. By the same argument as the proof of Claim 2.12, we see that $\kappa(S) = -\infty$. Therefore S is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. We also infer that $\mathrm{rank}\mathcal{E} = 2$ because $\dim X = 3$. So we see from [20, Corollary (2.11)] that $(S, \mathcal{E})$ is one of the following.

- $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} \cong (p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(1)) \oplus (p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(2))$, where p_i is the ith projection.
- S is the blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E} = (p^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E)^{\oplus 2}$, where $p : S \to \mathbb{P}^2$ is the morphism and E is the exceptional divisor of p.

But here we assume that $g(X, L) \geq 4$, so these cases do not occur.

(D.III) Next we consider the case (D.3). First we note the following.

Claim 2.13. *$\kappa(S) = -\infty$.*

Proof. Let $p : X \to \mathbb{P}^1$ be the projection map. If $\pi_{F_p} : F_p \to S$ is finite for a fiber F_p of p, then $\kappa(S) = -\infty$ because $\kappa(F_p) = -\infty$. If $\pi_{F_p} : F_p \to S$ is not finite for any fiber F_p of p, then there exists a fiber F_π of π such that $p(F_\pi)$ is a point. So by [2, Lemma 4.1.13] there exists a surjective morphism $r : S \to \mathbb{P}^1$ such that $p = r \circ \pi$. Since the irregularity of a general fiber of p is zero, so is the irregularity of a general fiber of r. Therefore $\kappa(S) = -\infty$. □

In this case we have $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 8}$. Hence $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 7}$. Since $h^1(\mathcal{O}_S) = 0$, we have $K_S^2 = 3$. On the other hand we have $g(S, c_1(\mathcal{E})) = g(X, L) = 4$ and $c_1(\mathcal{E})^2 = H(\mathcal{E})^3 + c_2(\mathcal{E}) = 12$. Hence $K_S c_1(\mathcal{E}) = -6$. Hence we have $(K_S c_1(\mathcal{E}))^2 = 36 = (K_S^2)(c_1(\mathcal{E})^2)$. By the Hodge index theorem we have $c_1(\mathcal{E}) \equiv -2K_S$, that is, S is a Del Pezzo surface of degree 3. Since $\mathrm{rank}\mathcal{E} = 2$, we see from [20, Corollary (3.14)] that $\mathcal{E} \cong \mathcal{O}(-K_S)^{\oplus 2}$. This is the type (vi) in Theorem 2.9.

(D.IV) Next we consider the case (D.4). Let $\mu : X \to \mathbb{Q}^3$ be the reduction map. Then μ is not the identity map because $L^3 = 9$ and $\mathcal{O}_{\mathbb{Q}^3}(2)^3 = 16$. Hence there exists an effective divisor E on X such that $E \cong \mathbb{P}^2$. If $\pi(E) \neq S$, then $\pi(E)$ is a point. But this is impossible because π is a $\mathbb{P}^1$-bundle. Hence $\pi(E) = S$ holds. Moreover $\pi_E : E \to S$ is finite because $E \cong \mathbb{P}^2$. Hence we see that $\kappa(S) = -\infty$ and $h^1(\mathcal{O}_S) = 0$. Here we prove the following.

Claim 2.14. $S \cong \mathbb{P}^2$.

Proof. Assume that $S \not\cong \mathbb{P}^2$. Then there exists a surjective morphism $p : S \to \mathbb{P}^1$. Hence $p \circ \pi_E : E \to \mathbb{P}^1$ is surjective. But this is impossible because $E \cong \mathbb{P}^2$. □

Therefore we see that $K_S^2 = 9$. We also have $c_1(\mathcal{E})^2 = 12$ and $g(S, c_1(\mathcal{E})) = g(X, L) = 5$. Therefore $K_S c_1(\mathcal{E}) = -4$. But this is impossible because of the Hodge index theorem.

(D.V) Next we consider the case (D.6). By the same argument as the proof of Claim 2.12, we have $\kappa(S) = -\infty$.

In this case we have $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 3}$. Hence $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 2}$. Since $h^1(\mathcal{O}_S) = 0$, we have $K_S^2 = 8$. On the other hand we have $g(S, c_1(\mathcal{E})) = g(X, L) = 5$ and $c_1(\mathcal{E})^2 = 12$. Hence $K_S c_1(\mathcal{E}) = -4$. But this is impossible because of the Hodge index theorem.

(D.VI) Next we consider the case (D.7). In this case there exists an effective divisor E on X such that $E \cong \mathbb{P}^2$. Then we see that $\pi_E : E \to S$ is finite, $\kappa(S) = -\infty$ and $h^1(\mathcal{O}_S) = 0$ by the same reason as the case (D.4). By the same argument as the proof of Claim 2.14 we see that $S \cong \mathbb{P}^2$. Therefore we have $K_S^2 = 9$. We also have $c_1(\mathcal{E})^2 = 12$ and $g(S, c_1(\mathcal{E})) = g(X, L) = 6$. Therefore $K_S c_1(\mathcal{E}) = -2$. But this is impossible because of the Hodge index theorem.

(D.VII) Next we consider the case (D.8). Then by the proof of [5, Proposition (2.5)], there exists a Del Pezzo fibration $f : X \to \mathbb{P}^1$. In particular $K_X + L$ is nef.

Claim 2.15. *$\kappa(S) = -\infty$ holds.*

Proof. Let F_f be a fiber of f. If $\pi(F_f) \neq S$ for a general fiber F_f of f, then F_f contains a fiber of π and by [2, Lemma 4.1.13] there exists a morphism $\delta : S \to \mathbb{P}^1$ such that $f = \delta \circ \pi$. Since the irregularity of a general fiber of f is 0, we see that any general fiber of δ is $\mathbb{P}^1$. Hence we get the assertion. So we may assume that $\pi(F_f) = S$ for any general fiber F_f of f. If $\pi_{F_f} : F_f \to S$ is not a finite morphism, then F_f contains a fiber of π and we get the assertion by the same argument as above. So we may assume that $\pi_{F_f} : F_f \to S$ is a finite morphism. Since $\kappa(F_f) = -\infty$, we have $\kappa(S) = -\infty$. □

Let D be a general member of $|L|$. Then D is a smooth projective surface and $\kappa(D) \geq 0$ because $K_X + L$ is nef. But since $\pi_D : D \to S$ is birational, this is a contradiction.

(D.VIII) Next we consider the case (D.10). In this case $\kappa(X) = 1$, see [1, 8) in Table I, pg 53]. But this is impossible.

(D.IX) Next we consider the case (D.11). Let $D \in |L|$ be a general member. Then D is a smooth projective surface and $\pi_D : D \to S$ is birational. Hence $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_S)$. By 9) in Table I of [1, Page 53], we have $\chi(\mathcal{O}_D) = 4$. On the other hand since $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_S)$, we have $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S) = 4$. But this is impossible because $\chi(\mathcal{O}_X) = 1$, see 9) in Table I of [1, Page 53].

(D.X) Next we consider the case (D.2). Let $f : X \to \mathbb{P}^1$ be the fibration. If $n \geq 4$, then $\pi(F_f)$ is a point for a general fiber F_f of f because $\mathrm{Pic}(F_f) \cong \mathbb{Z}$. Hence by [2, Lemma 4.1.13] there exists a morphism $\delta : \mathbb{P}^1 \to S$ such that $\pi = \delta \circ f$. But this is impossible because π is surjective and $\dim S = 2$. So we may assume that $n = 3$. Let $F_f = aH(\mathcal{E}) + \pi^*(B)$, where $B \in \mathrm{Pic}(S)$. Then we have

$$\begin{aligned} 0 &= F_f^3 = 9a^3 + 3a^2 c_1(\mathcal{E})B + 3aB^2, & (1)\\ 0 &= LF_f^2 = 9a^2 + 2ac_1(\mathcal{E})B + B^2, & (2)\\ 2 &= L^2 F_f = 9a + c_1(\mathcal{E})B. & (3) \end{aligned}$$

By (1) and (2) we get $a^2 c_1(\mathcal{E})B + 2aB^2 = 0$.

If $a \neq 0$, then $B^2 = -\frac{a}{2}c_1(\mathcal{E})B$. Hence by (2) we have $c_1(\mathcal{E})B = -6a$. Therefore by (3) we get $2 = 9a + c_1(\mathcal{E})B = 3a$. But this is impossible because a is an integer. Hence $a = 0$ and $F_f = \pi^*(B)$. In particular a fiber of π is contained in a fiber of f. So by [2, Lemma 4.1.13] there exists a morphism $h : S \to \mathbb{P}^1$ such that $f = h \circ \pi$. Since $h^1(\mathcal{O}_{F_f}) = 0$, we see that $h^1(\mathcal{O}_{F_h}) = 0$ for any general fiber F_h of h. So we infer that any general fiber of h is $\mathbb{P}^1$. We note that $B = F_h$ for a fiber F_h of h. In particular we see from (3) that $F_h c_1(\mathcal{E}) = 2$ for any fiber F_h of h. On the other hand since $\mathcal{E}$ is an ample vector bundle of rank two, we infer that any fiber of h is $\mathbb{P}^1$ and therefore S is relatively minimal and S is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Let C_0 be the minimal section and let $e := -C_0^2$. Since $F_h c_1(\mathcal{E}) = 2$, we can write $c_1(\mathcal{E})$ as $c_1(\mathcal{E}) \equiv 2C_0 + bF_h$. Hence $c_1(\mathcal{E})^2 = 4(b-e)$. On the other hand $c_1(\mathcal{E})^2 = H(\mathcal{E})^3 + c_2(\mathcal{E}) = 12$. So we get $b - e = 3$. Since $c_1(\mathcal{E})$ is ample, by [16, Theorem 2.12 and Corollary 2.18 in Chapter V] we have $e \geq 0$ and $b > 2e$. Therefore $3 = b - e > 2e - e = e \geq 0$, namely we get $(b, e) = (3, 0), (4, 1), (5, 2)$. We also note that $2 \leq c_1(\mathcal{E})C_0$ because $C_0 \cong \mathbb{P}^1$. Hence $2 \leq c_1(\mathcal{E})C_0 = -2e + b$ and $(b, e) = (5, 2)$ is impossible. So by Ishihara's result [20, Corollary (2.11)] we have

- $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{E} \cong (p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(1)) \oplus (p_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(2))$, where p_i is the ith projection.
- S is a blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E} = (p^*(\mathcal{O}_{\mathbb{P}^2}(2)) - E)^{\oplus 2}$, where $p : S \to \mathbb{P}^2$ is the morphism and E is the exceptional divisor of p.

But we see that $g(X, L) \leq 3$ in these cases, and these cases are ruled out.

(D.XI) Next we consider the case (D.5). By the same argument as the proof of Claim 2.12, we have $\kappa(S) = -\infty$.

In this case we have $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 7}$. Hence $\mathrm{Pic}(S) \cong \mathbb{Z}^{\oplus 6}$. Since $h^1(\mathcal{O}_S) = 0$, we have $K_S^2 = 4$. On the other hand we have $g(S, c_1(\mathcal{E})) = g(X, L) = 5$ and $c_1(\mathcal{E})^2 = H(\mathcal{E})^3 + c_2(\mathcal{E}) = 12$. Hence $K_S c_1(\mathcal{E}) = -4$. But this is impossible because of the Hodge index theorem.

(D.XII) Finally we consider the case (D.12). Let $p : X \to Y$ be the projection, where Y is a minimal K3 surface. Then there exists a very ample line bundle H on Y and a smooth member $B \in |H|$ such that $g(B) \geq 2$ and $p^*(B) =: V$ is a smooth projective surface with $\kappa(V) = -\infty$.

(i) Assume that $\pi_V : V \to S$ is surjective. Then by the same argument as the proof of Claim 2.12, we have $\kappa(S) = -\infty$. We note that $h^1(\mathcal{O}_S) = 0$.

If $S \cong \mathbb{P}^2$, then since $\mathrm{rank}\mathcal{E} = 2$ we see from [20, Corollary (4.7)] that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$ or $T_{\mathbb{P}^2}$. But in these cases we have $g(X, L) = g(S, c_1(\mathcal{E})) \leq 3$ and this contradicts the assumption.

If $S \not\cong \mathbb{P}^2$, then there exists a surjective morphism $h : S \to \mathbb{P}^1$ such that any general fiber of h is $\mathbb{P}^1$. Let F be a general fiber of $h \circ \pi$. If $p_F : F \to Y$ is not

finite, then there exists a fiber F_p of p such that F_p is contained in F. Then by [2, Lemma 4.1.13] there exists a morphism $g : Y \to \mathbb{P}^1$ such that $g \circ p = h \circ \pi$. But since Y is a minimal K3 surface, we infer that p is an elliptic fibration and this is impossible because any general fiber of h is $\mathbb{P}^1$. Therefore $p_F : F \to Y$ is finite. But this is impossible because $\kappa(F) = -\infty$ and $\kappa(Y) = 0$.
(ii) Assume that $\pi_V : V \to S$ is not surjective. Then there exists a fiber F_π of π such that F_π is contained in V. Moreover $p(F_\pi)$ is a point because $g(B) \geq 2$ and $F_\pi \cong \mathbb{P}^1$. So by [2, Lemma 4.1.13] there exists a morphism $r : S \to Y$ such that $p = r \circ \pi$. Furthermore since p and π have connected fibers, we see that r is birational. Since p and π are $\mathbb{P}^1$-bundles, we see that r is finite. Hence r is an isomorphism and S is a minimal K3 surface. Since

$$8 = g(X, L) = g(S, c_1(\mathcal{E})) = 1 + \frac{c_1(\mathcal{E})^2}{2},$$

we have $c_1(\mathcal{E})^2 = 14$. Therefore $c_2(\mathcal{E}) = c_1(\mathcal{E})^2 - H(\mathcal{E})^3 = 14 - 9 = 5$, and this is impossible.

□

3. Main Theorem

THEOREM 3.1. *Let (X, L) be a polarized manifold of dimension $n \geq 3$ and let (M, A) be a reduction of (X, L). Assume that L is very ample. If $b_2(X, L) = h^2(X, \mathbb{C}) + 2$, then (X, L) is one of the following types.*

(i) *$(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is a very ample vector bundle on S with $c_2(\mathcal{E}) = 3$. In particular $(S, \mathcal{E})$ is described in Theorem 2.9.*

(ii) *(M, A) is a Del Pezzo fibration over a smooth curve C with $n = 3, 4$. Let $f : M \to C$ be its morphism. In this case there exists an ample line bundle H on C such that $K_M + (n-2)A = f^*(H)$, and we have $(g(C), \deg H) = (1, 1)$, $b_2(M, A) = 14$ and $h^2(M, \mathbb{C}) = 12$.*

(iii) *(M, A) is a quadric fibration over a smooth surface S with $n = 3, 4$. Let $f : M \to S$ be its morphism. In this case there exists an ample line bundle H on S such that $K_M + (n-2)A = f^*(K_S + H)$, and (S, H) is one of the following types:*

(iii.1) *S is a $\mathbb{P}^1$-bundle, $p : S \to B$, over a smooth elliptic curve B, and $H = 3C_0 - F$, where C_0 (resp. F) denotes the minimal section of S with $C_0^2 = 1$ (resp. a fiber of p). In this case $b_2(M, A) = 12$ and $h^2(M, \mathbb{C}) = 10$.*

(iii.2) *S is an abelian surface, $H^2 = 2$, and $h^0(H) = 1$. In this case $b_2(M, A) = 14$ and $h^2(M, \mathbb{C}) = 12$.*

(iii.3) *S is a hyperelliptic surface, $H^2 = 2$, and $h^0(H) = 1$. In this case $b_2(M, A) = 10$ and $h^2(M, \mathbb{C}) = 8$.*

Proof. First we note that the following hold.

- $b_2(X, L) = 2g_2(X, L) + h_2^{1,1}(X, L)$ by Remark 2.3 (ii.1) and (ii.3).
- $g_2(X, L) \geq h^2(\mathcal{O}_X)$ by Remark 2.3 (iii.3).
- $h_2^{1,1}(X, L) \geq h^{1,1}(X)$ by Remark 2.3 (iii.2).
- $h^2(X, \mathbb{C}) = 2h^2(\mathcal{O}_X) + h^{1,1}(X)$ by the Hodge theory.

Hence we see from $b_2(X, L) = h^2(X, \mathbb{C}) + 2$ that one of the following holds.

(A) $g_2(X, L) = h^2(\mathcal{O}_X)$ and $h_2^{1,1}(X, L) = h^{1,1}(X) + 2$.

(B) $g_2(X, L) = h^2(\mathcal{O}_X) + 1$ and $h_2^{1,1}(X, L) = h^{1,1}(X)$.

(A) First we consider the case (A). Since L is very ample and $g_2(X, L) = h^2(\mathcal{O}_X)$, by [10, Corollary 3.5] we infer that (X, L) is one of the types from (1) to (7.4) in Theorem 2.7. Since $b_2(X, L) = h^2(X, \mathbb{C}) + 2$, by using [13, Example 3.1], we see that (X, L) is one of the following types as possibility.

(a) $(\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$, where p_i is the ith projection.

(b) $(\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}), H(T_{\mathbb{P}^2}))$, where $T_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.

(c) A hyperquadric fibration over a smooth curve.

(d) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is a very ample vector bundle on S with $c_2(\mathcal{E}) = 3$.

(e) A reduction (M, A) of (X, L) is a Veronese fibration over a smooth curve C, that is, M is a $\mathbb{P}^2$-bundle over C and $A|_F = \mathcal{O}_{\mathbb{P}^2}(2)$ for every fiber F of it.

(A.1) The case (a) (resp. (b)) corresponds to the case (i.1) (resp. (i.2)) in Theorem 3.1.

(A.2) Next we consider the case (c) and we use Notation 2.8 (1). Here we note that $h^2(X, \mathbb{C}) = 2$ in this case (see Definition 2.4 (3)). Since $b_2(X, L) = h^2(X, \mathbb{C}) + 2$ and $h^2(X, \mathbb{C}) = 2$, we see from [13, Example 3.1 (5)] that $2e + 3b = 2$. On the other hand, from the fact that $L^n = 2e + b > 0$ and $2e + (n+1)b \geq 0$ by [7, (3.3)], we get the following.

Claim 3.2. *$(e, b) = (1, 0)$ or $(4, -2)$. Moreover $n = 3$ if $(e, b) = (4, -2)$.*

Proof. If $b > 0$, then $2e + 3b = 2e + b + 2b \geq 3$ and this is impossible. So we have $b \leq 0$. If $b = 0$, then $e = 1$. So we assume that $b < 0$. Then $2 = 2e + 3b = 2e + (n+1)b - (n-2)b \geq -(n-2)b \geq -b$ because $n \geq 3$ and $b < 0$. So we have $b = -2$ or -1. If $b = -1$, then $2 = 2e + 3b = 2e - 3$. But this is impossible because e is an integer. Hence $b = -2$ and we see from the above inequality that $n = 3$. We also note that $2e + 3b = 2$ implies $e = 4$. $\square$

(A.2.1) If $(e, b) = (1, 0)$, then $L^n = 2e + b = 2$. Therefore we see that $(X, L) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ because L is very ample. Since $n \geq 3$, we have $\mathrm{Pic}(X) \cong \mathbb{Z}$. But this is impossible because (X, L) is a hyperquadric fibration over a smooth curve.
(A.2.2) Assume that $(e, b) = (4, -2)$. In this case $n = 3$ by Claim 3.2. Therefore $\mathrm{rank}\mathcal{E} = 4$. On the other hand we see from $L^3 = 6$ that $h^1(\mathcal{O}_X) = 0$ holds by Ionescu's result [17]. Hence $C = \mathbb{P}^1$. Therefore by the Riemann-Roch theorem we have

$$h^0(L) = h^0(\mathcal{E}) = \deg \mathcal{E} + (\mathrm{rank}\mathcal{E})\, \chi(\mathcal{O}_C) = 8$$

and X is embedded in $\mathbb{P}^7$. We see from the list of [17] that (X, L) is a Del Pezzo manifold, but this is impossible because $\mathcal{O}(K_X + (n-1)L) \neq \mathcal{O}_X$ in the case (c).

(A.3) Next we consider the case (e). We use Notation 2.8 (2). From [13, Example 3.1 (7.4)] we have

$$2e + 3b = 2. \tag{4}$$

Here we note that by [13, Remark 2.6]

$$g_1(M, A) = 2e + 2b + 1. \tag{5}$$

We also note that $g_1(M, A) \geq 2$ in this case because $K_M + 2A$ is ample. Hence by (5) we have

$$2e + 2b \geq 1. \tag{6}$$

Moreover by [13, Remark 2.6]

$$e + 2b + 2g(C) - 2 = 0. \tag{7}$$

Hence we see from (4) and (7)

$$b = 2 - 4g(C), \tag{8}$$

$$e = 6g(C) - 2. \tag{9}$$

By (6), (8) and (9), we get $2g(C) = b + e \geq \frac{1}{2}$, that is, $g(C) \geq 1$.

Then we have $L^3 \leq A^3 = 8e + 12b = 8$. Since L is very ample and $n = 3$, we have $h^0(L) \geq 4$. Assume that $h^0(L) = 4$. Then X is a 3-dimensional projective space. But this is impossible because X is a fiber space over a smooth curve. Next we consider the case $h^0(L) = 5$. Then X is a hypersurface in $\mathbb{P}^4$ and we

have $\operatorname{Pic}(X) \cong \mathbb{Z}$ in this case. But this is also impossible. So we may assume that $h^0(L) \geq 6$.
(A.3.1) If $L^3 \leq 5$, then $L^3 \geq 2\Delta(X,L)+1$ and $g(X,L) \geq 2 \geq L^3 - 3 \geq \Delta(X,L)$. Hence we see from [9, (3.5) Theorem] that $h^1(\mathcal{O}_X) = 0$. But this is a contradiction because $g(C) \geq 1$.
(A.3.2) Assume that $L^3 = 6$. If $h^0(L) \geq 7$, then $L^3 = 6 > 5 \geq 2\Delta(X,L)+1$ and $g(X,L) \geq 2 \geq \Delta(X,L)$. Hence we see from [9, (3.5) Theorem] that $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.

If $h^0(L) = 6$, then X is embedded in $\mathbb{P}^5$ and by Ionescu's result [17] we have $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.
(A.3.3) Assume that $L^3 = 7$. If $h^0(L) \geq 7$, then $L^3 = 7 \geq 2\Delta(X,L)+1$ and $g(X,L) = g(M,A) = 2e + 2b + 1 = 4g(C) + 1 \geq 5 > 3 \geq \Delta(X,L)$. Hence we see from [9, (3.5) Theorem] that $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.

If $h^0(L) = 6$, then X is embedded in $\mathbb{P}^5$ and by Ionescu's result [17] we have $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.
(A.3.4) Assume that $L^3 = 8$. If $h^0(L) \geq 8$, then $L^3 = 8 > 7 \geq 2\Delta(X,L)+1$ and $g(X,L) = g(M,A) = 2e + 2b + 1 = 4g(C) + 1 \geq 5 > 3 \geq \Delta(X,L)$. Hence we see from [9, (3.5) Theorem] that $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.

If $h^0(L) = 7$ (resp. 6), then X is embedded in $\mathbb{P}^6$ (resp. $\mathbb{P}^5$) and by Ionescu's result [19] we have $h^1(\mathcal{O}_X) = 0$. But this is a contradiction.

(A.4) Next we consider the case (d). In this case, since $\mathcal{E}$ is a very ample vector bundle with $c_2(\mathcal{E}) = 3$, we see from Theorem 2.9 that $(S, \mathcal{E})$ is one of the types from (i.1) to (i.6) in Theorem 3.1.

(B) Next we consider the case (B). Let (M,A) be a reduction of (X,L). Since L is very ample and $g_2(X,L) = h^2(\mathcal{O}_X) + 1$, by [10, Theorem 3.6] and [12, Theorem 1] we infer that (X,L) is one of the following types.

(f) (M,A) is a Mukai manifold.

(g) (M,A) is a Del Pezzo fibration over a smooth curve C. Let $f : M \to C$ be its morphism. In this case there exists an ample line bundle H on C such that $K_M + (n-2)A = f^*(H)$ and $(g(C), \deg H) = (1,1)$.

(h) (M,A) is a quadric fibration over a smooth surface S. Let $f : M \to S$ be its morphism. In this case there exists an ample line bundle H on S such that $K_M + (n-2)A = f^*(K_S + H)$ and (S,H) is one of the following types:

  (h.1) S is a $\mathbb{P}^1$-bundle, $p : S \to B$, over a smooth elliptic curve B, and $H = 3C_0 - F$, where C_0 (resp. F) denotes the minimal section of S with $C_0^2 = 1$ (resp. a fiber of p).

  (h.2) S is an abelian surface, $H^2 = 2$, and $h^0(H) = 1$.

(h.3) S is a hyperelliptic surface, $H^2 = 2$, and $h^0(H) = 1$.

First we note that $b_2(X, L) - h^2(X, \mathbb{C}) = b_2(M, A) - h^2(M, \mathbb{C})$ by [13, Remark 2.2 (3)].

(B.1) First we consider the case (f). Then we see from [10, Example 2.10 (7)] that $(K_M + (n-2)A)^2 A^{n-2} = 0$, $h^1(\mathcal{O}_M) = 0$ and $g_2(M, A) = 1$ holds. Hence by [13, Proposition 3.1] we have

$$h_2^{1,1}(M,A)=10(1-h^1(\mathcal{O}_M)+g_2(M,A))-(K_M+(n-2)A)^2A^{n-2}+2h^1(\mathcal{O}_M)=20.$$

Therefore $b_2(M, A) = 2g_2(M, A) + h_2^{1,1}(M, A) = 22$.

Next we calculate $h^2(M, \mathbb{C})$. Since L is very ample, there exist $n-3$ members $D_1, \ldots, D_{n-3}$ of $|A|$ such that $M_{n-3} := D_1 \cap \cdots \cap D_{n-3}$ is a smooth projective variety of dimension 3 and $\mathcal{O}(K_{M_{n-3}} + A_{M_{n-3}}) = \mathcal{O}_{M_{n-3}}$. By a classification of 3-dimensional Fano manifolds (see [21]), we see that $h^2(M_{n-3}, \mathbb{C}) \leq 10$ and by the Lefschetz theorem we get $h^2(M, \mathbb{C}) \leq 10$. Therefore $b_2(X, L) - h^2(X, \mathbb{C}) = b_2(M, A) - h^2(M, \mathbb{C}) > 2$ and this case is ruled out.

(B.2) Next we consider the case (g). We note that $g_2(M, A) = 1$, $h^1(\mathcal{O}_M) = 1$ and $(K_M + (n-2)A)^2 A^{n-2} = 0$ in this case. Hence by [13, Proposition 3.1] we have

$$h_2^{1,1}(M,A)=10(1-h^1(\mathcal{O}_M)+g_2(M,A))-(K_M+(n-2)A)^2A^{n-2}+2h^1(\mathcal{O}_M)=12.$$

Therefore $b_2(M, A) = 2g_2(M, A) + h_2^{1,1}(M, A) = 14$.

Next we calculate $h^2(M, \mathbb{C})$. First we note that $\tau(A) = n-2$ in this case, where $\tau(A)$ is the nef value of A. Assume that $n \geq 5$. Then

$$\tau(A) = n - 2 > \frac{n}{2} = \frac{n - \dim C + 1}{2}.$$

Hence by the proof of [3, (3.1.1) Theorem] we see that there exists a nonbreaking dominating family T of lines relative to A such that for any $t \in T$ the curve l_t corresponding to t satisfies $(K_M + (n-2)A)l_t = 0$.

(B.2.1) If $n \geq 6$, then $\tau(A) = n - 2 \geq \frac{n}{2} + 1$ holds. Hence by (3.1.1.2) in [3, (3.1.1) Theorem] we see that f is an elementary contraction because $\dim C = 1$. In particular $\rho(M) = \rho(C) + 1 = 2$ and we get $h^2(M, \mathbb{C}) = 2$, where $\rho(M)$ (resp. $\rho(C)$) is the Picard number of M (resp. C). Therefore $b_2(X, L) - h^2(X, \mathbb{C}) = b_2(M, A) - h^2(M, \mathbb{C}) > 2$ and this case is ruled out.

(B.2.2) Next we consider the case $n = 5$. Let l be a line on M relative to A such that l is the curve corresponding to a point of T and let $\nu := -K_M l - 2$. Since $(K_M + (n-2)A)l = 0$, we have $-K_M l = 3$. Hence $\nu = 1$. On the other hand $\tau(A) = n - 2 = 3$. So we get $\nu = 1 \geq 1 = \frac{n-3}{2}$ and $\nu = 1 = \tau(A) - 2$. Hence by [3, (2.5) Theorem] we see that either (2.5.1) or (2.5.2) in [3, (2.5) Theorem] holds because $\dim C = 1$.

If (2.5.1) in [3, (2.5) Theorem] holds, then f is an elementary contraction and $\rho(M) = \rho(C)+1 = 2$. So we get $h^2(M, \mathbb{C}) = 2$. Therefore $b_2(X, L) - h^2(X, \mathbb{C}) = b_2(M, A) - h^2(M, \mathbb{C}) > 2$ and this case is ruled out.

If (2.5.2) in [3, (2.5) Theorem] holds, then there exist two morphism $\phi : M \to W$ and $\pi : W \to C$ such that ϕ is a $\mathbb{P}^2$-bundle over a smooth projective variety W of dimension 3, π is a $\mathbb{P}^2$-bundle over C and $f = \pi \circ \phi$. In this case $\rho(M) = \rho(W) + 1 = \rho(C) + 2 = 3$. So we get $h^2(M, \mathbb{C}) = 3$. Therefore $b_2(X, L) - h^2(X, \mathbb{C}) = b_2(M, A) - h^2(M, \mathbb{C}) > 2$ and this case is ruled out.

(B.3) Finally we consider the case (h). In this case, $g_2(M, A) = h^2(\mathcal{O}_M) + 1 = h^2(\mathcal{O}_S) + 1$ and $(K_M + (n-2)A)^2 A^{n-2} = 2(K_S + H)^2$. So we get

$$\begin{aligned} h_2^{1,1}(M, A) &= 10(1-h^1(\mathcal{O}_M)+g_2(M, A)) - (K_M+(n-2)A)^2A^{n-2} + 2h^1(\mathcal{O}_M) \\ &= 10(\chi(\mathcal{O}_S) + 1) - 2(K_S + H)^2 + 2h^1(\mathcal{O}_S). \end{aligned}$$

(B.3.1) We consider the case (h.1). Then $(K_S + H)^2 = 1$, $h^2(S, \mathbb{C}) = 2$, $h^1(\mathcal{O}_S) = 1$ and $h^2(\mathcal{O}_S) = 0$. Hence $g_2(M, A) = 1$, $h_2^{1,1}(M, A) = 10$ and $b_2(M, A) = 2g_2(M, A) + h_2^{1,1}(M, A) = 12$.

(B.3.2) We consider the case (h.2). Then $(K_S + H)^2 = 2$, $h^2(S, \mathbb{C}) = 6$, $h^1(\mathcal{O}_S) = 2$ and $h^2(\mathcal{O}_S) = 1$. Hence $g_2(M, A) = 2$, $h_2^{1,1}(M, A) = 10$ and $b_2(M, A) = 2g_2(M, A) + h_2^{1,1}(M, A) = 14$.

(B.3.3) We consider the case (h.3). Then $(K_S + H)^2 = 2$, $h^2(S, \mathbb{C}) = 2$, $h^1(\mathcal{O}_S) = 1$ and $h^2(\mathcal{O}_S) = 0$. Hence $g_2(M, A) = 1$, $h_2^{1,1}(M, A) = 8$ and $b_2(M, A) = 2g_2(M, A) + h_2^{1,1}(M, A) = 10$.

Next we calculate $h^2(M, \mathbb{C})$. First we note that $\tau(A) = n - 2$ in this case. Assume that $n \geq 4$. Then

$$\tau(A) = n - 2 > \frac{n-1}{2} = \frac{n - \dim S + 1}{2}.$$

Hence by [3, (3.1.1) Theorem] we see that there exists a non-breaking dominating family of lines relative to A such that for any $t \in T$ the curve l_t corresponding to t satisfies $(K_M + (n-2)A)l_t = 0$.

If $n \geq 6$, then $\tau(A) = n - 2 \geq \frac{n}{2} + 1$ holds. Hence by (3.1.1.2) in [3, (3.1.1) Theorem] we see that f is an elementary contraction because $\dim S = 2$. In particular $\rho(M) = \rho(S) + 1$ and we get $h^2(M, \mathbb{C}) = h^2(S, \mathbb{C}) + 1$. Therefore $b_2(X, L) - h^2(X, \mathbb{C}) = b_2(M, A) - h^2(M, \mathbb{C}) > 2$ for each case and the case where $n \geq 6$ is ruled out.

Next we consider the case $n = 5$. Let l be a line on M relative to A such that l is the curve corresponding to a point of T and let $\nu := -K_M l - 2$. Since $(K_M + (n-2)A)l = 0$, we have $-K_M l = 3$. Hence $\nu = 1$. On the other hand $\tau(A) = n - 2 = 3$. So we get $\nu = 1 \geq 1 = \frac{n-3}{2}$ and $\nu = 1 = \tau(A) - 2$. Hence by [3, (2.5) Theorem] we see that (2.5.1) in [3, (2.5) Theorem] holds because $\dim S = 2$. Then f is an elementary contraction and $\rho(M) = \rho(S) + 1$. So we get $h^2(M, \mathbb{C}) = h^2(S, \mathbb{C}) + 1$. Therefore $b_2(X, L) - h^2(X, \mathbb{C}) = b_2(M, A) - h^2(M, \mathbb{C}) > 2$ and the case where $n = 5$ is also ruled out.

Therefore we get the assertion. $\square$

Corollary 3.3. *Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that L is very ample. If $b_2(X, L) = h^2(X, \mathbb{C}) + 2$, then $n = 3$ or 4.*

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A note on secants of Grassmannians

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Abstract. *Let $\mathbb{G}(k,n)$ be the Grassmannian of k-subspaces in an n-dimensional complex vector space, $k \geq 3$. Given a projective variety X, its s-secant variety $\sigma_s(X)$ is defined to be the closure of the union of linear spans of all the s-tuples of independent points lying on X. We classify all defective $\sigma_s(\mathbb{G}(k,n))$ for $s \leq 12$.*

Keywords: Grassmannians, secant varieties, Terracini lemma
MS Classification 2010: 14M15, 15A69

1. Introduction

Let $X \subset \mathbb{P}^N$ be a non-degenerate projective variety. The *s-th secant variety* $\sigma_s(X)$ is defined to be the closure of the union of linear spans of all the s-tuples of independent points lying on X.

Let $\mathbb{G}(k,n)$ denote the *Grassmannian* parametrizing k-subspaces in an n-dimensional complex vector space. It is embedded in $\mathbb{P}^N = \mathbb{P}(\Lambda^k \mathbb{C}^n)$ via the Plücker map, where $N = \binom{n}{k} - 1$.

Consider the secant variety $\sigma_s(\mathbb{G}(k,n))$. Its dimension is bounded by:

$$\dim \sigma_s(\mathbb{G}(k,n)) \leq \min\{sk(n-k) + s - 1, N\}. \tag{1}$$

We say that $\sigma_s(\mathbb{G}(k,n))$ *has the expected dimension* if equality holds in (1). Otherwise $\sigma_s(\mathbb{G}(k,n))$ is called *defective* and its *defect* is the difference between the right and left hand side in (1).

This short note is a contribution to the classification of defective $\sigma_s(\mathbb{G}(k,n))$. There is an extensive literature related to this highly non-trivial problem—not only on Grassmannians, but on many other homogeneous varieties, such as Veronese varieties [3], Segre products [1], Lagrangian Grassmannians [6] and Spinor varieties [4]. The first one is the only case where the classification is complete. For a recent survey on the subject and its applications we refer the reader to [8].

Using a clever linear algebra observation and Terracini Lemma we prove the following classification result:

THEOREM 1.1. *If $k \geq 3$ then $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension for every $s \leq 12$, except for the cases $(k,n;s) = (3,7;3), (4,8;3), (4,8;4)$ and $(3,9;4)$.*

If $k = 2$ then $\mathbb{G}(k,n)$ is a Grassmannian of (projective) lines and $\sigma_s(\mathbb{G}(k,n))$ is almost always defective—it corresponds to the locus of skew-symmetric matrices of rank at most $2s$. Thus throughout the paper we assume $k \geq 3$. Only four defective cases are known then, and in [5, Conjecture 4.1] it is hypothesized that they are the only ones. Indeed this conjecture can be implicitly found in previous works, for example in [7]. In [5] the authors use a computational technique to check that the conjecture holds true for $n \leq 15$. (The same result for $n \leq 14$ can be found in [9].)

In [7] explicit bounds on $(k,n;s)$ were found for $\sigma_s(\mathbb{G}(k,n))$ to have the expected the dimension. Improving these bounds and using the monomial technique Abo, Ottaviani and Peterson showed that the conjecture is true for $s \leq 6$ [2]. Exploiting the quoted results from [2], together with the explicit computations perforemed in [5], Theorem 1.1 can be strenghtened; this is done in Theorem 3.6, which concludes this note.

2. A lemma on tangent spaces

Let $V \simeq \mathbb{C}^n$ be a complex vector space of dimension n. The Grassmannian $\mathbb{G}(k,V) = \mathbb{G}(k,n)$ is the variety parametrizing k-subspaces in V. The Grassmannian $\mathbb{G}(k,V)$ embeds in $\mathbb{P}(\bigwedge^k V)$ via Plücker map. Remark that if we identify points in $\mathbb{P}^N$ with general skew-symmetric tensors, then points in $\mathbb{G}(k,V)$ correspond to decomposable skew-symmetric tensors.

We start by describing the affine tangent space to the Grassmannian. (Recall that the affine tangent space $\hat{T}X$ is the tangent space to the affine cone of the variety X.)

LEMMA 2.1. *Let $E = e_1 \wedge \ldots \wedge e_k$ be a point of $\mathbb{G}(k,V)$, where $e_i \in V$. The affine tangent space to $\mathbb{G}(k,V)$ at E is:*

$$\hat{T}_E\mathbb{G}(k,V) = \sum_{j=1}^{k} e_1 \wedge \ldots \wedge e_{j-1} \wedge V \wedge e_{j+1} \wedge \ldots \wedge e_k.$$

Using compact notation we can write: $\hat{T}_E\mathbb{G}(k,V) = \bigwedge^{k-1} E \wedge V$.

The proof of Lemma 2.1 is an immediate consequence of Leibniz rule. Using this description we can prove the following result.

LEMMA 2.2. *For $i = 1 \ldots s$, let $E_i = e_{i,1} \wedge \ldots \wedge e_{i,k}$ be points of $\mathbb{G}(k,V)$ such that the spaces $\hat{T}_{E_i}\mathbb{G}(k,V)$ are linearly independent in $\bigwedge^k V$. (Where $(e_{i,j})_{j=1\ldots k}$ are elements of V.)*

Let W be a complex vector space of dimension $m > n$, and consider $V \hookrightarrow W$ any immersion. Then the spaces $\hat{T}_{E_i}\mathbb{G}(k, W)$ are linearly independent in $\bigwedge^k W$. (We keep the notation E_i for the image of the subspaces E_i inside W.)

Proof. The spaces:

$$\begin{aligned}\hat{T}_{E_i}\mathbb{G}(k, W) &= \textstyle\bigwedge^{k-1} E_i \wedge W \\ &= \textstyle\bigwedge^{k-1} E_i \wedge (V \oplus W/V) \\ &= \left(\textstyle\bigwedge^{k-1} E_i \wedge V\right) \oplus \left(\textstyle\bigwedge^{k-1} E_i \wedge W/V\right)\end{aligned}$$

live inside:

$$\textstyle\bigwedge^k W = \bigwedge^k (V \oplus W/V) = \displaystyle\bigoplus_{h=0}^{k} \textstyle\bigwedge^{k-h} V \otimes \bigwedge^h (W/V),$$

and more precisely the situation is:

$$\begin{array}{ccccc} \hat{T}_{E_i}\mathbb{G}(k, V) & = & \bigwedge^{k-1} E_i \wedge V & \oplus & \bigwedge^{k-1} E_i \wedge W/V \\ \cap & & \cap & & \cap \\ \bigwedge^k W & \subseteq & \bigwedge^k V & \oplus & \bigwedge^{k-1} V \otimes (W/V) \end{array} \tag{2}$$

The pieces $\bigwedge^{k-1} E_i \wedge V$ in the first summand of (2) are linearly independent by our assumption, and since the sum is direct, the result follows if we prove the linear independence of the pieces $\bigwedge^{k-1} E_i \wedge W/V$ in the second summand of (2). Elements of $\bigwedge^{k-1} E_i \wedge W/V$ are of the form:

$$\sum_{j=1}^{k} a_{i,j}(e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge w \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k}),$$

for some coefficients $a_{i,j}$ and some nonzero element $w \in W/V$. Without loss of generality we ignore these coefficients in what follows. Linear dependence would mean that there exist $\alpha_1, \ldots, \alpha_s$ not all zero such that:

$$\begin{aligned} 0 &= \sum_{i=1}^{s} \alpha_i \left(\sum_{j=1}^{k} e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge w \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k} \right) \\ &= \left(\sum_{i=1}^{s} \sum_{j=1}^{k} (-1)^{\epsilon} \alpha_i (e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k}) \right) \wedge w, \end{aligned}$$

where we use $(-1)^{\epsilon}$ as a reminder that there might be a sign change. (That can also be ignored without losing any generality.) Since $w \neq 0$ we get that:

$$\sum_{i=1}^{s} \sum_{j=1}^{k} (-1)^{\epsilon} \alpha_i (e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k}) = 0$$

in $\bigwedge^{k-1} V$. Now let $\mu \in V$ be any vector and consider:

$$\sum_{i=1}^{s} \sum_{j=1}^{k} \alpha_i (e_{i,1} \wedge \ldots \wedge e_{i,j-1} \wedge \mu \wedge e_{i,j+1} \wedge \ldots \wedge e_{i,k}) = 0.$$

The linear combination is now in $\bigwedge^k V$; hence we have found a contradiction, and this concludes the proof. □

3. Results

Recall from the introduction that given $X \subset \mathbb{P}^N$ a non-degenerate projective variety, its s-th secant variety $\sigma_s(X)$ is defined to be the closure of the union of linear spans of all the s-tuples of independent points lying on X:

$$\sigma_s(X) = \overline{\bigcup_{p_1,\ldots,p_s \in X} \langle p_1, \ldots, p_s \rangle}.$$

If X is non-degenerate and $\dim X = d$, then

$$\dim \sigma_s(X) \leq \min\{sd + s - 1, N\}. \tag{3}$$

If equality holds in (3) we say that $\sigma_s(X)$ *has the expected dimension*, otherwise we call $\sigma_s(X)$ *defective*, and define its *defect* to be the difference between the two numbers. If $\dim \sigma_s(X) = N$ we say that $\sigma_s(X)$ *fills the ambient space.*

We want to classify all defective $\sigma_s(\mathbb{G}(k,n))$. Since $\dim \mathbb{G}(k,n) = k(n-k)$ note that (3) reduces to (1).

We recall the main tool to compute the dimension of secant varieties, Terracini Lemma. (For a proof we refer to [10, Proposition 1.10].)

LEMMA 3.1 (Terracini Lemma). *Let $p_1, \ldots, p_s$ be general points in X and let z be a general point of $\langle p_1, \ldots, p_s \rangle$. Then the affine tangent space to $\sigma_s(X)$ at z is given by*

$$\hat{\mathrm{T}}_z \sigma_s(X) = \hat{\mathrm{T}}_{p_1} X + \cdots + \hat{\mathrm{T}}_{p_s} X$$

where $\hat{\mathrm{T}}_{p_i} X$ denotes the affine tangent space to X at p_i.

LEMMA 3.2. *If $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension and does not fill the ambient space, then $\sigma_s(\mathbb{G}(k,m))$ has the expected dimension for every $m \geq n$.*

Proof. The statement follows from the computation of Lemma 2.2 together with Terracini Lemma 3.1. □

THEOREM 3.3. *If $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension and does not fill the ambient space, then $\sigma_s(\mathbb{G}(k+t,n+t))$ has the expected dimension for every $t \geq 0$.*

Proof. This is a consequence of the duality of Grassmannians: $\mathbb{G}(k,V) \simeq \mathbb{G}(n-k,V^*)$. If $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension, so does $\sigma_s(\mathbb{G}(n-k,n))$. Then using Lemma 3.2 also $\sigma_s(\mathbb{G}(n-k,n+t))$ has the expected dimension for every $t \geq 0$. Since $\mathbb{G}(n-k,n+t) \simeq \mathbb{G}(n+t-(n-k),n+t) = \mathbb{G}(k+t,n+t)$, the statement follows. □

We are now ready to give a proof of Theorem 1.1 from the Introduction.

Proof of Theorem 1.1. The proof is now an easy consequence of Theorem 3.3 together with the computational evidence provided in [5]. Duality of Grassmannians allows us to assume that $k \leq \frac{n}{2}$. The case $n \leq 15$ has been checked in [5]. Now take $\sigma_s(\mathbb{G}(k,n))$, with k,s as required and $n > 15$. Since for the given values of s the secant variety $\sigma_s(\mathbb{G}(3,15))$ has the expected dimension and does not fill the ambient space, using Lemma 3.2 we can conclude that the statement is true for $\sigma_s(\mathbb{G}(3,n-(k-3))$. For our choice of range of s,k and n we can also claim that $\sigma_s(\mathbb{G}(3,n-(k-3))$ does not fill the ambient space. Theorem 3.3 with $t = k-3$ then implies that the statement is true for $\sigma_s(\mathbb{G}(3+(k-3),n-(k-3)+(k-3)) = \sigma_s(\mathbb{G}(k,n))$. □

Remark 3.4. *Theorem 1.1 can be restated in terms of the conjecture by Baur, Draisma and De Graaf [5, Conjecture 4.1] quoted in the Introduction.*

Remark that all defective cases mentioned in the conjecture have $\sigma_s(\mathbb{G}(k-1,n-1))$ that is either defective or fills the ambient space, so Theorem 3.3 is no contradiction to the conjecture.

To the detriment of its clean statement, Theorem 1.1 can be strengthened using all of values of k in the computational results of [5] on $\mathbb{G}(k,15)$. For a more complete statement, we also include bounds on $(k,n;s)$ proved in [2] using the monomial technique. The result is in fact an extension of [7, Theorem 2.1].

Theorem 3.5. *[2, Theorem 3.3] If $3(s-1) \leq n-k$ and $k \geq 3$ then $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension and does not fill the ambient space.*

We conclude with this stronger statement. Its proof is immediate from the proof of Theorem 1.1, Theorem 3.5 and an explicit computation of the maximal $s = s(k)$ such that the secant $\sigma_s(\mathbb{G}(k,15))$ does not fill the ambient space.

Theorem 3.6. *If $k \geq 3$, $k \leq \frac{n}{2}$ then $\sigma_s(\mathbb{G}(k,n))$ has the expected dimension:*

1. *for $n \leq 15$, all k and s, except $(k,n;s) = (3,7;3), (4,8;3), (4,8;4), (3,9;4)$;*
2. *for $n > 15$, $k \geq 7$, $s \leq \max\{111, \frac{n-k+3}{3}\}$;*
3. *for $n > 15$, $3 \leq k \leq 6$, s as follows:*

(a) $k = 3$, $s \leq \max\{12, \frac{n}{3}\}$
(b) $k = 4$, $s \leq \max\{30, \frac{n-1}{3}\}$
(c) $k = 5$, $s \leq \max\{59, \frac{n-2}{3}\}$
(d) $k = 6$, $s \leq \max\{90, \frac{n-3}{3}\}$.

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Increasing chains and discrete reflection of cardinality[1]

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Abstract. *Combining ideas from two of our previous papers ([26] and [27]), we refine Arhangel'skii Theorem by proving a cardinal inequality of which this is a special case: any increasing union of strongly discretely Lindelöf spaces without uncountable free sequences and with countable pseudocharacter has cardinality at most continuum. We then give a partial positive answer to a problem of Alan Dow on reflection of cardinality by closures of discrete sets.*

Keywords: discrete set, free sequence, elementary submodel, strongly discretely Lindelöf, Arhangel'skii Theorem
MS Classification 2010: 54A25

1. Introduction and notation

All spaces are assumed to be Hausdorff. A set is discrete if each one of its points is isolated in the relative topology. While structurally very simple, discrete sets play an important role in Set-theoretic Topology. For example, by an old result of De Groot, the cardinality of every topological space where discrete sets are countable cannot exceed $2^{\mathfrak{c}}$, where $\mathfrak{c}$ denotes the cardinality of the continuum.

If discrete sets have a strong influence on cardinal properties of topological spaces, their closure are often true *mirrors* of global properties of a topological space (see [1] and [5]). A classical result of Tkachuk [28] states that a topological space X is compact if and only if the closures of its discrete sets are compact. Whether this remains true when compactness is replaced by the Lindelöf property is a well-known open question of Arhangel'skii [3]. Partial answers to this question have been provided in [3], [4] and [24].

Another well-studied open problem, also due to Arhangel'skii [2], is whether closures of discrete sets reflect cardinality in compact spaces. More precisely, Arhangel'skii asked whether $|\overline{D}| = |X|$ for every compact space X and discrete set $D \subset X$. Dow provided consistent counterexamples to this question in [12], while Efimov [13] proved that compact dyadic spaces reflect cardinality. In

[1] The content of this paper was presented at ItEs2012 (Italia - España 2012).

answer to a question of Alan Dow, Juhász and Szentmiklóssy [20] proved that under a slight weakening of the GCH, compact spaces of countable tightness also reflect cardinality.

Aurichi noted in [5], that if X is an L-space, left separated in order type ω_1, then $|\overline{D}| < |X|$, for every discrete set $D \subset X$, so, by Justin Moore's construction of a ZFC L-space, there are non-discretely reflexive Tychonoff spaces in ZFC. But as far as we know, the ZFC existence of a non-discretely reflexive compact space is still open.

Arhangel'skii's question continues to inspire attempts at partial positive solutions. In particular, the following question of Alan Dow is still open.

Problem 1.1: ([12]) Is $g(X) = |X|$ for every compact separable space X?

Where $g(X)$ is defined as the supremum of the cardinalities of the closures of discrete sets in X. We will provide a partial positive answer to the above question in the final part of our paper.

One of the most central results in the theory of cardinal invariants is Arhangel'skii's Theorem, which solved a 50 year old question of Alexandroff (see [17] for a survey).

Theorem 1.2. *Let X be a Lindelöf first-countable space. Then $|X| \leq \mathfrak{c}$.*

Arhangel'skii's original proof of his theorem made use of a particularly strong kind of discrete set called *free sequence.* A set $\{x_\alpha : \alpha < \kappa\}$ is called a *free sequence* if for every $\beta < \kappa$ we have $\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha \geq \beta\}} = \emptyset$. In [27] we showed how the supremum of the sizes of free sequences in the space X ($F(X)$) could replace the tightness in a generalization of the Arhangel'skii Theorem due to Juhász. With some additional help from the technique of elementary submodels, this resulted in a considerably shorter proof of Juhász's Theorem.

Theorem 1.3. *([27]) Let $\{X_\alpha : \alpha < \lambda\}$ be an increasing chain of topological spaces such that $F(X_\alpha) \cdot L(X_\alpha) \cdot \psi(X_\alpha) \leq \kappa$, for every $\alpha < \lambda$. Then $|\bigcup_{\alpha<\lambda} X_\alpha| \leq 2^\kappa$.*

Given a topological space (X, τ), $L(X)$ (the *Lindelöf number* of X) is the minimum cardinal κ such that every cover of X has a subcover of cardinality κ and $\psi(X)$ (the *pseudocharacter* of X) is defined as follows: $\psi(X) = \sup\{\psi(x, X) : x \in X\}$, and $\psi(x, X) = \min\{\kappa : (\exists \mathcal{U} \in [\tau]^\kappa)(\bigcap \mathcal{U} = \{x\})\}$.

The above theorem has been generalized by various authors, especially with the aim of improving it in a non-regular setting and to provide bounds for the cardinality of power-homogeneous spaces (see, for example, [6], [7] and [9] and [10]). Here we present a new refinement in a completely different direction. Putting together ideas from [26] and [27] we are able to replace the Lindelöf number with its supremum on closures of free sequences ($FL(X)$) in Theorem 1.3. As a byproduct we obtain that the cardinality of the union of an increasing chain of a strongly discretely Lindelöf spaces of countable pseudocharacter with countable free sequences does not exceed the continuum. Although in [4], Arhangel'skii and Buzyakova proved that $L(X) \leq F(X) \cdot FL(X)$ for every Tychonoff space X, their proof uses the Tychonoff separation axiom in an essential way (they consider a compactification of X), while we are only assuming X to be Hausdorff. Notation and terminology follow [14] for Topology and [21] for Set Theory.

Kunen's book [21] contains a good introduction on elementary submodels submodel. Dow's article [11] is the most comprehensive survey on applications of elementary submodels to Topology. Other good introductions to this last topic are [15] [16], [17] and [29].

2. Closures of discrete sets and increasing chains

The proof of Theorem 2.1 does not present significant changes from that of the case $\lambda = 1$ in Theorem 1.3, as presented, for example, in [25]. We nevertheless include it, for the reader's convenience.

THEOREM 2.1. *(Juhász, essentially) Let (X, τ) be a space. Then*

$$|X| \leq 2^{FL(X) \cdot \psi(X) \cdot F(X)}$$

Proof. Let $FL(X) \cdot \psi(X) \cdot F(X) = \kappa$ and M be a κ-closed elementary submodel of $H(\theta)$ where θ is a large enough regular cardinal, such that $X, \tau, \kappa \in M$, $\kappa \subset M$ and $|M| = 2^\kappa$.

We claim that $X \subset M$. Suppose this is not the case and let $p \in X \setminus M$. For every $x \in X \cap M$ use the fact that $\psi(x, X) \leq \kappa$ to pick a κ-sized family $\mathcal{U}_x \in M$ such that $\bigcap \mathcal{U}_x = \{x\}$. We actually have $\mathcal{U}_x \subset M$ (see, for example, Theorem 1.6 of [11]), and we can use that to pick $U_x \in \mathcal{U}_x$ such that $x \in U_x$ and $p \notin U_x$.

Let $\mathcal{U} = \{U \in M \cap \tau : x \in U \wedge p \notin U\}$. Then $\mathcal{U}$ covers $X \cap M$. Suppose that for some $\beta < \kappa^+$ we have constructed points $\{x_\alpha : \alpha < \beta\} \subset X \cap M$ and subcollections $\{\mathcal{U}_\alpha : \alpha < \beta\}$ of $\mathcal{U}$ such that $|\mathcal{U}_\alpha| \leq \kappa$ for every $\alpha < \beta$ and $\overline{\{x_\alpha : \alpha < \gamma\}} \subset \bigcup \bigcup_{\alpha \leq \gamma} \mathcal{U}_\alpha$ for every $\gamma < \beta$.

Let $A \subset X$ be a κ-sized free sequence. Note that $|\overline{A}| \leq 2^\kappa$. Indeed, the set $RC(X)$ of all regular closed sets of $\overline{A}$ has cardinality at most 2^κ. The closed pseudocharacter of a Hausdorff space is bounded by the product of the pseudocharacter and the Lindelöf number, so $\psi_c(\overline{A}) \leq \kappa$. Now, for every $x \in \overline{A}$ choose a κ-sized family $\mathcal{U}_x \subset RC(X)$ such that $x \in Int(F)$ for every $F \in \mathcal{U}_x$ and $\bigcap \mathcal{U}_x = \{x\}$. The map $x \to \mathcal{U}_x$ is injective and hence $|\overline{A}| \leq (2^\kappa)^\kappa = \kappa$. From this observation it follows that if $A \in M$ and $|A| \leq \kappa$ then $\overline{A} \subset M$.

In particular, since M is κ-closed we have that $\{x_\alpha : \alpha < \beta\} \in M$ and hence $\overline{\{x_\alpha : \alpha < \beta\}} \subset M$. Therefore, we can choose a κ sized subcollection

$\mathcal{U}_\beta$ of $\mathcal{U}$ covering $\overline{\{x_\alpha : \alpha < \beta\}}$. If $\bigcup_{\alpha \leq \beta} \mathcal{U}_\alpha$ does not cover $X \cap M$ pick a point $x_\beta \in X \cap M \setminus \bigcup_{\alpha \leq \beta} \mathcal{U}_\alpha$. If we didn't stop before reaching κ^+, then $\{x_\alpha : \alpha < \kappa^+\}$ would be a free sequence of size κ^+ in X. Therefore, there is $\mathcal{V} \subset \mathcal{U}$ of size κ such that $X \cap M \subset \bigcup \mathcal{V}$. Note that since M is κ-closed we have $\mathcal{V} \in M$.Therefore $M \models X \subset \bigcup \mathcal{V}$ and hence $H(\theta) \models X \subset \bigcup \mathcal{V}$. So there is $V \in \mathcal{V}$ such that $p \in V$, which is a contradiction. □

The proof of the increasing chain version of Theorem 2.1 relies on the following Lemmas.

Lemma 2.2. *Let X be a space such that $FL(X) \leq \kappa$ and $\mathcal{U}$ be an open cover for X. Then there is a free sequence $F \subset X$ and a subcollection $\mathcal{V} \subset \mathcal{U}$ such that $|\mathcal{V}| = |F| \cdot \kappa$ and $X = \overline{F} \cup \bigcup \mathcal{V}$.*

Proof. Suppose you have constructed, for some ordinal β, a free sequence $\{x_\alpha : \alpha < \beta\}$ and κ-sized subcollections $\{\mathcal{U}_\alpha : \alpha < \beta\}$ of $\mathcal{U}$ such that $\overline{\{x_\alpha : \alpha < \gamma\}} \subset \bigcup_{\alpha \leq \gamma} \bigcup \mathcal{U}_\alpha$ for every $\gamma < \beta$.

Let $\mathcal{U}_\beta$ be a κ-sized subcollection of $\mathcal{U}$ covering the subspace $\overline{\{x_\alpha : \alpha < \beta\}}$ and, if you can, pick a point $x_\beta \in X \setminus \bigcup_{\alpha \leq \beta} \bigcup \mathcal{U}_\beta$. Let μ be the least ordinal such that

$$\overline{\{x_\alpha : \alpha < \mu\}} \cup \bigcup_{\alpha < \mu} \bigcup \mathcal{U}_\alpha = X.$$

Then $F = \{x_\alpha : \alpha < \mu\}$ is a free sequence and if we set $\mathcal{V} = \bigcup_{\alpha < \kappa} \bigcup \mathcal{U}_\alpha$ we have $|\mathcal{V}| = |F| \cdot \kappa$. □

Lemma 2.3. *For every $x \in X$ we have that $FL(X \setminus \{x\}) \leq FL(X) \cdot \psi(X)$.*

Proof. Set $\kappa = FL(X) \cdot \psi(X)$ and let $F \subset X \setminus \{x\}$ be a free sequence in $X \setminus \{x\}$. Let $\mathcal{U}$ be a κ-sized family of open neighbourhood of x such that $\bigcap \mathcal{U} = \{x\}$. Note that $F \subset \bigcup \{X \setminus U : U \in \mathcal{U}\}$, $F \setminus U$ is a free sequence in $X \setminus U$, and $FL(X \setminus U) \leq \kappa$ for every $U \in \mathcal{U}$. Now $Cl_{X \setminus \{x\}}(F) = \bigcup_{U \in \mathcal{U}} \overline{F} \setminus U$. Therefore $L(Cl_{X \setminus \{x\}}(F)) \leq \kappa$ and we are done. □

Theorem 2.4. *Let (X, τ) be a topological space and $\{X_\alpha : \alpha < \lambda\}$ be an increasing chain of subspaces of X such that $X = \bigcup_{\alpha < \lambda} X_\alpha$ and $FL(X_\alpha) \cdot F(X_\alpha) \cdot \psi(X_\alpha) \leq \kappa$. Then $|X| \leq 2^\kappa$.*

Proof. If $\lambda \leq 2^\kappa$ then we are done by Theorem 2.1, so we can assume that $\lambda = (2^\kappa)^+$.

Let μ be a large enough regular cardinal and choose an elementary submodel $M \prec H(\mu)$ such that $[M]^\kappa \subset M$, $|M| = 2^\kappa$, and $\{X, \tau, \kappa, \lambda\} \cup \kappa \subset M$.

Call a set $C \subset X$ *bounded* if $|C| \leq 2^\kappa$.

Claim 1. If $C \in [X \cap M]^{\leq \kappa}$, then $\overline{C}$ is bounded.

Proof of Claim 1. Claim 1 will be proved if we can show that $\overline{C} \subset X \cap M$. So, suppose that this is not true and choose $p \in \overline{C} \setminus M$. Choose θ large enough, so that $\overline{C} \cap M \subset X_\theta$. By $\psi(X_\theta) \leq \kappa$ we can find open neighbourhoods $\{U_\alpha : \alpha < \kappa\}$ of the point p such that $X_\theta \setminus \{p\} = \bigcup_{\alpha<\kappa} X_\theta \setminus U_\alpha$. By Lemma 2.2 we can find a free sequence $D_\alpha \subset X_\theta \setminus U_\alpha$ and relative open sets $\{V_{\alpha\beta} : \beta < \kappa\}$ in $X_\theta \setminus U_\alpha$ such that $X_\theta \setminus U_\alpha \subset \overline{D_\alpha} \cup \bigcup_{\beta<\kappa} V_{\alpha\beta}$ for every $\alpha < \kappa$. By $FL(X_\theta \setminus U_\alpha) \leq \kappa$ we can find relative open sets $\{G_{\alpha\beta} : \beta < \kappa\}$ in $X_\theta \setminus U_\alpha$ such that $\overline{D_\alpha} \subset \bigcup_{\beta<\kappa} G_{\alpha\beta}$, for every $\alpha < \kappa$.

Note that $p \notin \overline{V_{\alpha\beta}} \cup \overline{G_{\alpha\beta}}$, for every $\alpha, \beta < \kappa$. Setting $C_{\alpha\beta} = V_{\alpha\beta} \cap C$ and $E_{\alpha\beta} = G_{\alpha\beta} \cap C$ we then have:

$$\overline{C} \cap X_\theta \setminus \{p\} = \bigcup_{\alpha,\beta<\kappa} (\overline{C_{\alpha,\beta}} \cup \overline{E_{\alpha,\beta}}) \cap X_\theta$$

Note now that by κ-closedness of M, $C_{\alpha\beta} \in M$ and $E_{\alpha\beta} \in M$, for every α, β and θ.
We have:

$$\overline{C} \cap M = \bigcup_{\alpha,\beta<\kappa} (\overline{C_{\alpha,\beta}} \cup \overline{E_{\alpha,\beta}}) \cap M$$

So:

$$M \models \overline{C} = \bigcup_{\alpha,\beta<\kappa} (\overline{C_{\alpha,\beta}} \cup \overline{E_{\alpha\beta}})$$

Which implies:

$$H(\mu) \models \overline{C} = \bigcup_{\alpha,\beta<\kappa} (\overline{C_{\alpha\beta}} \cup \overline{E_{\alpha\beta}})$$

But that is a contradiction, because:

$$H(\mu) \models p \in \overline{C} \setminus \bigcup_{\alpha,\beta<\kappa} (\overline{C_{\alpha,\beta}} \cup \overline{E_{\alpha,\beta}})$$

□

Now we claim that $X \subset M$. Suppose not and choose $p \in X \setminus M$.
Claim 2. The collection $\mathcal{U} = \{U \in M \cap \tau : p \notin U\}$ is an open cover of $X \cap M$.

Proof of Claim 2. Fix $x \in X \cap M$ and let $\mathcal{V} = \{V \in \tau : x \notin \overline{V}\}$. Note that $\mathcal{V} \in M$ and $\mathcal{V}$ covers $X \setminus \{x\}$. Suppose you have constructed subcollections $\{\mathcal{V}_\alpha : \alpha < \beta\}$ of $\mathcal{V}$ such that $\mathcal{V}_\alpha \in M$, $|\mathcal{V}_\alpha| \leq \kappa$ for every $\alpha < \beta$ and a free sequence $\{x_\alpha : \alpha < \beta\} \subset X \cap M$ such that $Cl_{X\setminus\{x\}}(\{x_\alpha : \alpha < \gamma\}) \subset \bigcup_{\alpha<\gamma} \mathcal{V}_\alpha$ for every $\gamma < \beta$. The set $Cl_{X\setminus\{x\}}(\{x_\alpha : \alpha < \beta\})$ is bounded, so we can find an ordinal $\lambda_\beta < \lambda$ such that $Cl_{X\setminus\{x\}}(\{x_\alpha : \alpha < \beta\}) \subset X_{\lambda_\beta}$. Since $FL(X_{\lambda_\beta}) \cdot \psi(X) \leq \kappa$, by Lemma 2.3 we have that the Lindelöf number of $Cl_{X\setminus\{x\}}(\{x_\alpha : \alpha < \beta\})$ is at most κ and hence we can pick a family $\mathcal{V}_\beta \in [\mathcal{V}]^{\leq\kappa}$ such that $Cl_{X\setminus\{x\}}(\{x_\alpha : \alpha < \beta\}) \subset \bigcup \mathcal{V}_\beta$. If $\bigcup_{\alpha\leq\beta} \mathcal{V}_\beta$ covers $X \setminus \{x\}$ we stop, otherwise we pick $x_\beta \in (X \setminus \{x\} \cap M) \setminus \bigcup_{\alpha\leq\beta} \mathcal{V}_\beta$. If we carried this on for κ^+ many steps, then $F = \{x_\alpha : \alpha < \kappa^+\}$ would be a free sequence of cardinality κ^+ in $X \setminus \{x\}$. Since F is bounded, we can choose $\theta < \lambda$ such that $F \subset X_\theta$. So $L(Cl_{X_\theta}(F)) \leq \kappa$. But F cannot converge to x, because every set of cardinality κ^+ of a space of Lindelöf number κ has a complete accumulation point. Therefore there is an open neighbourhood U of x which misses κ^+ many points of F. Therefore $F \setminus U$ is a free sequence in X of cardinality κ^+, but that contradicts $F(X_\theta) \leq \kappa$.

So there is a family $\mathcal{W} \in [\mathcal{U}]^{\leq\kappa}$ such that $X \setminus \{x\} \subset \bigcup \mathcal{W}$. By elementarity, we can assume that $\mathcal{W} \in M$ and hence $\mathcal{W} \subset M$. Let now $W \in \mathcal{W}$ be such that $p \in W$. Then the set $U := X \setminus \overline{W} \in M$ is a neighbourhood of x which misses p. △

Suppose that for some $\beta < \kappa^+$ we have constructed a free sequence $\{x_\alpha : \alpha < \beta\} \subset X \cap M$ and subcollections $\{\mathcal{U}_\alpha : \alpha < \beta\}$ of $\mathcal{U}$ such that $\mathcal{U}_\alpha \in M$, $|\mathcal{U}_\alpha| \leq \kappa$ and $\overline{\{x_\gamma : \gamma < \alpha\}} \subset \bigcup\bigcup_{\gamma<\alpha} \mathcal{U}_\alpha$, for every $\alpha < \beta$. Since $\overline{\{x_\alpha : \alpha < \beta\}}$ is bounded, we have that $L(\overline{\{x_\alpha : \alpha < \beta\}}) \leq \kappa$ and hence we can find a subcollection $\mathcal{U}_\beta$ of $\mathcal{U}$ of size κ such that $\overline{\{x_\alpha : \alpha < \beta\}} \subset \bigcup \mathcal{U}_\beta$. If $\bigcup_{\alpha\leq\beta} \mathcal{U}_\alpha$ does not cover $X \cap M$ we can find a point $x_\beta \in X \cap M \setminus \bigcup_{\alpha\leq\beta} \mathcal{U}_\alpha$. If we didn't stop before reaching κ^+, then $\{x_\alpha : \alpha < \kappa^+\}$ would be a κ^+-sized free sequence in X. But this can't happen, because $\{x_\alpha : \alpha < \kappa^+\}$ is bounded. So there is a $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $X \cap M \subset \bigcup \mathcal{V}$. But since M is κ-closed we have that $\mathcal{V} \in M$ and hence $M \models X \subset \bigcup \mathcal{V}$. Therefore $H(\mu) \models X \subset \bigcup \mathcal{V}$, and hence there is $V \in \mathcal{V}$ such that $p \in V$, which is a contradiction. □

As a corollary, we find a result related to discrete reflection of cardinality, which will be the main subject of the next section.

Lemma 2.5. *[26] Let κ be an infinite cardinal and X be a space where $|\overline{D}| \leq \kappa$ for every discrete $D \subset X$. Then $\psi(X) \leq \kappa$.*

Proof. Let $x \in X$. Now let $\mathcal{V} = \{V \subset X : V \text{ is open and } x \notin \overline{V}\}$. Then $\mathcal{V}$ covers $X \setminus \{x\}$ and hence we can find a discrete $D \subset X \setminus \{x\}$ and a subcollection $\mathcal{U} \subset \mathcal{V}$ with $|\mathcal{U}| = |D|$ such that $X \setminus \{x\} \subset \bigcup \mathcal{U} \cup \overline{D}$. So $(\bigcap_{x\in\overline{D}\setminus\{x\}} X \setminus \{x\}) \cap (\bigcap_{U\in\mathcal{U}} X \setminus \overline{U}) = \{x\}$, which implies that $\psi(x, X) \leq \kappa$. □

Corollary 2.6. *Let $\{X_\alpha : \alpha < \lambda\}$ be an increasing chain of spaces such that $|\overline{D}| \leq \kappa$ for every discrete set $D \subset X_\alpha$ and every $\alpha < \lambda$. Then $|\bigcup_{\alpha<\lambda} X_\alpha| \leq 2^\kappa$.*

3. A reflection theorem for hereditarily normal spaces

In [11], Dow asked whether compact separable spaces reflect cardinality. Even the following special case is at present unknown. Suppose that in some compact space X, the closure of every discrete set has size bounded by the continuum. Is then $|X| \leq \mathfrak{c}$? We are going to prove that this is the case if X is hereditarily normal. As a matter of fact, the only feature of compactness that we need is the fact that pseudocharacter equals character at every point, and separability can be relaxed to the ccc.

A *cellular family* is a family of pairwise disjoint non-empty open sets. The *cellularity* of X is defined as follows: $c(X) = \sup\{|\mathcal{U}| : \mathcal{U}$ is a cellular family in $X\}$. Recall that a π-base in a topological space X is a set $\mathcal{P}$ of non-empty open sets such that for every open set $U \subset X$ there is $P \in \mathcal{P}$ with $P \subset U$. The π-*weight* of X ($\pi w(X)$) is defined as the minimum cardinality of a π-base for X.

Given a cardinal μ, the logarithm of μ is defined as follows $\log(\mu) = \min\{\kappa : 2^\kappa \geq \mu\}$. We need a well-known, often used and easily proven lemma of Shapirovskii.

Lemma 3.1. *(Shapirovskii) Let X be a space and $\mathcal{U}$ be a cover of X. Then there is a discrete set $D \subset X$ and a subcollection $\mathcal{V} \subset \mathcal{U}$ such that $|D| = |\mathcal{V}|$ and $X = \overline{D} \cup \bigcup \mathcal{V}$.*

Theorem 3.2. *Let X be a hereditarily normal space such that $\psi(x, X) = \chi(x, X)$ for every point $x \in X$ and $|\overline{D}| \leq 2^{c(X)}$ for every discrete set $D \subset X$. Then $|X| \leq 2^{c(X)}$*

Proof. Set $\kappa = \log(2^{c(X)})^+$. Let M be a $< \kappa$-closed elementary submodel of $H(\theta)$, for large enough regular θ such that $|M| = 2^{c(X)}$ and M contains everything we need.

Claim 1. For every $x \in X \cap M$ we have $\chi(x, X) \leq 2^\kappa$.

Proof of Claim 1. Fix $x \in X \cap M$. Subclaim: *for every $p \in X \setminus M$ we can find an open $U \in M$ such that $x \in U$ and $p \notin U$.* If that were true, then we could find a family $\mathcal{S}$ of open neighbourhoods of x such that $|\mathcal{S}| \leq 2^\kappa$ and $\bigcap \mathcal{S} \subset X \cap M$. Now $|X \cap M| \leq 2^\kappa$, so x would have pseudocharacter 2^κ in X, and since pseudocharacter and character in X we would be done. To prove the subclaim, let $\mathcal{U}$ be the set of all open sets $U \subset X$ such that $x \notin \overline{U}$. Then $\mathcal{U} \in M$ and $\mathcal{U}$ covers $X \setminus \{x\}$. By Shapirovskii's lemma we can find a subcollection $\mathcal{W} \subset \mathcal{U}$ and a discrete set $D \subset X \setminus \{x\}$ such that $\mathcal{W} \in M$, $D \in M$ $|\mathcal{W}| = |D| \leq 2^\kappa$ and $X \setminus \{x\} \subset \overline{D} \cup \bigcup \mathcal{W}$. Observe that $\overline{D} \in M$ and $|\overline{D}| \leq 2^\kappa$ and hence $\overline{D} \subset X \cap M$. Therefore $p \notin \overline{D}$ and hence there is $W \in \mathcal{W}$ such that $p \in W$. Now $\overline{W} \in M$ and $x \notin \overline{W}$ therefore $X \setminus \overline{W} \in M$ is a neighbourhood of x which misses p. △

Claim 2. The set $X \cap M$ is dense in X.

Proof of Claim 2. Suppose that is not the case. Then there is an open set $V \subset X$ such that $\overline{V} \cap X \cap M = \emptyset$. Let now $x \in X \cap M$ and choose an open set $U_0 \in M$ such that $U_0 \cap \overline{V} = \emptyset$. Suppose we have constructed, for some $\beta \in \kappa^+$ a cellular family $\{U_\alpha : \alpha < \beta\} \subset M$ such that $U_\alpha \cap \overline{V} = \emptyset$ for every $\alpha < \beta$. Then $X \setminus \overline{\bigcup_{\alpha<\beta} U_\alpha} \in M$ and given $y \in X \setminus \overline{\bigcup_{\alpha<\beta} U_\alpha} \cap M$ we can find an open set neighbourhood U_β of y such that $U_\beta \cap \overline{V} = \emptyset$. Now replace U_β with its intersection with $X \setminus \overline{\bigcup_{\alpha<\beta} U_\alpha}$, which is still in M as the intersection of two elements of M. Eventually, $\{U_\alpha : \alpha \in \kappa^+\}$ would be a κ^+-sized cellular family in X, which is a contradiction. △

Putting together Claim 1 and Claim 2 we get that $\pi w(X) \leq 2^\kappa$.

We now claim that $X \subset M$. Indeed, suppose that this is not the case and let $p \in X \setminus M$.

Claim 3. For every $x \in X \cap M$, there is an open set $V \in M$ such that $x \in V$ and $p \notin V$.

Proof of Claim 3. Fix $x \in X \cap M$ and let $\mathcal{U} = \{V \in M : x \notin \overline{V}\}$. The set $\mathcal{U}$ covers $X \setminus \{x\}$. Use Shapirovskii's Lemma to find a discrete set $D \subset X \cap M$ such that $X \setminus \{x\} \subset \overline{D} \cup \bigcup\{U_x : x \in D\}$. By Shapirovskii's bound for the number of regular open sets (see [19] or [22] or [8] for a *game-theoretic* proof) we have that $\rho(X) \leq \pi w(X)^{c(X)} \leq (2^\kappa)^\kappa = 2^\kappa$. Moreover, since by Jones Lemma $\rho(X) \geq 2^{|D|}$ in every hereditarily normal space X, we must have $|D| < \kappa$ and hence $D \in M$. Therefore $\overline{D} \in M$. From $|\overline{D}| \leq 2^{c(X)}$ we get that $\overline{D} \subset X \cap M$ and thus $p \notin \overline{D}$. This implies that there is $x \in D$ such that $p \in U_x$. By letting $V = X \setminus \overline{U_x}$ we get that V is a neighbourhood of x such that $V \in M$ and $p \notin V$. △

If we now let $\mathcal{V} = \{U \in M : p \notin U\}$, we see that $\mathcal{V}$ is an open cover of $X \cap M$. Using Shapirovskii's Lemma again, we obtain the existence of a discrete set $E \subset X \cap M$ such that $X \cap M \subset \overline{E} \cup \bigcup\{U_x : x \in E\}$, where $U_x \in \mathcal{V}$ and $x \in U_x$. By the same reasoning as in the proof of the Claim we have that $\overline{E} \subset X \cap M$. The closure property of M implies that $\overline{E} \cup \bigcup\{U_x : x \in E\} \in M$ and hence $M \models X \subset \overline{E} \cup \bigcup\{U_x : x \in E\}$. By elementarity, we get that $H(\theta) \models X \subset \overline{E} \cup \bigcup\{U_x : x \in E\}$ and therefore there is $x \in E$ such that $p \in U_x$, but that contradicts the definition of $\mathcal{V}$.

Therefore $X \subset M$ and we are done. □

Recall the definition of the *depth* of X: $g(X) = \{|\overline{D}| : D \subset X \text{ discrete}\}$.

Corollary 3.3. *Let X be a compact hereditarily normal ccc space such that $g(X) \leq \mathfrak{c}$. Then $|X| \leq \mathfrak{c}$,* Note that there are consistent examples of compact hereditarily normal hereditarily separable spaces of cardinality $2^{\mathfrak{c}}$ (for example, Fedorchuk's compact S-space), and this shows that the condition about the depth is essential in Corollary 3.3.

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Recent progress on characterizing lattices $C(X)$ and $U(Y)$[1]

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Abstract. *Our effort to weaken algebraic assumptions led us to obtain characterizations of $C(X)$ as Riesz spaces, real ℓ-groups, semi-affine lattices and real lattices by using different techniques. We present a unified approach valid for any "convenient" category. By setting equivalent conditions to equi-uniform continuity, we obtain a characterization of the lattice $U(Y)$ in parallel with that of $C(X)$.*

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1. Introduction

In the early forties and starting mainly by Yosida [20], the topology community was very interested in obtaining internal conditions under which an object is isomorphic to the set $C(X)$ of all the real valued continuous functions on some topological space X.

The problem essentially depends on the algebraic structure in which we are interested, the weaker assumption the more difficult the answer. Whenever $X \neq \varnothing$, the set $C(X)$ endowed with its pointwise defined order becomes a distributive lattice containing all the constant functions into $\mathbb{R}$, thus a copy of $\mathbb{R}$ as a sublattice. Henceforth, our basic starting structure on $C(X)$ will be that of the real lattices (Definition 2.1).

At the crux of most attempts the following conditions on a real lattice L somehow are needed: (a) L embeds into some $C(X)$ and (b) the lattice of bounded elements L^* is isomorphic to $C^*(X)$. Without loosing generality we may assume that X is a Tychonoff space and even realcompact (since $C(X)$ is lattice-ordered algebra unit preserving isomorphic to $C(\upsilon X)$).

The only contribution appearing in the literature for the more general case is that of Jensen [15] as a refinement of that of Anderson [1], but by assuming

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richer compatible algebraic structures, namely for Φ-algebras. However, no effort was made to extend these results to more general situations.

Under this general context, Birkhoff [4] proposed explicitly in his venerable *Lattice Theory* the open problem 81 by asking for an internal characterization of $C(X)$ with X a compact Hausdorff space only as a lattice. The problem was solved by several authors, by making a special emphasis in the Anderson-Blair's solution [2]. Supported by this outstanding result (Lemma 4.3), verification of above condition (b) can be done by using the Urysohn's method on constructing a separating function (Definition 4.11), and embedding condition (a) can be established in any convenient subcategory of the real lattices, where among other requirements, morphisms should be defined by means of operations (Definition 2.3).

Still some conditions are needed to complete the characterization. We shall present different approaches to this aim, namely: 2-universal completeness (Definition 5.2), local uniform completeness (Definition 5.4) and pointwise completeness (Definition 5.8).

Similarly to $C(X)$, an internal characterization of the real lattice $U(Y)$ of real uniformly continuous functions on some uniform space Y will be obtained by determining equi-uniformly continuous sequences (Definition 4.19) and by setting equi-uniform completeness (Definition 5.10).

This paper has a survey character aiming recent contributions by the authors to the problem. All the technical proofs are avoided refering the readers to their respective original sources.

2. Representation in convenient categories

We start denoting by $\boldsymbol{T}$ the category of the topological Hausdorff spaces with their continuous maps $\mathrm{Hom}_{\boldsymbol{T}}$, and by $\boldsymbol{U}$ the category of Hausdorff uniform spaces with their uniformly continuous maps $\mathrm{Hom}_{\boldsymbol{U}}$.

As usual, $C(X) = \mathrm{Hom}_{\boldsymbol{T}}(X, \mathbb{R})$ and $U(Y) = \mathrm{Hom}_{\boldsymbol{U}}(Y, \mathbb{R}) \subseteq C(Y)$ are the sets of real continuous functions on $X \in \boldsymbol{T}$ and real uniformly continuous functions on $Y \in \boldsymbol{U}$ respectively. Our basic structure both on $C(X)$ and $U(Y)$ is that of a distributive lattice by assuming its pointwise defined order relationship:

$$f \leq g \text{ iff } f(x) \leq g(x), \text{ for all } x \in X \text{ or } Y \quad (f, g \in C(X) \text{ or } U(Y)).$$

Notice that whenever $X \neq \varnothing$, the set $\mathbb{R}$ of constant functions becomes a sublattice of $C(X)$. This requirement can be stated in terms of the lattice structure since every densely-ordered countable chain of a distributive lattice is isomorphic to the chain $\mathbb{Q}$ of the rational numbers (Birkhoff [4]).

DEFINITION 2.1. *A real lattice is a distributive lattice containing the conditional completion R of a fixed densely-ordered countable chain by removing the first and the last element.*

In the sequel, we shall make no distinction between R and the chain $\mathbb{R}$ of the real numbers. We are denoting by $\boldsymbol{L}$ the category of the real lattices with their lattice homomorphisms $\mathrm{Hom}_{\boldsymbol{L}}$ being identity on $\mathbb{R}$.

One of the stronger reasons of starting with a real lattice L is that we may work with its real sublattice

$$L^* = \{f \in L : r \leq f \leq s \text{ for some } r, s \in \mathbb{R}\} \in \boldsymbol{L}$$

of *bounded elements*.

On the other hand, morphisms in $\boldsymbol{L}$ are defined by means of "operations". Let us formally generalize this framework: A *signature* is a nonempty set O endowed with a mapping $a : O \to \mathbb{Z}_+$ called *arity*.

Every signature O defines a category $\boldsymbol{O}$ called *universal algebra* whose objects L satisfy that for any $o \in O$, there are subsets $L_1^o, \ldots, L_{a(o)}^o \subseteq L$ and a mapping

$$o_L : L_1^o \times \cdots \times L_{a(o)}^o \to L, \quad (f_1, \ldots, f_{a(o)}) \mapsto o_L(f_1, \ldots, f_{a(o)})$$

called *$a(o)$-ary operation*.

Their homomorphisms $\mathrm{Hom}_{\boldsymbol{O}}$ are the mappings $x : L \to C$ preserving operations:

$$x\left(o_L(f_1, \ldots, f_{a(o)})\right) = o_C\left(x(f_1), \ldots, x(f_{a(o)})\right),$$

for every $o \in O$, $f_1 \in L_1^o, \ldots, f_{a(o)} \in L_{a(o)}^o$.

In the sequel the inclusion symbol among categories refers whenever to be a subcategory (for instance, $\boldsymbol{C} \subseteq \boldsymbol{L}$ means that the $\boldsymbol{C}$-objects and $\boldsymbol{C}$-morphisms becomes at least real lattices and real lattice morphisms). Some technical considerations will be required on setting up a suitable representation theory.

DEFINITION 2.2. *A category $\boldsymbol{C}$ is said to be appropriate if it is a full subcategory of some universal algebra $\boldsymbol{O} \subseteq \boldsymbol{L}$, and L^* is a $\boldsymbol{C}$-subobject of L whenever $L \in \boldsymbol{C}$.*

Denote by $\boldsymbol{K}$ and $\boldsymbol{X}$ the full subcategories of $\boldsymbol{T}$ consisting of compact and realcompact Hausdorff spaces respectively.

DEFINITION 2.3. *A subcategory $\boldsymbol{C} \subseteq \boldsymbol{L}$ is said to be convenient if it is appropriate and satisfies:*

(a) $\{C(X) : X \in \boldsymbol{T}\} \subseteq \boldsymbol{C}$;

(b) *$C(X)$ is $\boldsymbol{C}$-isomorphic to $C(\upsilon X)$ ($\upsilon X \in \boldsymbol{X}$ is the Hewitt-Nachbin realcompactification of X);*

(c) *if $X, Y \in \boldsymbol{X}$, then $T \in \mathrm{Hom}_{\boldsymbol{C}}(C(X), C(Y))$ iff there exists $t \in \mathrm{Hom}_{\boldsymbol{T}}(Y, X)$ such that $Tf = f \circ t$;*

(d) *$C^*(X)$ is $\boldsymbol{C}$-isomorphic to $C(\beta X)$ ($\beta X \in \boldsymbol{K}$ is the Čech-Stone compactification of X).*

Almost all the algebraic structures appearing in the literature regarding characterizations of $C(X)$ are convenient in the above sense, namely: Φ-algebras (see [6, 14, 18]), Archimedean Riesz spaces with a designated weak order unit (see [13, 17, 19]), real ℓ-groups (see [7, 20]) and semi-affine lattices ([9]).

In order to characterize $U(Y)$, first we must define uniform spaces topologically equivalent to realcompact spaces.

Definition 2.4. *A uniform space is called* realcomplete *if it is both complete and uniformly homeomorphic to a subspace of a power of $\mathbb{R}$. In the sequel $\boldsymbol{Y}$ denotes the full subcategory of $\boldsymbol{U}$ of realcomplete Hausdorff uniform spaces.*

Given $Y \in \boldsymbol{U}$, we set

$$cY = \big\{\{f(x)\}_{f \in U(Y)} : x \in Y\big\} \subset \mathbb{R}^{U(Y)},$$

the *prerealcomplete modification* of Y, and $\gamma cY \in \boldsymbol{Y}$ its completion in $\mathbb{R}^{U(Y)}$.

Definition 2.5. *A subcategory $\boldsymbol{C} \subseteq \boldsymbol{L}$ is said to be* uniformly convenient *if it is appropriate and satisfies*

(a) *$\big\{U(Y) : Y \in \boldsymbol{U}\big\} \subseteq \boldsymbol{C}$;*

(b) *$U(Y)$ is $\boldsymbol{C}$-isomorphic to $U(\gamma cY)$ ($\gamma cY \in \boldsymbol{Y}$ is the* realcompletion *of Y);*

(c) *if $X, Y \in \boldsymbol{Y}$, then $T \in \mathrm{Hom}_{\boldsymbol{C}}(U(X), U(Y))$ iff there exists $t \in \mathrm{Hom}_{\boldsymbol{U}}(Y, X)$ such that $Tf = f \circ t$;*

(d) *$U^*(Y)$ is $\boldsymbol{C}$-isomorphic to $U(sY)$ ($sY \in \boldsymbol{K}$ is the* Samuel compactification *of Y).*

In the sequel $\boldsymbol{C}$ denotes either a convenient or uniformly convenient category according either to the topological or uniform case.

Definition 2.6. *The* spectrum (*resp.* uniform spectrum) *of a given object $L \in \boldsymbol{C}$ is the set $X_L^{\boldsymbol{C}} = \mathrm{Hom}_{\boldsymbol{C}}(L, \mathbb{R})$ equipped with the subspace topology* (*resp. $Y_L^{\boldsymbol{C}} = \mathrm{Hom}_{\boldsymbol{C}}(L, \mathbb{R})$ equipped with the subspace uniformity*) *of $\mathbb{R}^L$.*

It is not difficult to prove that $X_L^{\boldsymbol{C}} \in \boldsymbol{X}$, $Y_L^{\boldsymbol{C}} \in \boldsymbol{Y}$ and that both $X_{L^*}^{\boldsymbol{C}}, Y_{L^*}^{\boldsymbol{C}} \in \boldsymbol{K}$. Topological and uniform version of next theorem can be found in [11] and [10] respectively, and it is the key of our representation theory.

Theorem 2.7. *The functors*

$$\boldsymbol{C} \to \boldsymbol{X},\ L \rightsquigarrow X_L^{\boldsymbol{C}}, \quad \textit{and} \quad \boldsymbol{X} \to \boldsymbol{C},\ X \rightsquigarrow C(X),$$

$$\boldsymbol{C} \to \boldsymbol{Y},\ L \rightsquigarrow Y_L^{\boldsymbol{C}}, \quad \textit{and} \quad \boldsymbol{Y} \to \boldsymbol{C},\ Y \rightsquigarrow U(Y),$$

form adjoint situations in convenient and uniformly convenient categories $\boldsymbol{C}$ *respectively.*

As a consequence $X \in \boldsymbol{X}$ iff $X_{C(X)}^{\boldsymbol{C}} = X$, and $Y \in \boldsymbol{Y}$ iff $Y_{U(Y)}^{\boldsymbol{C}} = Y$. Moreover, for $L \in \boldsymbol{C}$ there are reflections

$$\eta_L^{\boldsymbol{C}} \in \mathrm{Hom}_{\boldsymbol{C}}\big(L, C(X_L^{\boldsymbol{C}})\big),\ x \mapsto \eta_L^{\boldsymbol{C}}(f)(x) = x(f) \quad (x \in X_L^{\boldsymbol{C}}, f \in L),$$

$$\mu_L^{\boldsymbol{C}} \in \mathrm{Hom}_{\boldsymbol{C}}\big(L, U(Y_L^{\boldsymbol{C}})\big),\ y \mapsto \mu_L^{\boldsymbol{C}}(f)(y) = y(f) \quad (y \in Y_L^{\boldsymbol{C}}, f \in L),$$

called *spectral* and *uniform spectral representation* of L respectively.

Question 1. *Which* $\boldsymbol{C}$*-stated conditions are required for an object* $L \in \boldsymbol{C}$ *in order to* $\eta_L^{\boldsymbol{C}} \in \mathrm{Iso}_{\boldsymbol{C}}\big(L, C(X_L^{\boldsymbol{C}})\big)$ *or* $\mu_L^{\boldsymbol{C}} \in \mathrm{Iso}_{\boldsymbol{C}}\big(L, U(Y_L^{\boldsymbol{C}})\big)$*?*

We shall proceed in three steps:

- *Embedding*: $L \subseteq C(X_L^{\boldsymbol{C}})$ or $L \subseteq U(Y_L^{\boldsymbol{C}})$;
- *Intermediateness*: $L^* = C^*(X_L^{\boldsymbol{C}})$ or $L^* = U^*(Y_L^{\boldsymbol{C}})$;
- *Completion*: $L = C(X_L^{\boldsymbol{C}})$ or $L = U(Y_L^{\boldsymbol{C}})$.

3. Embedding

The first task will consist on setting when the spectral representation is injective (we may use the notation $\eta_L^{\boldsymbol{C}}(L) = L$).

Let $\boldsymbol{V} \subset \boldsymbol{L}$ be the convenient category consisting of *Archimedean* (i.e. $nf \le g$ for all $n \in \mathbb{N}$ implies $f \le 0$) vector lattices with a designated *weak order unit* $e > 0$ (i.e. $f \wedge e > 0$ for every $f > 0$), and $\mathrm{Hom}_{\boldsymbol{V}}$ their vector lattices homomorphisms mapping weak order units in weak order units.

Luxemburg-Zaanen [17] showed that there is a one-to-one correspondence between $X_L^{\boldsymbol{V}}$ and the set of *real maximal ideals* of L (i.e. vector subspaces M of L not containing the weak order unit e, and which are maximal among those being *solid*, i.e. $|f| \le |g|$ for $g \in M$ implies $f \in M$).

Lemma 3.1. $\eta_L^{\boldsymbol{V}}(L) = L$ *iff the intersection of all the real maximal ideals of* L *is* $\{0\}$.

This condition is usually known as "semisimplicity" and can be generalized to any convenient category $\boldsymbol{C}$.

DEFINITION 3.2. *L is said to be $\boldsymbol{C}$-semisimple if for every $r \in \mathbb{R}$*

$$\bigcap_{x \in X_L^{\boldsymbol{C}}} x^{-1}(r) = \{r\}$$

Since morphisms in convenient categories are defined by means of operations, and reasoning as in [11], we may correspond elements of $X_L^{\boldsymbol{C}}$ with certain subsets of the real lattice L.

LEMMA 3.3. *Let $\boldsymbol{C}$ be a convenient category with signature O. There is a one-to-one correspondence between $X_L^{\boldsymbol{C}}$ and the set $R_L^{\boldsymbol{C}}$ consisting of real indexed families $R = \{R(r)\}_{r \in \mathbb{R}}$ of subsets of L having the following properties*

(a) *$\bigcup R(r) = L$ and $R(r) \cap R(s) = \varnothing$ if $r \neq s$;*

(b) *$R(r) \cap \mathbb{R} = \{r\}$ for every $r \in \mathbb{R}$;*

(c) *if $o_L(f_1, \dots, f_{a(o)}) \in R(r)$, then $f_1 \in R(r_1), \dots, f_{a(o)} \in R(r_{a(o)})$ for some $r_1, \dots, r_{a(o)} \in \mathbb{R}$ such that $o_{\mathbb{R}}(r_1, \dots, r_{a(o)}) = r$, for every $r \in \mathbb{R}$.*

Such families from $R_L^{\boldsymbol{C}}$ are called real-systems of L.

Recall that given $R \in R_L^{\boldsymbol{C}}$, the mapping

$$R^{-1} : L \to \mathbb{R},\ f \mapsto R^{-1}(f) = r, \text{ such that } f \in R(r)$$

belongs to $X_L^{\boldsymbol{C}}$, and conversely if $x \in X_L^{\boldsymbol{C}}$, then $\{x^{-1}(r)\}_{r \in \mathbb{R}} \in R_L^{\boldsymbol{C}}$.

COROLLARY 3.4. *L is $\boldsymbol{C}$-semisimple iff $\bigcap_{R \in R_L^{\boldsymbol{C}}} R(r) = \{r\}$ for every $r \in \mathbb{R}$.*

Unfortunately, $\boldsymbol{C}$-semisimplicity is not enough to ensure $\eta_L^{\boldsymbol{C}}(L) = L$. The well behavior of $\boldsymbol{V}$-semisimplicity responds to the embedding $\eta_{L^*}^{\boldsymbol{V}}(L^*) = L^*$ (from [20]), but without assuming linear structures this fact does not hold.

In [9] the convenient category $\boldsymbol{S}$ of semi-affine lattices is studied in details (roughly speaking, a semi-affine lattice of $C(X)$ is a sublattice which is closed under addition by $\mathbb{R}$ and multiplication by $\{0\} \cup \{w^n : n \in \mathbb{N}\}$ for some real number $w < -1$) where the following counterexample is produced:

EXAMPLE 3.5. *Let*

$$L = \left\{ (a,b,i) \in \mathbb{R}^2 \times \{-1,0,1\} \ : \ \begin{array}{l} \text{if } a < b, \text{ then } i \in \{-1,0\}, \\ \text{if } a = b, \text{ then } i = 0, \\ \text{if } a > b, \text{ then } i \in \{0,1\}, \end{array} \right\}$$

endowed with the following order, addition and multiplication:

$$(a,b,i) \leq (c,d,j) \text{ iff } (a,b) \leq (c,d), \text{ and either } i \leq j \text{ or } a \geq b, c \leq d\,,$$

$$r + (a,b,i) = (a+r, b+r, i)\,, \quad r * (a,b,i) = (ra, rb, \operatorname{sign}(ri))\,.$$

Then L is $\boldsymbol{S}$-semisimple but $\eta^{\boldsymbol{S}}_L(L) \neq L$ (since $\eta^{\boldsymbol{S}}_{L^*}(L^*) \neq L^*$).

Next definition from [3] ensures injectivity of $\eta^{\boldsymbol{L}}_{L^*}$.

Definition 3.6. *$L \in \boldsymbol{L}$ is said to be special if*

(a) *for every $r, s \in \mathbb{R}$ and $f \in L$:*

  (a.1) *$f \vee r \geq s > r$ implies $f \geq s$;*

  (a.2) *$f \wedge r \leq s < r$ implies $f \leq s$;*

(b) *for every pair $f < g$ in L there exists $r < s$ in $\mathbb{R}$ and $h \in L$ such that $f \wedge h \leq r$ and $g \wedge h \not\leq t$ for every $t < s$.*

Lemma 3.7. *L^* is special iff $\eta^{\boldsymbol{L}}_{L^*}(L^*) = L^*$.*

By adding speciality to semisimplicty, injectivity of the spectral representation yields in any convenient category (see [11]).

Theorem 3.8. *L is special and $\boldsymbol{C}$-semisimple iff $\eta^{\boldsymbol{C}}_L(L) = L$.*

4. Intermediateness

Yosida proved in [20] that L^* is uniformly dense in $C(X^{\boldsymbol{V}}_{L^*})$. However this fact does not work in weaker convenient categories, even by assuming speciality as one can see in the next counterexample extracted from [3].

Example 4.1. *Let $L = \{f \in C(\{0,1\}) : |f(0) - f(1)| < 1\}$. Then L is special and $\boldsymbol{L}$-semisimple, but L^* is not uniformly dense in $C(X^{\boldsymbol{L}}_{L^*})$.*

In order to solve this gap, Anderson-Blair introduced in [3] the notion of normality.

Definition 4.2. *$L \in \boldsymbol{L}$ is said to be normal if for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ with $\beta < \gamma$ and for every $f \in L^*$, there exist $g, h, k \in L^*$ such that $g \wedge h \leq \alpha$, $\beta \leq h \vee f$, $f \wedge k \leq \gamma$ and $\delta \leq k \vee g$.*

This condition allowed them to obtain a Stone-Weierstrass-like theorem in the category $\boldsymbol{L}$.

Lemma 4.3 (Stone-Weierstrass-like). *L^* is special and normal iff L^* is uniformly dense in $C(X^{\boldsymbol{L}}_{L^*})$.*

Once arrived at this point, our interest focuses on stating a Kakutani-like theorem in $\boldsymbol{C}$, i.e. to obtain conditions under which L^* is uniformly dense in $C^*(X_L^{\boldsymbol{C}})$. In general, it is false (even by assuming semisimplicity, speciality and normality) as next example from [13] shows.

Example 4.4. *Let* $L = \{f_{|\mathbb{R}} : f \in C(\overline{\mathbb{R}}, \overline{\mathbb{R}}),\ f(x) = \pm\infty \text{ iff } x = \pm\infty\}$. *Then* L *is* $\boldsymbol{V}$*-semisimple (of course* L^* *is both special and normal) but* L^* *is not uniformly dense in* $C^*(X_L^{\boldsymbol{V}})$.

Next lemma from [11] will be important to our aims.

Lemma 4.5 (Kakutani-like). *Under the assumption of speciality and normality,* L *is* $\boldsymbol{C}$*-semisimple iff* $X_{L^*}^{\boldsymbol{C}}$ *is a compactification of* $X_L^{\boldsymbol{C}}$. *As a consequence, if* L *is special, normal and* $\boldsymbol{C}$*-semisimple, the following are equivalent:*

(i) L^* *is uniformly dense in* $C^*(X_L^{\boldsymbol{C}})$;

(ii) L^* *separates disjoint zero-sets of* $X_L^{\boldsymbol{C}}$.

Functionally separated subsets can be described by means of the method of the famous Urysohn's lemma. We shall start by determining closed subsets:

Definition 4.6. *A real indexed family* $C = \{C(r)\}_{r\in\mathbb{R}}$ *of subsets of* L *is said to be a closed-system if there exists a class* $\{R_i\}_i \subseteq R_L^{\boldsymbol{C}}$ *of real-systems in* L *(defined as in Lemma 3.3) such that* $C(r) = \bigcap_i R_i(r)$ *for every* $r \in \mathbb{R}$.

If $F \neq \varnothing$ is a closed subset of $X_L^{\boldsymbol{C}}$, then the family $C_F = \{\bigcap_{x\in F} x^{-1}(r)\}_{r\in\mathbb{R}}$ becomes a closed-system, and conversely, if $C = \{\bigcap_i R_i(r)\}_{r\in\mathbb{R}}$ is a closed-system, then $F_C = \{R_i^{-1}\}_i$ becomes a closed subset of $X_L^{\boldsymbol{C}}$ (see in the comments below Lemma 3.3 how the morphisms R_i are constructed). Furthermore, $C_{X_L^{\boldsymbol{C}}} = \mathbb{R}$ and $F_{\mathbb{R}} = X_L^{\boldsymbol{C}}$.

Denote by $C_L^{\boldsymbol{C}}$ the set consisting of its closed-systems by adding L as one of its members under the assertion $C_\varnothing = L$ and $F_L = \varnothing$. As a consequence:

Corollary 4.7. *There is a one-to-one correspondence between* $C_L^{\boldsymbol{C}}$ *and the family of closed subsets of* $X_L^{\boldsymbol{C}}$.

Actually, $C_L^{\boldsymbol{C}}$ becomes a complete lattice with the first element $\mathbb{R}$ and the last element L, whenever we are setting for $B, D \in C_L^{\boldsymbol{C}}$ the lattice operations

$$B \wedge D = C_{(F_B \cup F_D)} \text{ and } B \vee D = C_{(F_B \cap F_D)}.$$

A description of cozero-sets was given by Kerstan [16].

Proposition 4.8. *A subset of a topological space* X *is a cozero-set iff it belongs to a family* $\mathscr{V}$ *of open sets having the property that for every its member* U *there exist two sequences* $\{U_n\}_n, \{V_n\}_n$ *in* $\mathscr{V}$ *such that*

$$U = \bigcup U_n\,,\ U_n \subset X \setminus V_n \subset U,\ \textit{for each } n.$$

Above result will be helpful in order to determine zero-sets.

DEFINITION 4.9. *A closed-system in L is said to be a zero-system provided it belongs to a class $\mathscr{X}$ of closed-systems in L having the property that for every its member B there are countable sequences $\{B_n\}_n$, $\{D_n\}_n$ from $\mathscr{X}$ such that*

$$B = \bigvee_n B_n, \ B_n \wedge D_n = \mathbb{R} \text{ and } D_n \vee B = L, \text{ for all } n.$$

Denote by $Z_L^{\boldsymbol{C}}$ the set consisting of zero-systems in L.

LEMMA 4.10. *There is a one-to-one correspondence between $Z_L^{\boldsymbol{C}}$ and the family of zero-sets of $X_L^{\boldsymbol{C}}$.*

Proof. Every zero-system belongs to a class $\mathscr{X}$ of closed-systems in L such that for every its member B there are sequences $\{B_n\}_n$,$\{D_n\}_n$ from $\mathscr{X}$ such that $B = \bigvee_n B_n$, $B_n \wedge D_n = \mathbb{R}$ and $D_n \vee B = L$, for every n. One derives that $F_B = \bigcap_n F_{B_n}$, $F_B \subset X_L^{\boldsymbol{C}} \setminus F_{D_n} \subset F_{B_n}$. Thus, F_C becomes a zero-set whenever $C \in Z_L^{\boldsymbol{C}}$.

Conversely, from Proposition 4.8, the complementary of a given zero-set Z from $X_L^{\boldsymbol{C}}$ belongs to a family $\mathscr{V}$ of open sets in X_L having the property that that for every its member U there are sequences sequences $\{U_n\}_n, \{V_n\}_n$ from $\mathscr{V}$ such that $U = \bigcup U_n$, $U_n \subset X_L^{\boldsymbol{C}} \setminus V_n \subset U$ for each n. By taking $\mathscr{X} = \{C_{X_L^{\boldsymbol{C}} \setminus U} : U \in \mathscr{V}\}$, the closed-systems $B_n = C_{X_L^{\boldsymbol{C}} \setminus U_n}$, $D_n = C_{X_L^{\boldsymbol{C}} \setminus V_n}$ satisfy $C_{X_L^{\boldsymbol{C}} \setminus U} = \bigvee_n B_n$, $B_n \wedge D_n = \mathbb{R}$ and $D_n \vee B = L$, for every n. As a consequence, $C_Z \in Z_L^{\boldsymbol{C}}$. □

DEFINITION 4.11. *L is said to be $\boldsymbol{C}$-separating provided $C(r) \cap D(s) \neq \varnothing$ for every pair of distinct reals $r \neq s$, and for any pair $C, D \in Z_L^{\boldsymbol{C}}$ which satisfies $C \vee D = L$.*

On the one hand, $C \vee D = L$ for $C, D \in Z_L^{\boldsymbol{C}}$ is equivalent to assert that F_C, F_D are disjoint zero-sets of $X_L^{\boldsymbol{C}}$. On the other hand, if $C = \{\bigcap_i R_i(r)\}_{r\in\mathbb{R}}$ and $D = \{\bigcap_j S_j(r)\}_{r\in\mathbb{R}}$, $f \in C(r) \cap D(s)$ iff $R_i^{-1}(f) = r$ and $S_j^{-1}(f) = s$ for every i, j, equivalently $f(F_C) = r$ and $f(F_D) = s$. As a consequence:

THEOREM 4.12 (Uryshon-like). *L is $\boldsymbol{C}$-separating iff L separates functionally separated subsets of $X_L^{\boldsymbol{C}}$.*

The isomorphism $L^* = C^*(X_L^{\boldsymbol{C}})$ can be currently obtained by assuming uniform completeness. However, to define this concept subtraction and absolute value operations are needed. A partial solution was proposed in [3].

DEFINITION 4.13. *A continuous ideal is a solid subset I of L which is closed under finite suprema and such that for any $0 < r \in \mathbb{R}$ there exists $0 < \alpha < \beta < r$ in $\mathbb{R}$, $k_1, k_2 \in L$ and $g \in I$ such that $I \leq k_1 \vee k_2$, and if $h \in I$, $g \leq h$ and $k_i \wedge \alpha \nleq h$, then $k_i \wedge h \leq \beta$ $(i = 1, 2)$.*

Next theorem constitutes the Anderson-Blair's solution [3] to the problem 81 of Birkhoff.

THEOREM 4.14. *L is special, normal and every continuous ideal in L^* has a supremum in L^* iff $L^* = C(X^{\boldsymbol{L}}_{L^*})$.*

We shall say:

DEFINITION 4.15. *L is $\boldsymbol{C}$-intermediate if:*

(a) *L is $\boldsymbol{C}$-semisimple;*

(b) *L is $\boldsymbol{C}$-separating;*

(c) *L is special, normal and every continuous ideal in L^* has a supremum in L^*.*

From all above mentioned, we get the following result:

COROLLARY 4.16. *L is $\boldsymbol{C}$-intermediate iff $C^*(X^{\boldsymbol{C}}_L) \subseteq L \subseteq C(X^{\boldsymbol{C}}_L)$.*

Once arrived at this point, we asked whether it would be possible to translate this intermediate situation to uniform spaces. Suppose $\boldsymbol{C}$ is a uniformly convenient category and $L \in \boldsymbol{C}$.

Given $\delta > 0$ and $g \in L$ we set

$$U_{\delta,g} = \big\{(R,S) \in R^{\boldsymbol{C}}_L \times R^{\boldsymbol{C}}_L : g \in R(r) \cap S(s) \text{ implies } |r-s| < \delta\big\}.$$

Notice that the definition is internal in character since the operation $|r-s| < \delta$ is in $\mathbb{R}$.

DEFINITION 4.17. *A sequence $\{f_n\}_n \subseteq L$ is said to be equi-uniformly $\boldsymbol{C}$-continuous if for any $\varepsilon > 0$ there are $g_1, \ldots, g_m \in L$ and $\delta > 0$ such that for all n*

$$f_n \in \bigcap \left\{ \left[\bigcup \{R(r) \cap S(s) : |r-s| < \varepsilon\} \right] : (R,S) \in U_{\delta,g_1} \cap \cdots \cap U_{\delta,g_m} \right\}.$$

If $Y \in \boldsymbol{Y}$, then $\{f_n\}_n \subseteq U(Y)$ is equi-uniformly $\boldsymbol{C}$-continuous iff for any $\varepsilon > 0$ there exists an entourage U of Y such that $|f_n(x) - f_n(y)| < \varepsilon$ for all n, $(x,y) \in U$.

We have recently obtained in [12] an equivalent condition to intermediateness by avoiding separation.

THEOREM 4.18. *Suppose L is special, normal and every continuous ideal in L^* has a supremum in L^* (recall from Theorem 4.14 that this is equivalent to $L^* = U(Y^{\boldsymbol{C}}_{L^*})$). The following are equivalent:*

(i) $L^* = U^*(Y_L^{\boldsymbol{C}})$;

(ii) *every equi-uniformly* $\boldsymbol{C}$*-continuous sequence* $\{f_n\}_n \subset L_+$ *bounded from above by a real number has a supremum* f *in* L *which satisfies* $f \wedge g = 0$ *whenever* $f_n \wedge g = 0$ *for all* n.

Now we may add a condition ensuring injectivity of $\mu_L^{\boldsymbol{C}}$ and furthermore determining uniform intermediateness (see [12]).

Definition 4.19. *L is uniformly* $\boldsymbol{C}$*-intermediate if:*

(a) *L is special, normal and every continuous ideal in* L^* *has a supremum in* L^*;

(b) *every equi-uniformly* $\boldsymbol{C}$*-continuous sequence* $\{f_n\}_n \subset L_+$ *bounded from above by a real number has a supremum* f *in* L *which satisfies* $f \wedge g = 0$ *whenever* $f_n \wedge g = 0$ *for all* n;

(c) *if* $f \neq g$ *from* L, *there exist* $n, k \in \mathbb{N}$ *such that* $(f \wedge (-k) \vee n) \neq (g \wedge (-k) \vee n)$.

Corollary 4.20. *L is uniformly* $\boldsymbol{C}$*-intermediate iff* $U^*(Y_L^{\boldsymbol{C}}) \subseteq L \subseteq U(Y_L^{\boldsymbol{C}})$.

5. Completion

We produced in [8] a $\boldsymbol{C}$-intermediate lattice not isomorphic to any $C(X)$.

Example 5.1. *Let* $L = C^*(\mathbb{N}) \cup \{f \in C(\mathbb{N}) : |f(n)| \leq n$ *starting from some* $n_0 \in \mathbb{N}\}$. *Then L is* $\boldsymbol{L}$*-intermediate, but L is not* $\boldsymbol{L}$*-isomorphic to* $C(\mathbb{N})$.

Next definition close to inversion closeness is due to Feldman-Porter [5].

Definition 5.2. *L is 2-universally* $\boldsymbol{C}$*-complete if any sequence* $\{f_n\}_n \subseteq L_+$ (*resp. in* L_-) *having some member* $f_m \notin R(0)$ *for every* $R \in R_L^{\boldsymbol{C}}$ *and satisfying that* $f_n \wedge f_k \neq 0$ (*resp.* $f_n \vee f_k \geq 0$) *for at most two indices k distinct from n, has a supremum* (*resp. infimum*) f *in* L.

Montalvo et al. obtained in [19] an internal characterization of $C(X)$ as a Riesz space.

Theorem 5.3. *The following are equivalent:*

(i) *L is* $\boldsymbol{V}$*-isomorphic to some* $C(X)$;

(ii) *L is* $\boldsymbol{V}$*-intermediate and 2-universally* $\boldsymbol{V}$*-complete.*

By taking into account that $|f - g| \leq \varepsilon$ on $\mathrm{coz}(h)$ iff $mh \wedge |f - g| \leq \varepsilon$ for all $m \in \mathbb{N}$, a "local uniform completeness" definition can be proposed.

Definition 5.4. *Let L be a $\boldsymbol{V}$-object with a designated weak order unit e. A sequence $\{f_n\}_n$ in L_+ is said to be* locally $\boldsymbol{V}$-Cauchy *if there exists a subset H of L_+ contained in no real maximal ideal and having the property: if $h \in H$, and $\varepsilon > 0$, then there exists $n_\varepsilon^h \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $mh \wedge |f_n - f_{n_\varepsilon^h}| \leq \varepsilon e$ whenever $n \geq n_\varepsilon^h$.*

L is said to be locally uniformly $\boldsymbol{V}$-complete *if for every locally $\boldsymbol{V}$-Cauchy sequence $\{f_n\}_n$ in L_+ there exists $f \in L_+$ such that for every $h \in H$ and $\varepsilon > 0$, $mh \wedge |f - f_{n_\varepsilon^h}| \leq \varepsilon e$ for all $m \in \mathbb{N}$.*

In [13] we have obtained recently an improvement of Theorem 5.3 by removing both uniform completeness and 2-universal completeness.

Theorem 5.5. *The following are equivalent:*

(i) *L is $\boldsymbol{V}$-isomorphic to some $C(X)$;*

(ii) *L is $\boldsymbol{V}$-semisimple, $\boldsymbol{V}$-separating and locally uniformly $\boldsymbol{V}$-complete.*

Furthermore, in the category $\boldsymbol{S}$ of semi-affine lattices, condition $mh \wedge |f - g| \leq \varepsilon$ for all $m \in \mathbb{N}$ is equivalent to both

$$(w^{2m} * h) \wedge g_+ - \varepsilon \leq (w^{2m} * h) \wedge f_+ \leq (w^{2m} * h) \wedge g_+ + \varepsilon, \text{ and}$$
$$(w^{2m-1} * h) \vee g_- - \varepsilon \leq (w^{2m-1} * h) \vee f_- \leq (w^{2m-1} * h) \vee g_- + \varepsilon.$$

By shifting the condition that H is contained in no real maximal ideal by that: for any $R \in R_L^{\boldsymbol{S}}$ there exists $f \in H$ such that $f \notin R(0)$, then local uniform $\boldsymbol{S}$-completeness yields, and we derive next characterization theorem (see [8]).

Theorem 5.6. *The following are equivalent:*

(i) *L is $\boldsymbol{S}$-isomorphic to some $C(X)$;*

(ii) *L is special, $\boldsymbol{S}$-semisimple, $\boldsymbol{S}$-separating and locally uniformly $\boldsymbol{S}$-complete.*

However, local uniform completeness can not be stated in $\boldsymbol{L}$, and Theorem 5.3 does not work, as next example from [8] shows.

Example 5.7. *Let $L = C^*(\mathbb{R}) \cup C_+(\mathbb{R}) \cup C_-(\mathbb{R})$. Then L is $\boldsymbol{L}$-intermediate and 2-universally $\boldsymbol{L}$-complete, but L is not $\boldsymbol{L}$-isomorphic to some $C(X)$.*

We develop a different approach.

Definition 5.8. *A sequence $\{f_n\}_n$ in L is said to be* pointwise $\boldsymbol{C}$-bounded *if for every $R \in R_L^{\boldsymbol{C}}$ there are $r < s$ in $\mathbb{R}$ with $f_n \wedge r \in R(r)$ and $f_n \vee s \in R(s)$ for each n.*

L is said to be pointwise $\boldsymbol{C}$-complete *if every increasing (resp. decreasing) pointwise $\boldsymbol{C}$-bounded sequence $\{f_n\}_n$ in L having the property that $f_n \wedge k = f_k$ (resp. $f_n \vee -k = f_k$) for every $n > k$, has a supremum (resp. infimum) f in L which satisfies $f \wedge n = f_n$ (resp. $f \vee -n = f_n$) for each n.*

In [11] we have obtained the most general characterization of $C(X)$ up to the date.

THEOREM 5.9. *The following are equivalent:*

(i) *L is $\boldsymbol{C}$-isomorphic to some $C(X)$;*

(ii) *L is $\boldsymbol{C}$-intermediate and pointwise $\boldsymbol{C}$-complete.*

On realizing a uniform version of previous theorem we had to impose different requirements.

DEFINITION 5.10. *$\{f_n\}_n$ is said to be weakly pointwise $\boldsymbol{C}$-bounded from above (resp. below) if for every $R \in R_L^{\boldsymbol{C}}$ there is n with $f_n \notin R(n)$ (resp. $f_n \notin R(-n)$).*

L is said to be equi-uniformly $\boldsymbol{C}$-complete if every equi-uniformly $\boldsymbol{C}$-continuous and weakly pointwise $\boldsymbol{C}$-bounded from above (resp. from below) sequence $\{f_n\}_n$ in L having the property that $f_n \wedge k = f_k$ (resp. $f_n \vee -k = f_k$) whenever $n > k$, has an upper bound (or a lower bound) f in L which satisfies $f \wedge n = f_n$ (or $f \vee -n = f_n$, resp.) for all n.

We obtained in [12]:

THEOREM 5.11. *The following are equivalent:*

(i) *L is $\boldsymbol{C}$-isomorphic to some $U(Y)$;*

(ii) *L is uniformly $\boldsymbol{C}$-intermediate and equi-uniformly $\boldsymbol{C}$-complete.*

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On a coefficient concerning an ill-posed Cauchy problem and the singularity detection with the wavelet transform

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Abstract. *We study the Cauchy problem for 2nd order weakly hyperbolic equations. F. Colombini, E. Jannelli and S. Spagnolo showed a coefficient giving a blow-up solution in Gevrey classes. In this paper, we get a simple representation of the coefficient degenerating at an infinite number of points, with which the Cauchy problem is ill-posed in Gevrey classes. Moreover, we also report numerical results of the singularity detection with wavelet transform for coefficient functions.*

Keywords: weakly hyperbolic equations; ill-posed Cauchy problem; Gevrey classes; wavelet transform
MS Classification 2010: 35L15; 65J20; 65T60

1. Introduction

We are concerned with the Cauchy problem on $[0,T]\times\mathbf{R}_x$

$$\begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = 0, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x). \end{cases} \tag{1}$$

Throughout this paper, we assume the weakly hyperbolic condition, i.e.,

$$a(t) \geq 0 \quad \text{for} \quad t \in [0,T].$$

We denote by $G^s(\mathbf{R})$ the space of Gevrey functions satisfying

$$\sup_{x\in K} |\partial_x^n g(x)| \leq C_K r_K^n n!^s \quad \text{for any compact set } K \subset \mathbf{R},\ n \in \mathbf{N}.$$

From the finite propagation property of hyperbolic equations, it is sufficient to consider compactly supported initial data u_0, u_1 and solution u (see [3], [6], [7], etc). Thanks to this fact, we may use the following Gevrey norm for the functions on the whole interval $\mathbf{R}$:

$$\|g\|_{s,r} = \sup_{n\in\mathbf{N}} \frac{\|\partial_x^n g\|_{L^\infty(\mathbf{R})}}{r^n n!^s}.$$

We say that the Cauchy problem (1) is well-posed in G^s, if for any $u_0, u_1 \in G^s$, there is a unique solution $u \in C^2([0,T]; G^s)$ satisfying the energy estimate:

$$\|u(t)\|_{s,R} + \|\partial_t u(t)\|_{s,R} \le C_T\Big(\|u_0\|_{s,r} + \|u_1\|_{s,r}\Big) \quad \text{for} \quad t \in [0,T], \tag{2}$$

where R is a constant greater than r, which implies that the derivative loss *possibly* occurs in a sense of the radius of the Gevrey class G^s. To know that the derivative loss *really* occurs, we have a great interest for the counterexample.

There are many kinds of results on the well-posedness for 2nd order weakly hyperbolic equations (see [2], [4], [5], [6], [9] etc). Let us denote by $C^{k,\alpha}[0,T]$ ($k \in \mathbf{N}$, $0 \le \alpha \le 1$) the space of functions having k-derivatives continuous, and the k-th derivative Hölder continuous with exponent α on $[0,T]$. Especially for the coefficient $a \in C^{k,\alpha}[0,T]$, F. Colombini, E. Jannelli and S. Spagnolo [4] proved the well-posedness in G^s for $1 < s < 1 + (k+\alpha)/2$. Moreover, they also showed an example of a coefficient $a(t)$ giving a blow-up solution u as follows:

Theorem 1.1. ([4]) *For every $T > 0$, $k \in \mathbf{N}$ and $0 \le \alpha \le 1$, it is possible to construct a function $a(t)$, C^∞ and strictly positive on $[0,T)$, zero at $t = T$, and solution u of (1) in a way that $a(t)$ belongs to $C^{k,\alpha}[0,T]$ and u belongs to $C^1([0,T), G^s)$ for $s > 1 + (k+\alpha)/2$, whereas $\{u(t,\cdot)\}$ is not bounded in $\mathcal{D}'$, as $t \uparrow T$.*

Remark 1.2. $a \in C^{k,\alpha}[0,T]$ means that the zero extension of $a(t)$ belongs to $C^{k,\alpha}[0,\infty)$.

Their prior work [5] showed an example of $a \in C^\infty[0,T]$ giving a blow-up solution $u \in C^1([0,T), C^\infty)$. The main task of the proof of Theorem 1.1 is to construct the coefficient $a(t)$ defined piecewise on an infinite number of intervals between $[0,T]$. The piecewise functions are connected at the endpoints of contiguous intervals with a smooth cut off function. For this reason, it would not be easy to represent such a function $a(t)$. The behavior of $a(t)$ is well controlled with the parameters ρ_j, ν_j and δ_j regarded as *dilation*, *frequency* and *degeneracy* respectively.

Remark 1.3. As for the strictly hyperbolic case, F. Colombini, E. De Giorgi and S. Spagnolo [3] showed an example of $a \in C^\alpha[0,T]$ giving a blow-up solution $u \in C^1([0,T), G^s)$ for $s > 1/(1-\alpha)$. In this case, the degeneracy parameter δ_j is not necessary, and the piecewise functions in $a(t)$ can be connected at the endpoints of contiguous intervals without a cut off function.

1.1. Main Results

We shall follow their brilliant method with the parameters, and change some parts of their construction in order to represent the coefficient in a simple form without a smooth cut off function. We also say that the Cauchy problem (1) is ill-posed in G^s if the Cauchy problem (1) is not wellposed in G^s, i.e., the energy inequality (2) breaks. For the equations with lower order terms (having an interaction between several coefficients), the ill-posedness can be proved with an energy based on the Lyapunov function (see [7], [8]).

We note that the coefficient $a(t)$ in Theorem 1.1 degenerates only at $t = T$ where its regularity becomes $C^{k,\alpha}$. For our purpose to represent the coefficient in a simple form, $a(t)$ must be allowed to have oscillations touching the t axis. In fact, the case degenerating at an infinite number of points is more difficult situation than the case degenerating only at one point in the construction of a counterexample with an energy inequality. Assuming that $k = 0, 1$, we can get the following representation of the coefficient degenerating at an infinite number of points:

Theorem 1.4. *Let* $s_0 = 1+(k+\alpha)/2$, $s > s_0$, $T_0 = 0$, $T_j = \sum_{n=1}^{j} 2^{(1-s/s_0)(n-1)^2/2}$ $(j \geq 1)$, *and* $T = \lim_{j\to\infty} T_j$. *Define*

$$a(t) = 2^{(s/s_0+1-2s)j^2}\Theta\Big(2^{(s/s_0+1)j^2/2}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0),$$

where

$$\Theta(\tau) = \frac{2 - 2\cos 2\pi\tau}{2 + 3\Gamma^3 \sin 2\pi\tau + (\Gamma - 9\Gamma^2)\cos 2\pi\tau}$$

and

$$\Gamma = (1 + 2\sqrt{7})^{1/3} - \frac{3}{(1 + 2\sqrt{7})^{1/3}}.$$

Then, the followings hold:

1. $a(t)$ *is non-negative and degenerates at* $t = T_j$ $(j \geq 0)$ *and* $t = T$.
2. $a(t)$ *belongs to* $C^{k,\alpha}[0, T]$ *for* $k = 0, 1$ *and* $0 \leq \alpha \leq 1$.
3. *The Cauchy problem (1) with* $a(t)$ *is ill-posed in* G^s.

Remark 1.5. Multiplying T_j by a constant, we can take an arbitrary small $T > 0$ as far as $s > s_0$. It is interesting that the life span T tends to infinity as s tends to s_0.

In Theorem 1.4 and its proof, the following parts are different from [4]:

- In §2.1, $\Theta(\tau)$ which is not same as the corresponding function in [4]. We require $\Theta(\tau)$ for which both minimum point and minimum value can be calculated. Therefore, in §2.2 we can construct $a(t)$ which has oscillations touching the t axis in an infinite number of points accumulating at $t = T$.

- In §2.4, the parameters ρ_j, ν_j and δ_j are uniformly taken as some powers of $2^{(j-1)^2}$. This choice of the parameters enables us to simplify the representation of the coefficient.

It would seem strange that $a(t)$ defined piecewise without a cut off function, is still smooth i.e., $C^{k,\alpha}[0,T]$. This is true due to our construction of $\Theta(\tau)$ and the additional assumption $k = 0, 1$ in Theorem 1.4 (the piecewise functions are connected at the minimum points). Therefore, we can remove a cut off function to represent the coefficient $a(t)$. In order to remove the restriction that $k = 0, 1$ form Theorem 1.4, we also need to modify the coefficient with a cut off function (see Corollary 2.19 in §2.6).

In the particular case that $a(t)$ does not belongs to $C^0[0,T]$, we can also get the following corollary:

Corollary 1.6. *Assume* $s > 1$, $T_0 = 0$, $T_j = \displaystyle\sum_{n=1}^{j} 2^{(1-s)(n-1)^2}$ $(j \geq 1)$, *and* $T = \lim_{j\to\infty} T_j$. *Define*

$$a(t) = \Theta\Big(2^{sj^2}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0).$$

Then, the followings hold:

1. $a(t)$ *is non-negative and degenerates at* $t = T_j$ $(j \geq 0)$ *and* $t = T$.
2. $a(t)$ *is not continuous at* $t = T$ *and belongs to* $L^\infty(0,T) \cap C^2[0,T)$.
3. *The Cauchy problem (1) with* $a(t)$ *is ill-posed in* G^s.

Remark 1.7. Let $s = q(q-1)^{-1}$ $(q > 1)$ and $T_j = \sum_{n=1}^{j} 2^{(1-q)^{-1}(n-1)^2}$ $(j \geq 1)$. Define

$$a(t) = \Theta\Big(2^{q(q-1)^{-1}j^2}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0).$$

For $t \in [T_j, T_{j+1}]$, we know that $(T-t) \sim \sum_{n=j+1}^{\infty} 2^{(1-q)^{-1}(n-1)^2} \sim 2^{(1-q)^{-1}j^2}$. While, we have $|a'(t)| \leq C2^{q(q-1)^{-1}j^2} \leq C(T-t)^{-q}$. Thus, Corollary 1.6 is also a simple counterexample of the ill-posedness in G^s for $s \geq q(q-1)^{-1}$ with $a(t) \in L^\infty(0,T) \cap C^1[0,T)$ satisfying $|a'(t)| \leq C(T-t)^{-q}$ (see [1], [2]).

It is known that the Cauchy problem for weakly hyperbolic equations is well-posed in the Analytic class ($s = 1$), even if $a \in L^1(0,T)$. The simple periodic function Θ proposed in this paper can be expected useful in study of the ill-posedness. Indeed, we shall present numerical results with this Θ in Appendix.

2. Proof of Theorem 1.4

We shall put the parameters ρ_j, ν_j and δ_j $(j \geq 1)$ as follows:

$$\rho_j = 2^{-X(j-1)^2}, \quad \nu_j = 2^{Y(j-1)^2}, \quad \delta_j = 2^{-Z(j-1)^2},$$

where X, Y and Z are all positive and determined later. We suppose that ν_j $(j \geq 1)$ are integers, by taking a integer Y later. Moreover, we define

$$T_0 = 0, \quad T_j = \sum_{n=1}^{j} \rho_n \ (j \geq 1) \text{ and } I_j = [T_{j-1}, T_j] \ (j \geq 1).$$

2.1. Construction of $\Theta(\tau)$

F. Colombini, E. Jannelli and S. Spagnolo [4] consider the following auxiliary Cauchy problem for the ordinary equation:

$$\begin{cases} W_\gamma''(\tau) + \Theta_\gamma(\tau) W_\gamma(\tau) = 0, \\ W_\gamma(0) = 0, \ W_\gamma'(0) = 1, \end{cases} \tag{3}$$

where $\Theta_\gamma(\tau)$ is a non-negative periodic function.

Remark 2.1. The Cauchy problem (3) can be also regarded as a terminal value problem. In §2.3 we use the negative part $\tau \leq 0$ for this problem.

By the Floquet theory, the solution has a form $W_\gamma(\tau) = P_\gamma(\tau)\exp\{\gamma\tau\}$ with $\gamma \in \mathbf{R}$ and a periodic function $P_\gamma(\tau)$. Now we don't solve (3), but we find $\Theta_\gamma(\tau)$ form the solution $W_\gamma(\tau)$ inversely. Thus, we get

$$\Theta_\gamma(\tau) = -\frac{W_\gamma''(\tau)}{W_\gamma(\tau)} = -\gamma^2 - \frac{P_\gamma''(\tau) + 2\gamma P_\gamma'(\tau)}{P_\gamma(\tau)}. \tag{4}$$

Since $P_\gamma(\tau)$ is periodic, $\Theta_\gamma(\tau)$ is periodic too. But, we have to choose suitable $\gamma \in \mathbf{R}$ and $P_\gamma(\tau)$ such that $\Theta(\tau) \geq 0$.

Remark 2.2. In fact, most of choices with random $\gamma \in \mathbf{R}$ and $P_\gamma(\tau)$ fail to satisfy $\Theta_\gamma(\tau) \geq 0$. [4] succeeds to find a rare case:

$$\gamma = \frac{1}{10} \text{ and } P_\gamma(\tau) = \sin\tau \exp\Big\{-\frac{\gamma}{2}\sin 2\tau\Big\}. \tag{5}$$

Furthermore, we shall change (5) by the following:

$$0 < \gamma \leq \Gamma \text{ and } P_\gamma(\tau) = \sin\tau\Big(1 - \frac{\gamma}{2}\sin 2\tau\Big), \tag{6}$$

where $\Gamma > 0$ is a sufficiently small constant such that $\Theta_\gamma(\tau) \geq 0$ for $0 < \gamma \leq \Gamma$. (see Lemma 2.5). Then by (4) and (6) we have

$$\Theta_\gamma(\tau) = \frac{2 + (\gamma^3 - 9\gamma)\sin 2\tau + 6\gamma^2 \cos 2\tau}{2 - \gamma \sin 2\tau}, \tag{7}$$

here we remark that $\Theta_\gamma(\tau)$ becomes only π-periodic, since $\sin\tau$ has been canceled. $\Theta_\gamma(\tau)$ given by (7) enables us to calculate the exact points of the minimum and the maximum as follows:

LEMMA 2.3. *Let*

$$p_\pm = p_\pm(\gamma) = \frac{3\gamma^2(8-\gamma^2) \pm 12\gamma\sqrt{-2\gamma^4 + 5\gamma^2 + 16}}{(\gamma^2+4)(\gamma^2+16)}.$$

Then, $\Theta_\gamma(\tau)$ $(0 \leq \tau \leq \pi)$ *has the maximum value and the minimum value*

$$\Theta_\gamma(\tau_\pm) = \frac{2 + (\gamma^3 - 9\gamma)\sqrt{1 - p_\pm^2} + 6\gamma^2 p_\pm}{2 - \gamma\sqrt{1 - p_\pm^2}} \tag{8}$$

at $\tau_+ = \frac{1}{2}\mathrm{Cos}^{-1} p_+$ *and* $\tau_- = \frac{1}{2}\mathrm{Cos}^{-1} p_-$ *respectively.*

Proof. Differentiating $\Theta_\gamma(\tau)$, we get

$$\Theta_\gamma'(\tau) = \frac{4\gamma\Big\{(\gamma^2 - 8)\cos 2\tau - 6\gamma \sin 2\tau + 3\gamma^2\Big\}}{(2 - \gamma\sin 2\tau)^2}.$$

To find the maximum and minimum values, we solve the equation

$$(\gamma^2 - 8)\cos 2\tau - 6\gamma\sin 2\tau + 3\gamma^2 = 0.$$

When $0 \leq \tau \leq \pi/2$, we put $p = \cos 2\tau$ $(-1 \leq p \leq 1)$ and get

$$(\gamma^2 - 8)p + 3\gamma^2 = 6\gamma\sqrt{1 - p^2}. \tag{9}$$

For small $\gamma > 0$, we see that p must be negative, since the signatures of both sides must coincide. Taking the square of both sides, we can reduce to the following quadratic equation in p:

$$(\gamma^2+4)(\gamma^2+16)p^2 - 6\gamma^2(8-\gamma^2)p + 9\gamma^2(\gamma^2 - 4) = 0. \tag{10}$$

Hence, we have a (unique) negative solution

$$p_- = p_-(\gamma) = \frac{3\gamma^2(8-\gamma^2) - 12\gamma\sqrt{-2\gamma^4 + 5\gamma^2 + 16}}{(\gamma^2+4)(\gamma^2+16)}. \tag{11}$$

When $\pi/2 \le \tau \le \pi$, we put $0 \le \tilde{\tau} = \pi - \tau \le \pi/2$ and $p = \cos 2\tilde{\tau}$ $(-1 \le p \le 1)$ and get

$$(\gamma^2 - 8)p + 3\gamma^2 = -6\gamma\sqrt{1-p^2}.$$

For small $\gamma > 0$, we see that p must be positive, since the signatures of both sides must coincide. Taking the square of both sides, we can reduce to the same quadratic equation (10). Hence, we have a (unique) positive solution

$$p_+ = p_+(\gamma) = \frac{3\gamma^2(8-\gamma^2) + 12\gamma\sqrt{-2\gamma^4 + 5\gamma^2 + 16}}{(\gamma^2+4)(\gamma^2+16)}.$$

We note that $p_- = \cos 2\tau$ in $0 \le \tau \le \pi/2$ and $p_+ = (\cos 2\tilde{\tau} =) \cos 2\tau$ in $\pi/2 \le \tau \le \pi$. Thus it follows that $\tau_- := \frac{1}{2}\mathrm{Cos}^{-1} p_-$ and $\tau_+ := \frac{1}{2}\mathrm{Cos}^{-1} p_+$ satisfy $0 < \tau_- < \tau_+ < \pi$ and give the minimum value and the maximum value respectively, since $\Theta'_\gamma(0) = 8\gamma(\gamma^2 - 2) < 0$. Substituting $\tau_\pm$ into $\Theta(\tau)$ we also have (8). □

Remark 2.4. $p_\pm$ are the simple roots of the quadratic equation (10). Therefore, $\Theta'_\gamma(\tau)$ changes the sign at $\tau = \tau_\pm$.

If $\gamma = 0$, $\Theta_0(\tau)$ is a positive constant, i.e., the ratio $\Theta_0(\tau_+)/\Theta_0(\tau_-) \equiv 1$. Obviously, it holds that $\Theta_\gamma(\tau_+)/\Theta_\gamma(\tau_-) > 1$ for small $\gamma > 0$. As $\gamma > 0$ becomes larger, $\Theta_\gamma(\tau_+)/\Theta_\gamma(\tau_-)$ tends to infinity as follows:

Lemma 2.5. *For* $\Gamma = (1 + 2\sqrt{7})^{1/3} - 3(1+2\sqrt{7})^{-1/3} (\sim 0.221)$, *we have*

$$\Theta_\gamma(\tau) > 0 \quad \textit{if} \;\; 0 < \gamma < \Gamma, \quad \Theta_\Gamma(\tau_-) = 0 \quad \textit{and} \quad \tau_- = \frac{1}{2}\mathrm{Cos}^{-1}(-3\Gamma^2). \tag{12}$$

Remark 2.6. We remark that $\pi/4 < \tau_- < \pi/2$, since $\tau_- = \frac{1}{2}\mathrm{Cos}^{-1}(-3\Gamma^2) \sim \frac{1}{2}\mathrm{Cos}^{-1}(-3 \times 0.221^2) \sim 0.858$. By numerical computations we observe that $\Theta_\Gamma(\tau_+) < 2$.

Proof. By (8), $\Theta_\Gamma(\tau_-) = 0$ means that

$$2 + (\Gamma^3 - 9\Gamma)\sqrt{1 - p_-^2} + 6\Gamma^2 p_- = 0.$$

Hence, by (9) with $p = p_-$ we have

$$\frac{6\Gamma^2 p_- + 2}{9\Gamma - \Gamma^3} = \frac{(\Gamma^2 - 8)p_- + 3\Gamma^2}{6\Gamma} \left(= \sqrt{1 - p_-^2} \right).$$

Therefore, Γ satisfies the equation

$$p_- = \frac{-3\Gamma^4 + 27\Gamma^2 - 12}{\Gamma^4 + 19\Gamma^2 + 72}. \tag{13}$$

On the other hand, $p_- = p_-(\Gamma)$ is defined in (11). Therefore, $\Gamma > 0$ is a solution to the equation

$$\frac{3\Gamma^2(8-\Gamma^2) - 12\Gamma\sqrt{-2\Gamma^4+5\Gamma^2+16}}{(\Gamma^2+4)(\Gamma^2+16)} = \frac{-3\Gamma^4+27\Gamma^2-12}{\Gamma^4+19\Gamma^2+72}.$$

Adding 3 on both sides and dividing both sides by 12, we get

$$\frac{7\Gamma^2+16-\Gamma\sqrt{-2\Gamma^4+5\Gamma^2+16}}{(\Gamma^2+4)(\Gamma^2+16)} = \frac{7\Gamma^2+17}{\Gamma^4+19\Gamma^2+72}.$$

Multiplying both sides by $(\Gamma^2+4)(\Gamma^2+16)(\Gamma^4+19\Gamma^2+72)$, we also get

$$-8\Gamma^4+20\Gamma^2+64 = \Gamma\sqrt{-2\Gamma^4+5\Gamma^2+16}(\Gamma^4+19\Gamma^2+72).$$

Moreover, dividing both sides by $\sqrt{-2\Gamma^4+5\Gamma^2+16}$, we have

$$4\sqrt{-2\Gamma^4+5\Gamma^2+16} = \Gamma(\Gamma^4+19\Gamma^2+72). \tag{14}$$

(14) is reduced to the equation of degree 10

$$\Gamma^{10}+38\Gamma^8+505\Gamma^6+2768\Gamma^4+5104\Gamma^2-256=0.$$

Fortunately, this can be divided by $(\Gamma^2+4)(\Gamma^2+16)$. Then we have the equation of degree 6

$$\Gamma^6+18\Gamma^4+81\Gamma^2-4=0. \tag{15}$$

Regarding this as a cubic equation with respect to Γ^2, we can find the solution

$$\Gamma = \left\{(29+4\sqrt{7})^{1/3}+(29-4\sqrt{7})^{1/3}-6\right\}^{1/2} = (1+2\sqrt{7})^{1/3} - \frac{3}{(1+2\sqrt{7})^{1/3}} \sim 0.221.$$

Using (14) again, we can change $p_-(\Gamma)$ defined in (11) into

$$\begin{aligned} p_-(\Gamma) & \left(\equiv \frac{3\Gamma^2(8-\Gamma^2)-12\Gamma\sqrt{-2\Gamma^4+5\Gamma^2+16}}{(\Gamma^2+4)(\Gamma^2+16)}\right) \\ = & \quad \frac{3\Gamma^2(8-\Gamma^2)-3\Gamma^2(\Gamma^4+19\Gamma^2+72)}{(\Gamma^2+4)(\Gamma^2+16)} = -3\Gamma^2. \end{aligned}$$

Hence, it holds that $\tau_- = \frac{1}{2}\mathrm{Cos}^{-1}p_-(\Gamma) = \frac{1}{2}\mathrm{Cos}^{-1}(-3\Gamma^2)$. □

At last, we define

$$\Theta(\tau) := \Theta_\Gamma(\pi\tau+\tau_-).$$

By (15) we see that $4(1-9\Gamma^4) = \Gamma^6-18\Gamma^4+81\Gamma^2$. Hence, we get

$$2\sqrt{1-9\Gamma^4} = \Gamma(9-\Gamma^2).$$

By (12) and Remark 2.6 it holds that $\cos 2\tau_- = -3\Gamma^2$ and $\sin 2\tau_- = +\sqrt{1-9\Gamma^4} = \Gamma(9-\Gamma^2)/2$. Therefore, by (7) we have the 1-periodic function

$$\begin{aligned}\Theta(\tau) &= \frac{2+(\Gamma^3-9\Gamma)\sin(2\pi\tau+2\tau_-)+6\Gamma^2\cos(2\pi\tau+2\tau_-)}{2-\Gamma\sin(2\pi\tau+2\tau_-)} \\ &= \frac{4-(\Gamma^3+9)^2\cos 2\pi\tau}{4+6\Gamma^3\sin 2\pi\tau+\Gamma(\Gamma^3-9\Gamma)\cos 2\pi\tau} \\ &= \frac{2-2\cos 2\pi\tau}{2+3\Gamma^3\sin 2\pi\tau+(\Gamma-9\Gamma^2)\cos 2\pi\tau},\end{aligned}$$

here we used by (15) $(\Gamma^3+9\Gamma)^2=4$, i.e., $\Gamma^3+9\Gamma=2$ and $\Gamma^3-9\Gamma=2-18\Gamma$.

2.2. Construction of $a(t)$

For the construction of the coefficient, we shall use $\Theta(\tau)$. At the 1st step, let us consider

$$\phi_1(t) = \Theta(t) \quad \text{for} \quad t \in [0,1].$$

There are only 1 maximum point and only 2 minimum points in the interval $[0,1]$. The graph of $\phi_1(t)$ starts from the minimum point ($t=0$) and ends at the minimum point ($t=1$). Next, we consider

$$\phi_j(t) = \Theta(\nu_j t) \quad \text{for} \quad t \in [0,1].$$

By the 1-periodicity there are ν_j maximum points and (ν_j+1) minimum points in the interval $[0,1]$. The graph of $\phi_j(t)$ starts from a minimum point ($t=0$) and ends at a minimum point ($t=1$).

At the 2nd step, let us consider

$$\varphi_j(t) = \Theta\left(\nu_j \frac{t-T_{j-1}}{\rho_j}\right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j].$$

There are ν_j maximum points and (ν_j+1) minimum points in the interval I_j. The graph of $\varphi_j(t)$ starts from a minimum point ($t=T_{j-1}$) and ends at a minimum point ($t=T_j$). Each $\varphi_j(t)$ can be regarded as the piecewise definition of the following function in the whole interval $[0,T]$:

$$\Phi(t) = \Theta\left(\nu_j \frac{t-T_{j-1}}{\rho_j}\right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j].$$

We observe that $\Phi(t)$ is continuous at $t=T_j$ ($j \geq 1$), since $\Phi(T_j)=0$.

At the 3rd step, we define that

$$a(t) = \delta_j \Theta\left(\nu_j \frac{t-T_{j-1}}{\rho_j}\right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j]. \tag{16}$$

We remark that $a(t)$ is continuous at the whole interval $[0,T]$. Furthermore, we shall show the following lemma:

Lemma 2.7. *If $k = 0, 1$ and there exists $\varepsilon_1 > 0$ such that*

$$\delta_j \left(\frac{\nu_j}{\rho_j}\right)^{k+\alpha} \le 2^{-\varepsilon(j-1)^2} \quad \text{for} \quad 0 < \varepsilon \le \varepsilon_1, \tag{17}$$

$a(t)$ belongs to $C^{k,\alpha}[0, T]$.

Remark 2.8. When we consider the proof of Corollary 1.6, the right hand side $2^{-\varepsilon(j-1)^2}$ is replaced by C.

Proof. We may check Hölder continuity in the right interval $t \in I_{j+1}$ and the left interval $t \in I_j$. Replacing j by $j + 1$ in (16) we obviously get

$$a(t) = \delta_{j+1}\Theta\left(\nu_{j+1}\frac{t - T_j}{\rho_{j+1}}\right) \quad \text{for} \quad t \in I_{j+1} = [T_j, T_{j+1}]. \tag{18}$$

By the 1-periodicity of Θ, the definition (16) can be rewritten as

$$a(t) = \delta_j\Theta\left(\nu_j\frac{t-(T_j-\rho_j)}{\rho_j}\right) = \delta_j\Theta\left(\nu_j\frac{t - T_j}{\rho_j}\right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j]. \tag{19}$$

In the case of $k = 0$, noting that Θ belongs to at least $C^\alpha[0, T]$, by (18) and (19) we get

$$\begin{aligned}
|a(t) - a(T_j)| &\le \begin{cases} \left|\delta_j\Theta\left(\nu_j\dfrac{t - T_j}{\rho_j}\right) - \delta_j\Theta(0)\right| \text{ if } t \in I_j \\ \left|\delta_{j+1}\Theta\left(\nu_{j+1}\dfrac{t - T_j}{\rho_{j+1}}\right) - \delta_{j+1}\Theta(0)\right| \text{ if } t \in I_{j+1} \end{cases} \\
&\le \begin{cases} M\delta_j\left|\nu_j\dfrac{t - T_j}{\rho_j}\right|^\alpha \le M\delta_j\left(\dfrac{\nu_j}{\rho_j}\right)^\alpha |t - T_j|^\alpha \text{ if } t \in I_j \\ M\delta_{j+1}\left|\nu_{j+1}\dfrac{t - T_j}{\rho_{j+1}}\right|^\alpha \le M\delta_{j+1}\left(\dfrac{\nu_{j+1}}{\rho_{j+1}}\right)^\alpha |t - T_j|^\alpha \text{ if } t \in I_{j+1} \end{cases} \\
&\le \begin{cases} M2^{-\varepsilon_1(j-1)^2}|t - T_j|^\alpha \text{ if } t \in I_j \\ M2^{-\varepsilon_1 j^2}|t - T_j|^\alpha \text{ if } t \in I_{j+1} \end{cases} \\
&\le M2^{-\varepsilon_1(j-1)^2}|t - T_j|^\alpha\Big(\le M|t - T_j|^\alpha\Big),
\end{aligned}$$

here we used (17), but we need not use the fact that $a(T_j) = 0$. Hence we see that $a(t)$ is α-Hölder continuous at $t = T_j$. As for $t = T$, since $a(T) = 0$ we also have

$$|a(t)-a(T)| = |a(t)| \leq \begin{cases} \big|a(t)-a(T_j)\big| + \displaystyle\sum_{n=j}^{\infty} \big|a(T_n)-a(T_{n+1})\big| \text{ if } t \in I_j \\ \big|a(t)-a(T_{j+1})\big| + \displaystyle\sum_{n=j+1}^{\infty} \big|a(T_n)-a(T_{n+1})\big| \text{ if } t \in I_{j+1} \end{cases}$$

$$\leq \left(\sum_{n=1}^{\infty} M2^{-\varepsilon_1(n-1)^2}\right)|t-T|^\alpha \leq M_{\varepsilon_1}|t-T|^\alpha.$$

This means that $a(t)$ is α-Hölder continuous at $t = T$.

In the case of $k = 1$, by (18) and (19) we have

$$a'(t) = \frac{\delta_{j+1}\nu_{j+1}}{\rho_{j+1}}\Theta'\left(\nu_{j+1}\frac{t-T_j}{\rho_{j+1}}\right) \quad \text{for} \quad t \in I_{j+1} = [T_j, T_{j+1}],$$

and

$$a'(t) = \frac{\delta_j\nu_j}{\rho_j}\Theta'\left(\nu_j\frac{t-T_j}{\rho_j}\right) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j].$$

To get the differentiability at $t = T_j$, the right derivative and the left derivative must coincide. The right derivative and the left derivative are respectively

$$a'(T_j) = \frac{\delta_{j+1}\nu_{j+1}}{\rho_{j+1}}\Theta'(0) \quad \text{and} \quad a'(T_j) = \frac{\delta_j\nu_j}{\rho_j}\Theta'(0),$$

that is, $a'(T_j) = 0$ ($\Theta'(0) = 0$) since $a(t)$ takes a minimum value in I_{j+1} and a minimum value in I_j at $t = T_j$ from our construction. Therefore, $a(t)$ is differentiable at $t = T_j$. As for $t = T$, we see that $\lim_{t\uparrow T}|a'(t)| = 0$, since by (17)

$$\lim_{j\to\infty}\frac{\delta_{j+1}\nu_{j+1}}{\rho_{j+1}} = \lim_{j\to\infty}\frac{\delta_j\nu_j}{\rho_j} = 0.$$

Hence the left derivative at $T = t$ is zero. Then we have $a'(T) = 0$ since by the zero extension the right derivative at $T = t$ is also zero. Thus, $a(t)$ belongs to $C^1[0,T]$. Similarly, noting that Θ belongs to at least $C^{1+\alpha}[0,T]$, we obtain the estimates $|a'(t)-a'(T_j)| \leq M\delta_j\big(\nu_j/\rho_j\big)^{1+\alpha}|t-T_j|^\alpha = M2^{-\varepsilon_1(j-1)^2}|t-T_j|^\alpha\Big(\leq M|t-T_j|^\alpha\Big)$ and $|a'(t)-a'(T)| \leq M_{\varepsilon_1}|t-T_j|^\alpha$. $\square$

Remark 2.9. In order to justify $a'(T_j)$ and $a'(T)$ we first showed that $a(t)$ belongs to $C^1[0,T]$. Then, we are allowed to consider $|a'(t)-a'(T_j)|$ and $|a'(t)-a'(T)|$.

Remark 2.10. We can not deal with $k = 2$, because the right 2nd derivative and the left 2nd derivative does not coincide at $t = T_j$. So, we can not justify $a''(T_j)$. Thus $a(t)$ does not belong to $C^2[0,T]$. But, $a(t)$ belongs to $C^{1,1}[0,T]$ which implies $a'(t) \in Lip[0,T]$.

2.3. Construction of Solutions

We consider a sequence of the solutions $\{u^{(J)}(t,x)\}_{J\geq 1}$ to the Cauchy problem on $[0,T]\times \mathbf{R}_x$

$$\begin{cases} \partial_t^2 u^{(J)} - a(t)\partial_x^2 u^{(J)} = 0, \\ u(0,x) = u_0^{(J)}(x),\ \partial_t u(0,x) = u_1^{(J)}(x). \end{cases} \tag{20}$$

Let us take the sequence $\{t_j\}_{j\geq 1}$ defined by

$$t_j := T_j - \frac{\rho_j \tau_-}{\pi \nu_j}. \tag{21}$$

We see that $t_j \in I_j = [T_{j-1}, T_j]$, since $\frac{\tau_-}{\pi\nu_j} \leq 1$. Now we shall devote ourselves to only the interval $[0,t_j]$ by separating into two parts $[T_{j-1},t_j]$ and $[0,T_{j-1}]$, where the Cauchy problems are solved in the inverse direction.

For the interval $[T_{j-1},t_j]$, we suppose that $u^{(J)}(t,x)$ has a form of

$$u^{(J)}(t,x) = \sum_{j=J}^{\infty} v_j(t)\cos h_j x, \tag{22}$$

where

$$h_j = \frac{\pi\nu_j}{\rho_j\sqrt{\delta_j}}, \tag{23}$$

and v_j solves the terminal value problem on $[T_{j-1},t_j]\subset I_j$

$$\begin{cases} v_j'' + h_j^2 a(t) v_j = 0, \\ v_j(t_j) = 0,\ v_j'(t_j) = 1. \end{cases} \tag{24}$$

Noting that by (19)

$$a(t) = \delta_j \Theta\left(\nu_j \frac{t-T_j}{\rho_j}\right) = \delta_j \Theta_\Gamma\left(\pi\nu_j \frac{t-T_j}{\rho_j} + \tau_-\right) \quad \text{for } \ t\in[T_{j-1},t_j]\subset I_j,$$

and putting

$$v_j(t) = \frac{\rho_j}{\pi\nu_j} W_\Gamma\left(\pi\nu_j\frac{t-T_j}{\rho_j} + \tau_-\right),$$

by the change of variable $\tau = \pi\nu_j\frac{t-T_j}{\rho_j} + \tau_-$ we have just (3). Therefore, by (6) it follows that

$$W_\Gamma(\tau) = \sin\tau\left(1 - \frac{\Gamma}{2}\sin 2\tau\right)e^{\Gamma\tau}.$$

Hence, noting Remark 2.1 we have

$$\begin{aligned} V_0 := v_j(T_{j-1}) &= \frac{\rho_j}{\pi\nu_j} W_\Gamma(-\pi\nu_j + \tau_-) \\ &= \frac{\rho_j}{\pi\nu_j} \sin\tau_- \Big(1 - \frac{\Gamma}{2}\sin 2\tau_-\Big) \exp\big\{ -\Gamma\pi\nu_j + \Gamma\tau_-\big\}, \qquad (25) \\ V_1 := v_j'(T_{j-1}) &= W_\Gamma'(-\pi\nu_j + \tau_-) \\ &= \Big(\cos\tau_- + \Gamma\sin\tau_- - \frac{\Gamma}{2}\sin 2\tau_- \cos\tau_- - \Gamma\cos 2\tau_- \sin\tau_- \\ &\qquad - \frac{\Gamma^2}{2}\sin\tau_- \sin 2\tau_-\Big) \exp\big\{ -\Gamma\pi\nu_j + \Gamma\tau_-\big\}. \qquad (26) \end{aligned}$$

By (25) and (26) it follows that

$$|V_0| \le C_0 \frac{\rho_j}{\nu_j} e^{-\Gamma\pi\nu_j}, \qquad |V_1| \le C_1 e^{-\Gamma\pi\nu_j}. \tag{27}$$

This fact plays an important role in the construction of the counterexample.

For the interval $[0, T_{j-1}]$ we suppose that $u^{(J)}(t,x)$ also has a form of (22) with v_j solving the terminal value problem on $[0, T_{j-1}] = \cup_{n=1}^{j-1} I_n$ $(j \ge 2)$

$$\begin{cases} v_j'' + h_j^2 a(t) v_j = 0, \\ v_j(T_{j-1}) = V_0, \ v_j'(T_{j-1}) = V_1. \end{cases} \tag{28}$$

We remark that the formula with W_Γ can not be obtained in this interval. Therefore, we shall use the energy method. Let us introduce the following proposition concerned with the energy method:

Proposition 2.11. *Let $h > 0$ and $a(t)$ be a non-negative C^1 function. Then, for the solution v satisfying $v'' + h^2 a(t) v = 0$, it holds that*

$$E(\sigma_1) \le E(\sigma_2) \exp\left[\left| \int_{\sigma_2}^{\sigma_1} \frac{\max\{a'(t), 0\}}{a(t) + \lambda^2 h^{2(1/s-1)}} dt \right| + |\sigma_1 - \sigma_2| \lambda h^{1/s} \right],$$

where $E(t) = |v'(t)|^2 + (h^2 a(t) + \lambda^2 h^{2/s})|v(t)|^2$.

Remark 2.12. We can apply the energy inequality also into the terminal value problem. Because we may take σ_1 and σ_2 such that $\sigma_1 \le \sigma_2$.

Proof. Differentiating $E(t)$, we have

$$\begin{aligned} E'(t) &= 2\Re\Big(v'(t), v''(t)\Big) + 2(h^2 a(t) + \lambda^2 h^{2/s})\Re\Big(v'(t), v(t)\Big) + h^2 a'(t)|v(t)|^2 \\ &\le h^2 a'(t)|v(t)|^2 + 2\lambda^2 h^{2/s}|v'(t)||v(t)| \\ &\le h^2 \max\{a'(t), 0\}|v(t)|^2 + \lambda^2 h^{2/s}\Big(\lambda^{-1} h^{-1/s}|v'(t)|^2 + \lambda h^{1/s}|v(t)|^2\Big) \\ &\le \left\{ \frac{\max\{a'(t), 0\}}{a(t) + \lambda^2 h^{2(1/s-1)}} + \lambda h^{1/s} \right\} E(t), \end{aligned}$$

which proves the proposition. $\square$

From the construction of the coefficient, we know that $a(t)$ has ν_{j-1} maximum points and $(\nu_{j-1}+1)$ minimum points in the interval $I_{j-1}=[T_{j-2},T_{j-1}]$. Using Proposition 2.11 with $\sigma_1=T_{j-2}$ and $\sigma_2=T_{j-1}$, by Remark 2.4 we get the estimate in the interval $I_{j-1}=[T_{j-2},T_{j-1}]$

$$\begin{aligned}
E_j(T_{j-2}) &\le E_j(T_{j-1})\exp\left[\left|\int_{T_{j-1}}^{T_{j-2}}\frac{\max\{a'(t),0\}}{a(t)+\lambda^2h_j^{2(1/s-1)}}dt\right|+|T_{j-2}-T_{j-1}|\lambda h_j^{1/s}\right]\\
&\le E_j(T_{j-1})\exp\left[\nu_{j-1}\log\left\{\lambda^{-2}h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\right\}+(T_{j-1}-T_{j-2})\lambda h_j^{1/s}\right],
\end{aligned}$$

where $E_j(t)=|v_j'(t)|^2+(h_j^2a(t)+\lambda^2h_j^{2/s})|v_j(t)|^2$. Combining all the energy inequalities in I_n $(n=1,2,\cdots,j-1)$, we have

$$E_j(0)\le E_j(T_{j-1})\exp\left[\sum_{n=1}^{j-1}\nu_n\log\left\{\lambda^{-2}h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\right\}+T_{j-1}\lambda h_j^{1/s}\right].$$

Noting that by (27)

$$E_j(T_{j-1})\le|V_1|^2+Ch_j^2|V_0|^2\le C_3\left(1+\frac{h_j^2\rho_j^2}{\nu_j^2}\right)\exp\{-2\Gamma\pi\nu_j\},$$

and taking $\lambda=\frac{1}{\pi T_{j-1}}$, we obtain

$$\begin{aligned}
E_j(0)\quad\le\quad C_3\left(1+\frac{h_j^2\rho_j^2}{\nu_j^2}\right)\exp&\left[\sum_{n=1}^{j-1}\nu_n\log\left\{\pi^2T_{j-1}^2h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\right\}\right.\\
&\left.+\frac{1}{\pi}h_j^{1/s}-2\Gamma\pi\nu_j\right].
\end{aligned}\tag{29}$$

Moreover, we need the following lemma:

Lemma 2.13. *If*

$$\rho_j\nu_j^{s-1}\sqrt{\delta_j}=1,\tag{30}$$

and there exists $\varepsilon_2>0$ *such that*

$$\sum_{n=1}^{j-1}\nu_n(\log j+\log\nu_j+3)\le\Big(\Gamma\pi-\frac{1}{2}-2\varepsilon\Big)\nu_j\quad\text{for}\quad 0<\varepsilon\le\varepsilon_2,\tag{31}$$

it holds that

$$\sum_{n=1}^{j-1}\nu_n\log\left\{\pi^2T_{j-1}^2h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\right\}+\frac{1}{\pi}h_j^{1/s}-2\Gamma\pi\nu_j\le-\varepsilon_2h_j^{1/s}.\tag{32}$$

Proof. By (23) and (30) we get $h_j^{1/s} = \left(\frac{\pi\nu_j}{\rho_j\sqrt{\delta_j}}\right)^{1/s} = \pi^{1/s}\nu_j(\geq 1)$. Hence, noting that $(1 \leq)T_{j-1} = \sum_{n=1}^{j-1}\rho_n \leq \sum_{n=1}^{j-1} 1 \leq j$, by Remark 2.6 and (31) we have

$$\begin{aligned}
& \sum_{n=1}^{j-1}\nu_n \log\Big\{\pi^2 T_{j-1}^2 h_j^{2(1-1/s)}\delta_j\Theta_\Gamma(\tau_+)+1\Big\} + \frac{1}{\pi}h_j^{1/s} - 2\Gamma\pi\nu_j \\
\leq\ & \sum_{n=1}^{j-1}\nu_n \log\Big\{\pi^2\cdot T_{j-1}^2\cdot\pi^{2(1-1/s)}\nu_j^{2(1-1/s)}\cdot 1\cdot 2+1\Big\} + \pi^{1/s-1}\nu_j - 2\Gamma\pi\nu_j \\
\leq\ & \sum_{n=1}^{j-1}\nu_n \log\left\{4\pi^4T_{j-1}^2\nu_j^2\right\} + \pi^{1/s-1}\nu_j - 2\Gamma\pi\nu_j \\
\leq\ & \sum_{n=1}^{j-1}\nu_n(2\log j + 2\log\nu_j + 6) + \pi^{1/s-1}\nu_j - 2\Gamma\pi\nu_j \\
\leq\ & 2\sum_{n=1}^{j-1}\nu_n(\log j + \log\nu_j + 3) + \nu_j - 2\Gamma\pi\nu_j \\
\leq\ & -4\varepsilon_2\nu_j = -\frac{4\varepsilon_2}{\pi^{1/s}}h_j^{1/s} \leq -\varepsilon_2 h_j^{1/s},
\end{aligned}$$

thus getting the conclusion. □

Consequently, by (29) and (32) it follows that

$$E_j(0) \leq C_3\left(1+\frac{h_j^2\rho_j^2}{\nu_j^2}\right)\exp\Big\{-\varepsilon_2 h_j^{1/s}\Big\}. \tag{33}$$

2.4. Choice of ρ_j, ν_j and δ_j

For our purpose, $\rho_j(=2^{-X(j-1)^2})$, $\nu_j(=2^{Y(j-1)^2})$ and $\delta_j(=2^{-Z(j-1)^2})$ satisfy (17), (30) and (31). Only the parameter Y must be an integer in order that ν_j becomes an integer. So, the simplest choice is $Y=1$. Then (31) means that there exists $\varepsilon_2 > 0$ such that

$$\sum_{n=1}^{j-1} 2^{(n-1)^2}(\log j + (j-1)^2 + 3) \leq \Big(\Gamma\pi - \frac{1}{2} - 2\varepsilon\Big)2^{(j-1)^2} \quad \text{for } 0 < \varepsilon \leq \varepsilon_2. \tag{34}$$

We remark that j is greater than or equal to J which tends to infinity later in §2.5. Thus, for large $j \geq 1$, the inequality (34) holds, since,

$$\begin{aligned}\sum_{n=1}^{j-1} 2^{(n-1)^2}(\log j + (j-1)^2 + 3) &\leq j^2 \sum_{n=1}^{j-1} 2^{(n-1)^2} \leq j^3 2^{(j-2)^2} \\ &\leq \frac{1}{10} e^{(j-1)^2}, \end{aligned} \tag{35}$$

and $\Gamma \sim 0.221$ and $1/10 \leq \Gamma\pi - 1/2 - 2\varepsilon$ for a sufficiently small $\varepsilon > 0$.

Remark 2.14. More generally, if we consider the functions $\rho_j(= 2^{-X(j-1)^r})$, $\nu_j(= 2^{Y(j-1)^r})$ and $\delta_j(= 2^{-Z(j-1)^r})$ with the parameter $r \geq 1$, we can not obtain the corresponding inequality of (35) just for $r = 1$.

Taking the binary logarithm and dividing by $(j-1)^2$ in (17) and (30), we may take X and Z such that

$$\begin{cases} (k+\alpha)X - Z + k + \alpha + \varepsilon_1 = 0, \\ -X - \dfrac{1}{2}Z + s - 1 = 0. \end{cases}$$

Hence, we get

$$X = \frac{s}{s_0} - 1 - \frac{\varepsilon_1}{2s_0} \quad \text{and} \quad Z = 2s\left(1 - \frac{1}{s_0}\right) + \frac{\varepsilon_1}{s_0}.$$

Since $s_0 \geq 1$, we see that $Z > 0$ for $\varepsilon_1 > 0$. In order to have $X > 0$, we may take $\varepsilon_1 = s - s_0$. Then we obtain

$$X = \frac{1}{2}\left(\frac{s}{s_0} - 1\right) \quad \text{and} \quad Z = 2s - \frac{s}{s_0} - 1.$$

Summing up, we have

$$\rho_j = 2^{-(s/s_0-1)(j-1)^2/2}, \ \nu_j = 2^{(j-1)^2} \text{ and } \delta_j = 2^{-(2s-s/s_0-1)(j-1)^2}, \tag{36}$$

and with (18) instead of (16)

$$a(t) = 2^{(s/s_0+1-2s)j^2}\Theta\Big(2^{(s/s_0+1)j^2/2}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}].$$

Remark 2.15. If we consider a discontinuous coefficient, we need not Lemma 2.7 anymore. So, we can take $\varepsilon_1 = 0$ and $Z = 0$ ($\delta_j = 1$) with $s_0 = 1$. Then, we also have $X = s - 1$ ($\rho_j = 2^{-(s-1)(j-1)^2}$) and

$$a(t) = \Theta\Big(2^{sj^2}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}],$$

which proves the proof of Corollary 1.6.

We also note that $h_j = \pi\nu_j^s = \pi 2^{s(j-1)^2} \geq 1$ and $\rho_j^2/\nu_j^2 = 2^{-(2+X)(j-1)^2} \leq 1$. By (33) it follows that

$$E_j(0) \leq C_3 \left(1 + \frac{h_j^2\rho_j^2}{\nu_j^2}\right) \exp\Big\{ -\varepsilon_2 h_j^{1/s}\Big\} \leq C_4 h_j^2 \exp\Big\{ -\varepsilon_2 h_j^{1/s}\Big\}.$$

Thus, we have

$$E_j(0) \leq C_5 \exp\Big\{ -\varepsilon h_j^{1/s}\Big\} \text{ for } 0 < \varepsilon < \varepsilon_2. \tag{37}$$

Remark 2.16. The Cauchy problem (28) is solved in the inverse direction. Therefore, we can also see that for all $0 \leq t \leq T_{j-1}$

$$E_j(t) \leq C_5 \exp\Big\{ -\varepsilon h_j^{1/s}\Big\} \text{ for } 0 < \varepsilon < \varepsilon_2.$$

In particular, if $j_1 < j_2$, it holds that for the point $t = t_{j_1} (\leq T_{j_1} \leq T_{j-1})$

$$E_j(t_{j_1}) \leq C_5 \exp\Big\{ -\varepsilon h_j^{1/s}\Big\} \text{ for } 0 < \varepsilon < \varepsilon_2. \tag{38}$$

2.5. Ill-posedness of the Cauchy problem

We finally show the ill-posedness by the contradiction. We suppose that the energy inequality for $u^{(J)}$ holds, i.e.,

$$\|u^{(J)}(t)\|_{s,R} + \|\partial_t u^{(J)}(t)\|_{s,R} \leq C_T\Big(\|u_0^{(J)}\|_{s,r} + \|u_1^{(J)}\|_{s,r}\Big) \text{ for } t \in [0,T]. \tag{39}$$

Let us note the point $(t,x) = (t_J, 0)$, where $t_J \in I_J$ defined by (21) with $j = J$. From the definition of the Gevrey norm, by (22) and (38) we have

$$\begin{aligned}
\|\partial_t u^{(J)}(t_J)\|_{s,R} \geq\ & \|\partial_t u^{(J)}(t_J)\|_{L^\infty} \geq |\partial_t u^{(J)}(t_J, 0)| = \left|\sum_{j=J}^{\infty} v_j'(t_J)\cos(h_j \cdot 0)\right| \\
&= \left|\sum_{j=J}^{\infty} v_j'(t_J)\right| \geq |v_J'(t_J)| - \sum_{j=J+1}^{\infty} |v_j'(t_J)| \\
&\geq |v_J'(t_J)| - \sum_{j=J+1}^{\infty} E_j(t_J) \geq |v_J'(t_J)| - \sum_{j=J+1}^{\infty} C_5 \exp\Big\{ -\varepsilon h_j^{1/s}\Big\} \\
&= 1 - C_5 \sum_{j=J+1}^{\infty} \exp\Big\{ -\varepsilon \pi^{1/s} 2^{(j-1)^2}\Big\},
\end{aligned} \tag{40}$$

here we used (24).

On the other hand, from the definition of the Gevrey norm, by (22), (37) and Stirling's formula we also have

$$\begin{aligned}
\|u_1^{(J)}\|_{s,r} &\leq \sum_{j=J}^{\infty} |v_j'(0)| \sup_{n\in\mathbf{N}} \frac{h_j^n}{r^n n!^s} \leq \sum_{j=J}^{\infty} E_j(0) \sup_{n\in\mathbf{N}} \frac{h_j^n}{r^n n!^s} \\
&\leq \sum_{j=J}^{\infty} C_5 \exp\Big\{-\varepsilon h_j^{1/s}\Big\} \sup_{n\in\mathbf{N}} \frac{h_j^n}{r^n (2n\pi)^{s/2} n^{sn} e^{-sn}} \\
&= \frac{C_5}{(2\pi)^{s/2}} \sum_{j=J}^{\infty} 2^{-(j-1)^2} \sup_{n\in\mathbf{N}} \frac{\exp\Big\{-\varepsilon\pi^{1/s} 2^{(j-1)^2}\Big\} 2^{(sn+1)(j-1)^2}}{n^{s/2}\big(\frac{r}{\pi e^s}\big)^n n^{sn}} \\
&\leq \frac{C_5}{(2\pi)^{s/2}} \sum_{j=J}^{\infty} 2^{-(j-1)^2} \sup_{n\in\mathbf{N}} \frac{\big(\frac{sn+1}{\varepsilon\pi^{1/s}}\big)^{sn+1} e^{-(sn+1)}}{n^{s/2}\big(\frac{r}{\pi e^s}\big)^n n^{sn}} \\
&= \frac{C_5}{e\varepsilon\pi^{1/s}(2\pi)^{s/2}} \sum_{j=J}^{\infty} 2^{-(j-1)^2} \sup_{n\in\mathbf{N}} \frac{(sn+1)^{sn+1}}{n^{s/2}(r\varepsilon^s)^n n^{sn}}.
\end{aligned}$$

here we used the inequality $e^{-\kappa\xi}\xi^\beta \leq \left(\frac{\beta}{\kappa}\right)^\beta e^{-\beta}$ with $\xi = 2^{(j-1)^2}$, $\kappa = \varepsilon\pi^{1/s}$ and $\beta = sn+1$. We note that

$$\begin{aligned}
\frac{(sn+1)^{sn+1}}{n^{s/2}(r\varepsilon^s)^n n^{sn}} &= \frac{sn+1}{n^{s/2}(r\varepsilon^s)^n} \cdot \left(s + \frac{1}{n}\right)^{sn} \\
&\leq \frac{sn+n}{1\cdot(r\varepsilon^s)^n} \cdot (s+1)^{sn} = n(s+1)\left(\frac{(s+1)^s}{r\varepsilon^s}\right)^n.
\end{aligned}$$

If we take $r>0$ such that $\frac{(s+1)^s}{r\varepsilon^s} < 1$, we see that $\sup_{n\in\mathbf{N}} \frac{(sn+1)^{sn+1}}{n^{s/2}(r\varepsilon^s)^n n^{sn}} \leq C_s$. Thus, we get

$$\|u_1^{(J)}\|_{s,r} \leq C_6 \sum_{j=J}^{\infty} 2^{-(j-1)^2}, \tag{41}$$

similarly,

$$\|u_0^{(J)}\|_{s,r} \leq C_7 \sum_{j=J}^{\infty} 2^{-(j-1)^2}. \tag{42}$$

If the energy inequality (39) with $t = t_J$ holds, by (40), (41) and (42) we have

$$\|u^{(J)}(t_J)\|_{s,R} + 1 - C_5 \sum_{j=J+1}^{\infty} \exp\Big\{-\varepsilon\pi^{\frac{1}{s}} 2^{(j-1)^2}\Big\} \leq (C_6 + C_7) \sum_{j=J}^{\infty} 2^{-(j-1)^2}.$$

If J tends to infinity, t_J tends to T and we get

$$\|u^{(J)}(T)\|_{s,R} + 1 \leq 0.$$

This implies that the energy inequality (39) breaks and that the derivative loss really occurs in a sense of the radius of the Gevrey class G^s.

2.6. Concluding Remarks

REMARK 2.17. For the well-posedness, the case degenerating only at one point is a better situation than the case degenerating at an infinite number of points in a sense of the derivative loss. While, for the ill-posedness one would think that the latter case included more factors that $a(t)$ causes a blow-up solution. But in fact, we can not find out such a factor in this construction. The proof of the ill-posedness also relays on the energy inequality in Proposition 2.11. This means that the case degenerating at an infinite number of points is not a better situation than the case degenerating only at one point.

REMARK 2.18. Let

$$g_\eta(t) = \begin{cases} e^{-\frac{1}{(\eta^2-4t^2)}} & \text{for } |t| < \eta/2, \\ 0 & \text{for } |t| \geq \eta/2, \end{cases} \quad \text{and} \quad \psi_\eta(t) = \frac{\int_{-\infty}^t g_\eta(\sigma)d\sigma}{\int_{-\infty}^\infty g_\eta(\sigma)d\sigma}.$$

We define that

$$\chi_\eta(t) = 1 - \psi_\eta\Big(t - \frac{\eta}{2}\Big)\psi_\eta\Big(t + \frac{\eta}{2}\Big).$$

We know that $\chi_\eta(t) \equiv 1$ for $|t| \geq \eta$ and $\chi_\eta(t)$ touches the t axis at $t = 0$. We pay attention to the degeneration of infinite order. Instead of (16) we define

$$a(t) = \delta_j \Theta\left(\nu_j \frac{t - T_{j-1}}{\rho_j}\right) \chi_\eta(t - T_{j-1})\chi_\eta(t - T_j) \quad \text{for} \quad t \in I_j = [T_{j-1}, T_j],$$

where η with a sufficiently small constant such that $T_{j-1} < T_{j-1} + \eta < t_j$. Thanks to degeneration of $\chi_\eta(t)$, we can remove the restriction that $k = 0, 1$ for the coefficient $a(t)$ (see Remark 2.10). Then, we may consider the terminal value problem (24) on $[T_{j-1} + \eta, t_j] \subset I_j$. Moreover, we insert the terminal value problem on $[T_{j-1}, T_{j-1} + \eta] \subset I_j$

$$\begin{cases} v_j'' + h_j^2 a(t) v_j = 0, \\ v_j(T_{j-1} + \eta) = \tilde{V}_0, \ v_j'(T_{j-1} + \eta) = \tilde{V}_1, \end{cases}$$

where $\tilde{V}_0$ and $\tilde{V}_1$ satisfy the estimates as (27). Similarly as (28), we have an energy inequality for this additional problem. Thus, we can also get the following:

COROLLARY 2.19. *There exists a coefficient $a(t)$ such that*

1. *$a(t)$ is non-negative and degenerates at an infinite number of points.*
2. *$a(t)$ belongs to $C^{k,\alpha}[0,T]$ for all $k \in \mathbf{N}$ and $0 \le \alpha \le 1$.*
3. *The Cauchy problem (1) with $a(t)$ is ill-posed in G^s for $s > 1+(k+\alpha)/2$.*

Appendix. Singularity Spectra of Coefficients

Theorem 1.4 with $s_0 = 1$ ($k = \alpha = 0$) suggests that there exists a continuous coefficient $a(t)$ such that the Cauchy problem is ill-posed in the non-analytic class, in other words, a solution may blow-up if we give the initial data which can not be represented as a Taylor series (an infinite sum). It will be practically useful to find a way to know such an unsuitable coefficient $a(t)$ in advance. The Fourier transform is the complete absence of information regarding the time. Meanwhile, the windowed Fourier transform:

$$(T_{w_\beta} f)(b,\xi) = \int_{\mathbf{R}} e^{-i\tau\xi} f(\tau)\overline{w_\beta(\tau - b)}d\tau \tag{43}$$

and the wavelet transform:

$$(W_\psi f)(b,a) = \frac{1}{\sqrt{a}}\int_{\mathbf{R}} f(\tau)\overline{\psi\left(\frac{\tau - b}{a}\right)}d\tau \tag{44}$$

can extract the local information in time. Here we remark that a function $g(t) \in L^2(\mathbf{R})$ such that $tg(t) \in L^2(\mathbf{R})$ is called *window*. In (43) and (44), w_β, ψ are window functions. In this paper, we shall utilize $w_\beta(t) = \chi_{(-\beta,\beta)}(t)\cos^2(10\pi t)$ in case 1 and case 2, $w_\beta(t) = \chi_{(-\beta,\beta)}(t)e^{-9t^2/5}$ in case 3, and $\psi(t) = \frac{2(1-t^2)}{\sqrt{3}\pi^{1/4}}e^{-t^2/2}$ for the windowed Fourier transform and the wavelet transform. The simplified representations of the coefficients in Theorem 1.4 and Corollary 1.6 make it possible to analyze coefficients with the windowed Fourier transform and the wavelet transform. Only in this section we shall write the coefficient function by the letter f instead of a in order to avoid a confusion with the parameter a in the wavelet transform.

Case 1: Let $0 < T < 1$ and $f(t)$ be a non-negative monotone function defined by

$$f(t) = \begin{cases} \dfrac{1}{-\log(T-t)} & \text{for } \ 0 \le t < T(<1), \\ 0 & \text{for } \ t \ge T. \end{cases} \tag{45}$$

$f(t)$ degenerates only at $t = T$. We find that $f(t)$ belongs to $C^0[0,\infty)$, but does not belong to $C^\alpha[0,\infty)$ for any $\alpha > 0$. Thanks to the monotonicity, we see that the Cauchy problem with (45) is C^∞ well-posed.

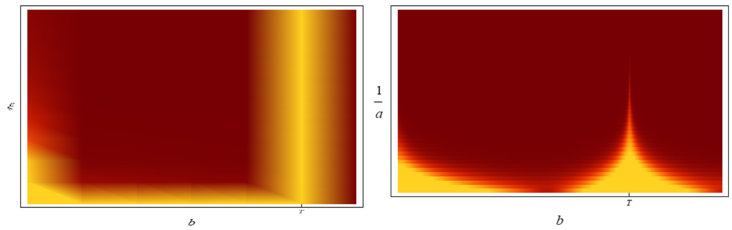

Figure 1: Graphs for windowed Fourier transform (left) and wavelet transform (right) of (45) with $T = 1/2$. Both figures show that the irregular point is $t(\equiv b) = T$. In particular, the wavelet transform (right) indicates that the high frequency (irregularity) increases toward the irregular point with a slope (the function (45) becomes irregular not rapidly but gradually).

Case 2: Let $0 < T < 1$ and $f(t)$ be a non-negative oscillating function defined by

$$f(t) = \begin{cases} \dfrac{1 - \cos\Big(-\log(T-t)\Big)}{-\log(T-t)} & \text{for} \quad 0 \le t < T(< 1), \\ 0 & \text{for} \quad t \ge T. \end{cases} \tag{46}$$

$f(t)$ degenerates at an infinite number of points. If we take $t_j = T - e^{-2j\pi}$ and $s_j = T - e^{-2j\pi-\pi/2}$, it holds that $|t_j - s_j| = e^{-2j\pi}|1 - e^{-\pi/2}| \sim e^{-2j\pi}$ and $|f(t_j) - f(s_j)| = (2j\pi + \pi/2)^{-1} \sim \frac{1}{j}$. Hence, we find that $f(t)$ belongs to $C^0[0,\infty)$, but does not belong to $C^\alpha[0,\infty)$ for any $\alpha > 0$. Noting that $f(t)$ satisfies $|f'(t)| \le C(T-t)^{-1}$, by [2] we see that the Cauchy problem with (46) is C^∞ well-posed.

Remark 2.20. In general, given functions are not always represented by the elementary periodic functions like sine and cosine. In this case,

$$\frac{1 - \cos\Big(-\log(T-t)\Big)}{-\log(T-t)} \equiv \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}\Big\{\log(T-t)\Big\}^{2n-1}.$$

If a function is given as the right hand side, it will be difficult to know the oscillations. The numerical analysis with the windowed Fourier transform and the wavelet transform can be available even for the function approximated by a finite sum

$$\tilde{f}(t) = \begin{cases} \displaystyle\sum_{n=1}^{100} \frac{(-1)^n}{(2n)!}\Big\{\log(T-t)\Big\}^{2n-1} & \text{for} \quad 0 \le t < T(< 1), \\ 0 & \text{for} \quad t \ge T. \end{cases} \tag{47}$$

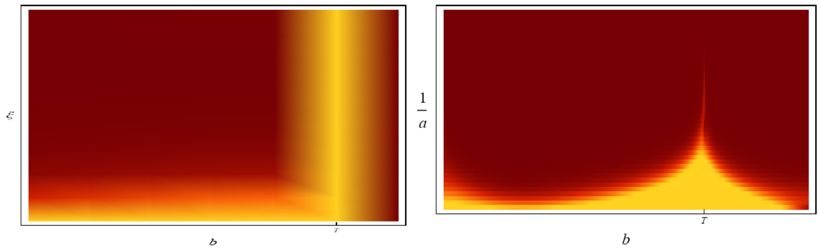

Figure 2: Graphs for windowed Fourier transform (left) and wavelet transform (right) of (46) with $T = 1/2$. Similarly as Figure 1, both figures show that the blow-up point is $t(\equiv b) = T$ and the wavelet transform (right) indicates that the high frequency (irregularity) increases toward the irregular point with a slope. Furthermore for the graph of the wavelet transform (right), we observe that the part of the slope becomes wider and higher since the oscillation influences on the irregularity in neighbourhood of $t(\equiv b) = T$.

Then, we observe that the figures for f and $\tilde{f}$ are almost same.

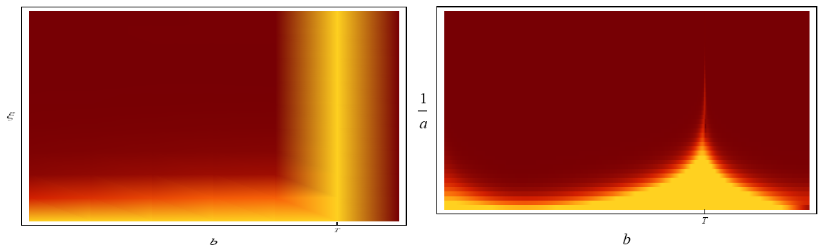

Figure 3: Graphs for windowed Fourier transform (left) and wavelet transform (right) of (47) with $T = 1/2$.

Case 3: Let $f(t)$ be a coefficient function in Theorem 1.4 with $s_0 = 1$ and $s = 11/10$, i.e., $T_j = \displaystyle\sum_{n=1}^{j} 2^{-(n-1)^2/20}$ $(j \geq 1)$ and

$$f(t) = 2^{-j^2/10}\Theta\Big(2^{21j^2/20}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0). \tag{48}$$

By Theorem 1.4 and its proof, $f(t)$ degenerates at an infinite number of points and belongs to $C^0[0,\infty)$. Then we see that the Cauchy problem with (48) is

$G^{11/10}$ ill-posed. For the ill-posedness it is possible to replace the function (48) by

$$f(t) = 2^{-j^r/10}\Theta(2^{11j^r/20}(t - T_j)$$

with $r > 1$ (see Remark 2.14). It is not so difficult to describe the figure of the wavelet transform even for a large r. Meanwhile, as r is larger, it would be more difficult to describe the figure of the windowed Fourier transform. For the simplicity, supposing that $r = 1$, we shall describe the figures of the following:

$$T_j = \sum_{n=1}^{j} 2^{-(n-1)/20} \ (j \geq 1)$$

and

$$f(t) = 2^{-j/10}\Theta\Big(2^{21j/20}(t - T_j)\Big) \quad \text{for} \quad t \in [T_j, T_{j+1}] \ (j \geq 0). \tag{49}$$

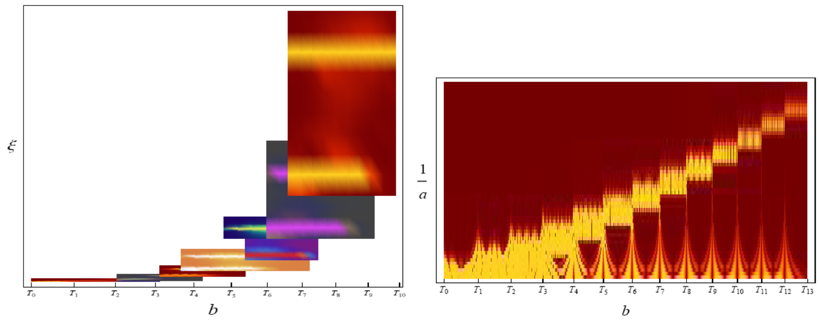

Figure 4: Graphs for windowed Fourier transform (left) and wavelet transform (right) of (49). In this case, the windowed Fourier transforms require 7 graphs to adjust the brightness of the spectrogram. On the other hand, such an arrangement is not necessary for the wavelet transform. In this sense the wavelet transform is convenient.

The degenerating and oscillating coefficients often appear in weakly hyperbolic equations. The amplitudes of oscillating coefficients are flattened by the degeneracy. In all above figures, the brightness shows a large value of windowed Fourier transform or wavelet transform, and the decay along the vertical axis denotes the smoothness of analyzed functions. For cases 1 and 2, from figures 1-3 we see that both the windowed Fourier transform and the wavelet transform detect the degenerations of analyzed functions at $t = T$. But, for case 3, to detect the variation of frequency with the windowed Fourier transform, we are forced to prepare some graphs according to the value of the windowed Fourier transform (its graph is obtained by pasting together). On the other hand, the wavelet transform is able to catch more information of low amplitudes with high-frequency oscillations in comparison with the windowed Fourier transform. Moreover, the multiplication by $1/\sqrt{a}$ in the definition of wavelet (44) makes the amplitudes more conspicuous. The slopes of figures in case 3 indicate that a peak moves toward the blow-up point $T > 0$ as the frequency increases, which possibly causes the ill-posedness. Thus, the wavelet transform can be used as a good screening test for coefficients giving the ill-posedness of the Cauchy problem.

Remark 2.21. Generally for a function $f(t) = F\left(\frac{t-b'}{a'}\right)$, the wavelet transform with $\psi\left(\frac{t-b}{a}\right)$ detects $a \sim a'$ and $b \sim b'$. Figure 4 means that $a \sim 2^{-21j/20}$ and $b = T_j$ are conspicuous since $f(t) = 20^{-j/10}\Theta\left(\frac{t-T_j}{2^{-21j/20}}\right)$.

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Metrizability of hereditarily normal compact like groups[1]

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Abstract. *Inspired by the fact that a compact topological group is hereditarily normal if and only if it is metrizable, we prove that various levels of compactness-like properties imposed on a topological group G allow one to establish that G is hereditarily normal if and only if G is metrizable (among these properties are locally compactness, local minimality and ω-boundedness). This extends recent results from [4] in the case of countable compactness.*

Keywords: locally compact group, locally minimal group, ω-bounded group, countably compact group, hereditarily normal topological group, metrizable group
MS Classification 2010: primary 22A05, 22C05; secondary 03E57, 54H11, 54D15, 54D30

1. Introduction

In this paper all topological spaces and topological groups are assumed to be Tychonov. The stronger separation axiom T_5, *hereditary normality*, will be the main point of the paper (recall that a topological space X is hereditary normal if every subspace of X is normal). Metrizable spaces are obviously hereditarily normal, while all countable spaces are T_5, but not necessarily metrizable.

Since compact topological spaces are always normal, one may expect that compact topological groups are often (sometimes) hereditarily normal. As the following example shows, this occurs precisely when the groups are metrizable.

Example 1.1. The hereditarily normal compact groups can be described making use of several classical theorems about compact groups and dyadic spaces (i.e., continuous images of the Cantor cubes $\{0,1\}^\kappa$).

(a) According to Hagler-Gerlits-Efimov's theorem, every compact group K of weight κ contains a Cantor cube $\{0,1\}^\kappa$ (see [12, 23]). Since $\{0,1\}^\kappa$ is T_5 precisely when $\kappa \leq \omega$, we deduce that K is T_5 if and only if K is metrizable.

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(b) Efimov [11] proved that T_5 dyadic spaces are metrizable (see also [14, 3.12.12(k)]). Since the compact groups are dyadic by the celebrated Kuz′minov theorem [18], we deduce again that the T_5 compact groups are metrizable.

In other words, one can resume the above observations in the following metrization criterion for compact groups:

Fact 1.2. *A compact group is T_5 if and only if it is metrizable.*

This fact does not remain true when compactness is replaced by the weaker property of countable compactness. Indeed, Hajnal and Juhasz [15] built, under the assumption of the Continuum Hypothesis (briefly, CH), a non-metrizable, countably compact, hereditarily normal subgroup of $\{0,1\}^{\mathfrak{c}}$ with some additional properties. Another example to this effect was produced by Tkachenko [25]. Under the assumption of CH, he proved that the free abelian group of size $\mathfrak{c}$ admits a non-metrizable, countably compact, hereditarily normal group topology, which is additionally connected, locally connected and hereditarily separable.

As item (a) of the next example shows, the validity of Fact 1.2 for countably compact groups is independent of ZFC.

Example 1.3. (a) Eisworth [13, Corollary 10] proved that under the Proper Forcing Axiom (briefly, PFA) all countably compact hereditary normal groups are metrizable.

(b) Further progress in this direction was achieved by Buzyakova [4], who reinforced (a) by showing that under PFA every countably compact subspace of a hereditarily normal topological group is metrizable [4, Corollary 2.6].

(c) A significant reinforcement of Fact 1.2 is available in the same paper [4, Theorem 2.9]: every compact *subspace* of a hereditarily normal topological group is metrizable.

(d) A variant of (c) for countable compactness is proposed in [4, Corollary 2.4] as well: every countably compact *subspace* of a hereditarily normal topological group containing non-trivial convergent sequences is metrizable.

The aim of this paper is to extend the "metrization criterion" 1.2 to other classes of compact like groups, among them locally compact groups, ω-bounded groups, locally minimal abelian groups, etc. (see §2.2 for the relevant definitions). To this end we essentially use the following theorem proved in [4]:

Theorem 1.4. [4, Theorem 2.3] *If G is a T_5 topological group with a non-trivial convergent sequence, then G has a G_δ-diagonal.*

From this theorem one can deduce the fact that every countably compact hereditarily normal group containing non-trivial convergent sequences is metrizable (this is [4, Corollary 2.5]), as well as item (d) of Example 1.3 (using Chaber's theorem about the metrizabilty of the countably compact spaces with a G_δ-diagonal).

The result for item (a) can be deduced also from the above result and the fact, established by Nyikos, L. Soukup, B. Veličković [21], that under PFA every countably compact hereditarily normal space is sequentially compact (so has non-trivial convergent sequences). For a further information on the impact of T_5 on compactness-like properties we recommend the nicely written outline of Nyikos [20].

This paper is organized as follows. In §2 we collect some properties of pseudo-character in topological groups, with particular emphasis on compact-like groups. In §3 come the main results. In order to keep our paper self-contained we include a proof of Theorem 1.4 in §3.1 (see Theorem 3.6) and give some immediate applications concerning ω-bounded groups and locally compact groups. Section 3.2 contains the main result of the paper, namely all hereditarily normal locally minimal abelian groups are metrizable (Theorem 3.11). We conclude with §4, containing some final comments and open questions.

2. Preliminaries

2.1. Properties of the pseudo-character of a topological group

We recall here the definitions of character and pseudo-character and some of their properties used in the paper.

Definition 2.1. *Let X be a topological space and $x \in X$.*

A local base at x is a filter-base of the filter of neighborhoods of x. Let $\chi(x, X)$ denote the character *of X at x, that is the maximum between ω and the minimal cardinality of a local base for x. Let $\chi(X) = \sup\{\chi(x, X) : x \in X\}$ be the* character *of X.*

A local pseudo-base at x is a family $\mathcal{F}$ of open neighborhoods such that $\bigcap \mathcal{F} = \{x\}$. Let $\psi(x, X)$ denote the pseudo-character *of X at x, that is the maximum between ω and the minimal cardinality of a local pseudo-base for x. Let $\psi(X) = \sup\{\psi(x, X) : x \in X\}$ be the* pseudo-character *of X.*

Remark 2.2. *Note that if G is a topological group, then for all $g \in G$*

$$\chi(g, G) = \chi(e, G) = \chi(G),$$
$$\psi(g, G) = \psi(e, G) = \psi(G).$$

Fact 2.3. *A group topology is metrizable if and only if it has a countable local base.*

Remark 2.4. *Let X be a topological space and $x \in X$. Then $\psi(x, X) = \omega$ if and only if $\{x\}$ is a G_δ-subset of X.*

Moreover if X is regular, then there exists a family $\{U_n : n \in \mathbb{N}\}$ that is a local pseudo-base at x such that $\bigcap_n \overline{U_n} = \bigcap_n U_n = \{x\}$.

If $Y \subseteq X$ is a G_δ-subset and $x \in Y$, then $\psi(x, X) = \omega$ if and only if $\psi(x, Y) = \omega$. Indeed, the necessity of this condition is obvious. Assume that $\psi(x, Y) = \omega$ and let this be witnessed by a countable family U_n of open neighborhoods of x in X such that $Y \cap (\bigcap_n U_n) = \{x\}$. If $Y = \bigcap_n W_n$ for some countable family of open sets in X, then the equality $(\bigcap_n W_n) \cap (\bigcap_n U_n) = \{x\}$ witnesses $\psi(x, Y) = \omega$.

If G is a set, let Δ_G denote the diagonal in $G \times G$, i.e. $\Delta_G = \{(g, g) : g \in G\} \subseteq G \times G$.

The next lemma is folklore (e.g., the implication (i) $\Leftarrow$(ii) was stated and proved in [4]), we give its proof for the sake of completeness.

Lemma 2.5. *Let G be a topological group. Then the following are equivalent:*

(i) Δ_G is a G_δ-subset of $G \times G$;

(ii) $\psi(G) = \omega$.

Proof. (i)$\Leftarrow$(ii). Let $\{e\} = \bigcap_n V_n$ with every V_n open neighborhood of e, so that for every $x \in G$ we have $\{x\} = \bigcap_{n\in\mathbb{N}} xV_n$. Hence $\{(x, x)\} = \bigcap_{n\in\mathbb{N}}(xV_n \times xV_n)$, and letting $U_n = \bigcup_{x\in G} xV_n \times xV_n$ we obtain $\Delta_G = \bigcap_{n\in\mathbb{N}} U_n$.

(i)$\Rightarrow$(ii). Let $\Delta_G = \bigcap_n U_n$ where every U_n is an open subset of $G \times G$. Then $(e, e) \in U_n$ for every $n \in \mathbb{N}$, so there exists an open subset $V_n \subseteq G$ such that $(e, e) \in V_n \times V_n \subseteq U_n$.

Now we verify that $\bigcap_{n\in\mathbb{N}} V_n = \{e\}$. If $g \in \bigcap_{n\in\mathbb{N}} V_n$, then $(g, e) \in \bigcap_{n\in\mathbb{N}}(V_n \times V_n) \subseteq \bigcap_{n\in\mathbb{N}} U_n = \Delta_G$, so $g = e$. □

2.2. Various levels of compactness

Let us recall several compactness-like properties of the topological spaces and topological groups. A space X is

(a) *pseudocompact*, if every real-valued function of X is bounded;

(b) ω-bounded, if every countable subset of X has a compact closure.

Obviously, ω-bounded spaces are countably compact, while countably compact spaces are pseudocompact.

A topological group G is *precompact*, if for every non-empty open set U of G there exists a finite set $F \subseteq G$ such that $FU = G$ (equivalently, if the completion of G is compact; in this case the two-sided completion coincides with Weil completion of G). Pseudocompact groups are precompact.

A topological group (G, τ) is *minimal* if for every Hausdorff group topology $\sigma \subseteq \tau$ on G one has $\sigma = \tau$ [8, 6].

The next notion in (b), proposed by Pestov and Morris [19] (see also T. Banakh [3]), is a common generalization of minimal groups, locally compact groups and normed spaces:

DEFINITION 2.6. *A topological group (G, τ) is* locally minimal *if there exists a neighborhood V of e such that whenever $\sigma \subseteq \tau$ is a Hausdorff group topology on G such that V is a σ-neighborhood of e, then $\sigma = \tau$.*

DEFINITION 2.7. *Let H be a subgroup of a topological group G. We say that H is* locally essential *in G if there exists a neighborhood V of e in G such that $H \cap N = \{e\}$ implies $N = \{e\}$ for all closed normal subgroups N of G contained in V.*

When necessary, we shall say H is locally essential with respect to V *to indicate also V. Note that if V witnesses local essentiality, then any smaller neighborhood of the neutral element does, too.*

We now recall a criterion for local minimality of dense subgroups.

THEOREM 2.8. [1, Theorem 3.5] *Let H be a dense subgroup of a topological group G. Then H is locally minimal if and only if G is locally minimal and H is locally essential in G.*

We will make use of the following fact from [1] connecting locally minimal groups and minimal groups in the abelian case.

PROPOSITION 2.9. [1, Proposition 5.13] *Every locally minimal abelian group contains a minimal G_δ-subgroup.*

2.3. Compact-like topological groups of countable pseudo-character

The next fact can be easily deduced from the proof of [1, Theorem 2.8]:

FACT 2.10. [1, Theorem 2.8] *If G is a locally minimal group with $\psi(G) = \omega$, then G is metrizable.*

LEMMA 2.11. *Let (G, τ) be an abelian topological group. If $\psi(G) = \omega$, then there exists a metrizable topology τ^* on G with $\tau^* \subseteq \tau$.*

Proof. Let $\bigcap_n U_n = \{0\}$ with U_n open neighborhood at 0 for every $n \in \mathbb{N}$. Without loss of generality we can assume $U_{n+1} \subseteq U_n$ and $U_n = -U_n$ for every $n \in \mathbb{N}$.

Let $V_0 = U_0$, and for $n \geq 1$ let $V_n = -V_n \in \tau$ be such that $V_n + V_n \subseteq U_n \cap V_{n-1}$. If τ^* is the group topology on G having the family $\{V_n \mid n \in \mathbb{N}\}$ as a local base, then $\tau^* \subseteq \tau$ and τ^* is metrizable. □

One can apply this lemma to obtain the following folklore fact about minimal abelian groups (see for example [8]).

COROLLARY 2.12. *Minimal abelian groups of countable pseudocharacter are metrizable.*

REMARK 2.13. *Minimal non-abelian groups of countable pseudocharacter need not be metrizable. Actually, their character may be arbitrarily large [22].*

LEMMA 2.14. *If G is a countably compact topological group and $\psi(G) = \omega$, then G is metrizable.*

Proof. Let $\{e\} = \bigcap_{n\in\mathbb{N}} \overline{U_n}$ with every U_n open neighborhood of e with $U_{n+1} \subseteq U_n$ for every $n \in \mathbb{N}$.

Assume for a contradiction that $\{U_n : n \in \mathbb{N}\}$ is not a local base. Then there exists an open neighborhood W of e such that $U_n \nsubseteq W$ for every $n \in \mathbb{N}$. Let $F_n = \overline{U_n} \setminus W$, and note that $F_{n+1} \subseteq F_n \neq \emptyset$. Moreover,

$$\bigcap_{n\in\mathbb{N}} F_n = \big(\bigcap_{n\in\mathbb{N}} \overline{U_n}\big) \setminus W = \{e\} \setminus W = \emptyset.$$

As G is countably compact, $F_n = \emptyset$ for some $n \in \mathbb{N}$, a contradiction. □

3. Hereditarily Normal Topological Groups

Let $(G, \cdot, e)$ be a monoid, equipped with a topology τ such that the pair (G, τ) is a *topological monoid*, i.e., the monoid operation $\mu\colon G \times G \to G$ is continuous with respect to the product topology.

Given a subset $X \subseteq G$, we let

$$X^{-1} = \{y \in G : yx = e \text{ for some } x \in X\}$$

(note that if G is a group, then the set X^{-1} consists of the inverses of the elements of X).

DEFINITION 3.1. *Let X be a topological space. A pair A, B of subsets of X is called* unseparable *in X if for every pair of open sets U, V of X such that $A \subseteq U$ and $B \subseteq V$, one has $U \cap V \neq \emptyset$.*

Clearly, the sets A and B in an unseparable pair are non-empty. A a topological space X with an unseparable pair of closed disjoint subsets is not normal.

The following lemma can be attributed to Katětov [17]. It relevance to questions related to hereditary normality in topological groups was discovered and exploited by Buzyakova [4].

Lemma 3.2 (Katětov). *Let S, R be two topological spaces, $r \in R$ and $s \in S$. If $\psi(r, R) > \omega$, s is a limit point of S and S is separable, then $Z = R \times S \setminus \{(r, s)\}$ is not normal. In particular, the pair formed by the closed, disjoint subsets $(\{r\} \times S) \setminus \{(r, s)\}$ and $(R \times \{s\}) \setminus \{(r, s)\}$ of Z is unseparable.*

Fact 3.3. *Let X, Y be topological spaces, and let $\varphi : X \to Y$ be a continuous map. If F_1 and F_2 is a pair of unseparable sets in X, then $\varphi(F_1)$, $\varphi(F_2)$ is a pair of unseparable sets in $\varphi(X)$.*

3.1. Hereditary normality versus countable pseudocharacter in topological groups

The proof of the next lemma (in the case of a topological group), as well as the proof of Theorem 3.6, are inspired by and follow the line of the proof of [4, Theorem 2.3]. In particular, we preferred to isolate the lemma from that proof in order to better enhance the idea triggered by Katětov's lemma.

Lemma 3.4. *Let G be a topological monoid. Assume that there exist two closed subset $S, R \subseteq G$ such that*

(i) S *is separable and* $e \in \overline{S \setminus \{e\}}^S$,

(ii) $\psi(R, e) > \omega$,

(iii) $R \cap S = \{e\}$,

(iv) $R \cap S^{-1} = \{e\}$.

Then G is not T_5.

Proof. Consider $Z = (R \times S) \setminus \{(e, e)\} \subseteq G \times G \setminus \{(e, e)\}$, and let

$$R_1 = (R \times \{e\}) \setminus \{(e, e)\} = (R \setminus \{e\}) \times \{e\} \subseteq Z,$$
$$S_1 = (\{e\} \times S) \setminus \{(e, e)\} = \{e\} \times (S \setminus \{e\}) \subseteq Z.$$

Note that $\mu Z = (R \cdot S) \setminus \{e\} \subseteq G \setminus \{e\}$ by (iv), and we are going to prove that $\mu Z \subseteq G$ is not normal, showing that the pair μR_1, μS_1 of closed subsets of μZ is unseparable.

Obviously, $\mu R_1 = R \setminus \{e\} \subseteq G \setminus \{e\}$ is closed in $G \setminus \{e\}$, and contained in μZ, so that μR_1 is a closed subset of μZ. Similarly, $\mu S_1 = S \setminus \{e\}$ is a closed subset of μZ. Moreover, μR_1 and μS_1 are disjoint by (iii).

Being $\mu Z \subseteq G \setminus \{e\}$, the restriction $\bar{\mu} = \mu \restriction_Z : Z \to G \setminus \{e\}$ is well defined and continuous. Then (i) and (ii) yield that the pair R_1, S_1 is unseparable in Z by Katětov's lemma, so that the pair μR_1, μS_1 is unseparable in μZ by Fact 3.3.

So μZ is not a normal space, and G is not T_5. □

From the above lemma, we immediately obtain the following result for topological groups.

Corollary 3.5. *Let be a topological group with two closed subset $S, R \subseteq G$ such that*

(i) $S = S^{-1}$ *is separable and* $e \in \overline{S \setminus \{e\}}^S$,

(ii) $\psi(R, e) > \omega$,

(iii) $R \cap S = \{e\}$,

Then G is not T_5.

Theorem 3.6. *Let G be a T_5 topological group. If there exists a non-trivial convergent sequence in G, then $\psi(G) = \omega$.*

Proof. Let $x_n \to e$ be a non trivial convergent sequence, and assume for a contradiction that $\psi(G) > \omega$. There exists an open neighborhood U_0 of e such that $\{x_0, x_0^{-1}\} \cap \overline{U_0} = \emptyset$. Thence for all $n \in \omega$ with $n > 0$ there exists an open neighborhood U_n of e such that $\overline{U_n} \subseteq U_{n-1}$ and $\{x_n, x_n^{-1}\} \cap \overline{U_n} = \emptyset$. Note that

$$R := \bigcap_n \overline{U_n} = \bigcap_n U_n \tag{1}$$

by the choice of U_n. Moreover, R is a closed G_δ-subset of G by (1). Hence, $\psi(R, e) > \omega$, in view of Remark 2.4.

Let $S = \{e\} \cup \{x_n : n \in \omega\} \cup \{x_n^{-1} : n \in \omega\}$. Obviously, S is a closed countable subset of G (as $e \in S$ is the only limit point of S), so S is separable. As $S = S^{-1}$ and $R \cap S = \{e\}$, G is not T_5 by Corollary 3.5, a contradiction. □

From this theorem and Lemma 2.5, one can deduce Theorem 1.4.

Note that if G is a countably compact group with a non-trivial convergent sequence, then G is T_5 if and only if it is metrizable (this is [4, Corollary 2.5]) in view of Theorem 3.6 (that yields $\psi(G) = \omega$) and Lemma 2.14. In other words, the "metrization criterion" 1.2 extends to countably compact group with a non-trivial convergent sequences. Moreover, since normal pseudocompact spaces are countably compact, 1.2 extends to pseudocompact groups with a non-trivial convergent sequence.

In the smaller class of ω-bounded groups one does not need to impose the blanket condition of existence of non-trivial convergent sequence.

COROLLARY 3.7. *Let G be an ω-bounded group. Then G is T_5 if and only if it is metrizable (hence compact).*

Proof. If G is finite, then there is nothing to prove, so assume from now on that G is an infinite ω-bounded group. Since ω-bounded groups are countably compact, it suffices to show that G has a non-trivial convergent sequence and then apply, as above, [4, Corollary 2.5].

Take a countably infinite subgroup of G. Then its closure K is an infinite compact group. Hence K contains an infinite Cantor cube, so K has non-trivial convergent sequences. $\square$

THEOREM 3.8. *A locally compact group G is T_5 if and only if it is metrizable.*

Proof. By a theorem of Davis [5], G is *homeomorphic* to a product $K \times \mathbb{R}^n \times D$, where K is a compact subgroup of G, $n \in \mathbb{N}$ and D is a discrete space. As K is a T_5 compact group, we deduce from Example 1.1 that K is metrizable. This immediately implies that G is metrizable as well.

Let us point out a second alternative proof that makes no recourse to Davis theorem. Let us recall first that the character and the pseudocharacter of a locally compact group coincide [16]. Since every locally compact group has non-trivial convergent sequences, Theorem 3.6 yields that $\psi(G) = \chi(G) = \omega$. $\square$

3.2. Metrizability of the hereditarily normal locally minimal abelian groups

For the proof of our main theorem 3.11, we need the following result which is of independent interest.

THEOREM 3.9. *Every locally minimal abelian group has an infinite metrizable subgroup.*

Proof. Let us consider first the case when G is precompact.

Let K denote the compact completion of G. By Theorem 2.8, G is locally essential in K. so there exists an open neighborhood V of 0 in K, such that every non-zero closed subgroup of V non-trivially meets G.

Suppose that for some prime p there exists a closed subgroup N of K isomorphic to the group $\mathbb{Z}_p$ of p-adic integers. Since $N \cap V$ is an open neighborhood of 0 in N, there exists $n \in \mathbb{N}$ such that $p^n N \subseteq N \cap V$. As $p^n N \cong N$ is a closed non-trivial subgroup of K contained in V, we deduce that it must non-trivially intersect G. Then $G \cap p^n N$ is an infinite metrizable subgroup of N.

Assume now that K contains subgroups isomorphic to the group $\mathbb{Z}_p$ of p-adic integers for no prime p. Then K is an *exotic torus* in terms of [7], i.e., $n = \dim K$ is finite and K contains a closed subgroup L such that $K/L \cong \mathbb{T}^n$ and $L = \prod_p B_p$, where each B_p is a compact p-group.

If B_p is infinite for some p, then its socle $S_p = \{x \in B_p : px = 0\}$ is an infinite closed subgroup of K (as B_p is a bounded p-group). Then $S_p \cong \mathbb{Z}(p)^\kappa$, where $\kappa = w(S_p)$. Hence, the topology of S_p has a local base at 0 formed by open subgroups. Therefore, the neighborhood $S_p \cap V$ of 0 in S_p contains an open subgroup H of S_p. Moreover, $H \neq 0$ as H is open and S_p is precompact. As $H \subseteq V$ is a non-trivial subgroup of K of exponent p, we deduce that $H \leq G$. Since $H \cong \mathbb{Z}(p)^\kappa$, we deduce that H contains an infinite metrizable subgroup.

Now assume that B_p is finite for all primes p, but the group L is infinite. Then L is a compact metrizable group having a local base at 0 formed by open subgroups. Let $\pi = \{p : B_p \neq \{0\}\}$. Then π is infinite, For each $p \in \pi$ fix an element $x_p \in B_p$ of order p and let H_p be the cyclic subgroup of B_p generated by x_p. The subgroup $L' = \prod_{p\in\pi} H_p$ is still an infinite compact metrizable group having a local base at 0 formed by open subgroups. Therefore, the neighborhood $L' \cap V$ of 0 in L' contains an open subgroup H of L'. Moreover, $H \neq 0$ as H is open and L' is compact and infinite. Using the Chinese remainder theorem one can easily prove that $H = \prod_{p\in\pi'} H_p$, where π' is an infinite subset of π. Pick $p \in \pi'$. Then $H_p \neq 0$ is closed subgroup of K of exponent p, with $H_p \subseteq V$. So $H_p \cap G \neq \{0\}$. Since H_p has no proper subgroups, we deduce that $H_p \leq G$. Therefore, $S = \bigoplus_{p\in\pi'} H_p$ is an infinite subgroup of G contained in L', hence S is metrizable.

Finally, assume that L is finite. Then the quotient map $K \to K/L \cong \mathbb{T}^n$ is a local homeomorphism. Since $\mathbb{T}^n$ is metrizable, we deduce that K and G are metrizable as well.

In the general case, the locally minimal abelian group G contains a minimal, G_δ-subgroup H of G. By a well-known theorem of Prodanov and Stoyanov [8, 6], every minimal abelian group is precompact. If H is infinite, then the above case allows us to claim that H contains an infinite metrizable subgroup. In case H is finite, obviously $\psi(H) \leq \omega$, so we can conclude that $\psi(G) = \omega$, by Remark 2.4. By Fact 2.10, G is metrizable. □

Corollary 3.10. *Every locally minimal abelian group has a non-trivial convergent sequence.*

Theorem 3.11. *Let G be a locally minimal abelian group. Then G is T_5 if and only if it is metrizable.*

Proof. By Corollary 3.10, G contains a non-trivial convergent sequence. By Theorem 3.6, $\psi(G) = \omega$. At this point we can deduce that G is metrizable from Fact 2.10. □

Corollary 3.12. *Let G be a minimal abelian group. Then G is T_5 if and only if it is metrizable.*

Proof. Since minimal groups are locally minimal, Theorem 3.11 applies.

For a direct alternative proof of this fact making no recourse to Theorem 3.11, we recall that every minimal abelian group contains a non-trivial convergent sequence (see for example [24]). By Theorem 3.6, $\psi(G) = \omega$. Now use the fact that minimal abelian groups of countable pseudocharacter are metrizable by Corollary 2.12. $\square$

4. Final comments and open questions

A topological group G is called *sequentially complete*, if G is sequentially closed in its two-sided completion [10]. Countably compact groups, as well as all complete groups, are sequentially complete. So sequential completeness can be considered as a simultaneous generalization of these two compactness-like properties. This explains the interest in the following example.

Example 4.1. Every countable abelian group G carries a precompact, sequentially complete group topology. Indeed, take the Bohr topology $\mathcal{P}_G$ of G (i.e., the maximum precompact topology on G). Following van Douwen, we denote the topological group $(G, \mathcal{P}_G)$ by $G^\#$. It is known that $G^\#$ has no convergent sequences, hence $G^\#$ is sequentially complete and non-metrizable. Since $G^\#$ is T_5 as every countable topological group, we deduce that a hereditarily normal precompact, sequentially complete group need not be metrizable.

It is worth noting that the group $G^\#$ is not normal whenever the group G is uncountable [26].

Our results leave open several questions.

Question 4.2. *Is every locally minimal T_5 group necessarily metrizable? What about countably compact locally minimal T_5 groups?*

In the non-abelian case, a minimal group need not contain any non-trivial convergent sequence ([24]) and minimal non-abelian groups of countable pseudocharacter need not be metrizable (Remark 2.13). Therefore, Corollary 3.12 leaves open the following question.

Question 4.3. *Is every minimal T_5 group necessarily metrizable? What about countably compact minimal T_5 groups?*

Note, that if the answer to the second question is positive, then every countably compact minimal T_5 group is compact metrizable. The answer depends on the answer of the following question from [9, Problem 23 (910)] which still remains open:

QUESTION 4.4. *Must an infinite, countably compact, minimal group contain a non-trivial convergent sequence ?*

Clearly, a positive answer to Question 4.4 yields a answer to the second part of Question 4.3.

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Compact groups with a dense free abelian subgroup[1]

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Abstract. *The compact groups having a dense infinite cyclic subgroup (known as monothetic compact groups) have been studied by many authors for their relevance and nice applications. In this paper we describe in full details the compact groups K with a dense free abelian subgroup F and we describe the minimum rank $r_t(K)$ of such a subgroup F of K. Surprisingly, it is either finite or coincides with the density character $d(K)$ of K.*

Keywords: compact group, dense subgroup, free abelian subgroup, topological generators, topological free rank, w-divisible group, divisible weight
MS Classification 2010: primary 22C05, 22A05; secondary 20K45, 54H11, 54D30

1. Introduction

Dense subgroups (with some additional properties) of compact groups have been largely studied for instance in [2, 3, 4, 7, 25]. Moreover, large independent families of dense pseudocompact subgroups of compact connected groups are built in [23], while potential density is studied in [13, 14, 15].

This note is dedicated mainly to the study of the class $\mathcal{F}$ of those Hausdorff topological groups that have a dense free abelian subgroup. These groups are necessarily abelian, so in this paper we are concerned exclusively with Hausdorff topological abelian groups, and we always use the additive notation. Moreover, we mainly consider the groups K in $\mathcal{F}$ that are also compact. The choice of compactness is motivated by the fact that many non-discrete topological abelian groups possess no proper dense subgroups at all (see [25] for a locally compact abelian group with this property), whereas every infinite compact group K admits proper dense subgroups. Indeed, it is known that $d(K) < |K|$, where $d(K)$ denotes the density character of K (i.e., the minimum cardinality of a dense subgroup of K). We recall also that

$$d(K) = \log w(K);$$

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here as usual $w(G)$ denotes the weight of a topological abelian group G, and for an infinite cardinal κ we let $\log \kappa = \min\{\lambda : 2^\lambda \geq \kappa\}$ the logarithm of κ.

Let us start the discussion on the class $\mathcal{F}$ from a different point of view that requires also some historical background.

The *topological generators* of a topological (abelian) group G have been largely studied by many authors in [1, 8, 10, 12, 16, 17, 18, 19]; these are the elements of subsets X of G generating a dense subgroup of G. In this case we say that G is *topologically generated* by X. In particular, a topological group having a finite topologically generating set X is called *topologically finitely generated* (*topologically s-generated*, whenever X has at most s element).

Usually, various other restraints apart finiteness have been imposed on the set X of topological generators. These restraints are mainly of topological nature and we collect some of them in the next example.

Example 1.1. *In the sequel G is a topological group with neutral element e_G and X is a topologically generating set of G.*

(a) When X is compact, G is called compactly generated*. This provides a special well studied subclass of the class of σ-compact groups.*

(b) The set X is called a suitable set *for G if $X \setminus \{e_G\}$ is discrete and closed in $G \setminus \{e_G\}$. This notion was introduced in 1990 by Hofmann and Morris. They proved that every locally compact group has a suitable set in [21] and dedicated the entire last chapter of the monograph [22] to the study of the minimum size $s(G)$ of a suitable set of a compact group G. Properties and existence of suitable sets are studied also in the papers [8, 10, 12, 16, 17, 18, 19].*

Clearly, a finite topologically generating set X is always suitable.

(c) The set X is called totally suitable *if X is suitable and generates a totally dense subgroup of G [19]. The locally compact groups that admit a totally suitable set are studied in [1, 19].*

(d) The set X is called a supersequence *if $X \cup \{e_G\}$ is compact, so coincides with the one-point compactification of the discrete set $X \setminus \{e_G\}$. Any infinite suitable set X in a countably compact group G is a supersequence converging to e_G [17]. This case is studied in detail in [12, 16, 26].*

Now, with the condition $G \in \mathcal{F}$ we impose a purely algebraic condition on the topologically generating set X of the topological abelian group G. Indeed, clearly a topological abelian group G belongs to $\mathcal{F}$ precisely when G has a topologically generating set X that is *independent*, i.e., X generates a dense free abelian subgroup of G. In case G is discrete, the *free rank* $r(G)$ of G is the maximum cardinality of an independent subset of G; in this paper we call it simply rank. Imitating the discrete case, one may introduce the following invariant measuring the minimum cardinality of an independent generating set X of G.

Definition 1.2. *For a topological abelian group $G \in \mathcal{F}$, the* topological free rank *of G is*

$$r_t(G) = \min\{r(F) : F \text{ dense free abelian subgroup of } G\}.$$

Let

$$\mathcal{F}_{\text{fin}} = \{G \in \mathcal{F} : r_t(G) < \infty\}.$$

Obviously, $d(G) \leq r_t(G)$ whenever $G \in \mathcal{F} \setminus \mathcal{F}_{\text{fin}}$, whereas always

$$d(G) \leq r_t(G) \cdot \omega \tag{1}$$

holds true.

We describe the compact abelian groups K in $\mathcal{F}$ in two steps, depending on whether the topological free rank $r_t(K)$ of K is finite.

We start with the subclass $\mathcal{F}_{\text{fin}}$, i.e., with the compact abelian groups K having a dense free abelian subgroup of finite rank. The complete characterization of this case is given in the next Theorem A, proved in Section 3. We recall that the case of rank one is that of monothetic compact groups, and the characterization of monothetic compact groups is well known (see [20]). Moreover, every totally disconnected monothetic compact group is a quotient of the universal totally disconnected monothetic compact group $M = \prod_p \mathbb{J}_p$, where $\mathbb{J}_p$ denotes the p-adic integers. Furthermore, an arbitrary monothetic compact group is a quotient of $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times M$ (see Proposition 3.4 below and [20, Section 25]).

For a prime p and an abelian group G, the p-socle of G is $\{x \in G : px = 0\}$, which is a vector space over the field $\mathbb{F}_p$ with p many elements; the *p-rank* $r_p(G)$ of G is the dimension of the p-socle over $\mathbb{F}_p$. Moreover, we give the following

Definition 1.3. *Let G be an abelian group. For p a prime, set*

$$\rho_p(G) = r_p(G/pG).$$

Moreover, let $\rho(G) = \sup_p \rho_p(G)$.

Several properties of the invariant ρ_p for compact abelian groups are given in Section 3.

In the next theorem we denote by $c(K)$ the connected component of the compact abelian group K.

Theorem A. *Let K be an infinite compact abelian group and $n \in \mathbb{N}_+$. Then the following conditions are equivalent:*

(a) $K \in \mathcal{F}$ *and* $r_t(K) \leq n$;

(b) $w(K) \leq \mathfrak{c}$ *and* $K/c(K)$ *is a quotient of* M^n;

(c) $w(K) \leq \mathfrak{c}$ *and* $\rho(K) \leq n$ *(i.e.,* $\rho_p(K) \leq n$ *for every prime* p*);*

(d) K *is a quotient of* $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times M^n$.

The case of infinite rank is settled by the next theorem characterizing the compact abelian groups that admit a dense free abelian subgroup, by making use of dense embeddings in some power of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (endowed with the quotient topology inherited from $\mathbb{R}$). Here $\widehat{K}$ denotes the Pontryagin dual of the compact abelian group K; in this case $\widehat{K}$ is a discrete abelian group, and

$$w(K) = |\widehat{K}|.$$

Note that, if K is an infinite compact abelian group and $K \in \mathcal{F}$, then there exists a dense free abelian subgroup of K of infinite rank, as $r(K) \geq \mathfrak{c}$.

Theorem B. *Let* K *be an infinite compact abelian group and* κ *an infinite cardinal. Then the following conditions are equivalent:*

(a) $K \in \mathcal{F}$ *and* $r_t(K) \leq \kappa$;

(b) $\widehat{K}$ *admits a dense embedding in* $\mathbb{T}^\lambda$ *for some* $\lambda \leq \kappa$;

(c) $d(K) \leq r(K)$ *and* $d(K) \leq \kappa$.

Theorem B has as an easy consequence the next characterization. To prove the second part of the corollary take $\kappa = d(K)$ in item (c) of Theorem B.

Corollary 1. *Let* K *be an infinite compact abelian group. Then* $K \in \mathcal{F}$ *if and only if* $d(K) \leq r(K)$*. In this case* K *has a dense free abelian subgroup of rank* $d(K)$.

According to Corollary 1, a compact abelian group K admits a dense free abelian subgroup of rank exactly $d(K)$. Hence, we see now in Corollary 2 that the inequality in (1) becomes an equality in case K is compact. Furthermore, as a consequence of Theorem A and Theorem B respectively, we can see that $r_t(K)$ is equal either to $\rho(K)$ or to $d(K)$ depending on its finiteness or infiniteness respectively.

Corollary 2. *Let* K *be an infinite compact abelian group. If* $K \in \mathcal{F}$*, then*

$$d(K) = r_t(K) \cdot \omega.$$

Moreover,

$$r_t(K) = \begin{cases} \rho(K) & \text{if } K \in \mathcal{F}_{\text{fin}}, \\ d(K) & \text{if } K \in \mathcal{F} \setminus \mathcal{F}_{\text{fin}}. \end{cases}$$

Roughly speaking, if K is an infinite compact abelian group in $\mathcal{F}$, Theorem B asserts that $d(K) \leq r(K)$. Moreover, if F is a dense free abelian subgroup of K of infinite rank, then $r(F)$ can range between $d(K)$ and $r(K)$. We underline that the maximum $r(K)$ can be reached by $r(F)$ since $r(K) \geq \mathfrak{c}$, and that also the minimum is a possible value of $r(F)$ by the equality in Corollary 2.

The proof of Theorem B is given in Section 4. It makes use of the following concepts introduced and studied in [9]; as usual, for an abelian group G we denote $mG = \{mx : x \in G\}$ and $G[m] = \{x \in G : mx = 0\}$ for $m \in \mathbb{N}_+$, where $\mathbb{N}_+$ denotes the set of positive natural numbers.

Definition 1.4. [9] *Let G be a topological abelian group.*

(i) The group G is w-divisible *if $w(mG) = w(G) \geq \omega$ for every $m \in \mathbb{N}_+$.*

(ii) The divisible weight *of G is $w_d(G) = \inf\{w(mG) : m \in \mathbb{N}_+\}$.*

This definition is different from the original definition from [9], where instead of $w(mG) = w(G) \geq \omega$ one imposes the stronger condition $w(mG) = w(G) > \omega$, which rules out all second countable groups. Since this is somewhat restrictive from the point of view of the current paper, we adopt this slight modification here.

So an infinite topological abelian group G is w-divisible if and only if $w(G) = w_d(G)$. In particular, an infinite discrete abelian group G is w-divisible if and only if $|mG| = |G|$ for every $m \in \mathbb{N}_+$. Moreover, it is worth to note here that every infinite monothetic group is w-divisible.

Another consequence of Theorem B is that the class $\mathcal{F}$ contains all w-divisible compact abelian groups, so in particular all connected and all torsion-free compact abelian groups:

Corollary 3. *If K is a w-divisible compact abelian group, then $K \in \mathcal{F}$.*

2. Some general properties of the class $\mathcal{F}$

The next lemma ensuring density of subgroups is frequently used in the sequel.

Lemma 2.1. [4] *Let G be a topological group and let N be a quotient of G with $h : G \twoheadrightarrow N$ the canonical projection. If H is a subgroup of G such that $h(H)$ is dense in N and H contains a dense subgroup of* $\ker h$*, then H is dense in G.*

In the next result we give two stability properties of the class $\mathcal{F}$.

Proposition 2.2. *The class $\mathcal{F}$ is stable under taking:*

(a) arbitrary direct products;

(b) extensions.

Proof. (a) Assume $G_i \in \mathcal{F}$ for all $i \in I$. Let F_i be a dense free abelian subgroup of G_i for every $i \in I$. Then the direct sum $F = \bigoplus_{i\in I} F_i$ is a dense free abelian subgroup of $\prod_{i\in I} G_i$.

(b) Assume that G is a topological abelian group with a closed subgroup $H \in \mathcal{F}$ such that also $G/H \in \mathcal{F}$. Let F and F_1 be dense free abelian subgroups of H and G/H, respectively. Let $q : G \to G/H$ be the canonical projection. Since q is surjective, we can find a subset X of G such that $q(X)$ is an independent subset of G/H generating F_1 as a free set of generators. Then X is independent, so generates a free abelian subgroup F_2 of G and the restriction $q\restriction_{F_2}: F_2 \to F_1$ is an isomorphism. In particular, $F_2 \cap \ker q = 0$, so $F_2 \cap F = 0$ as well. Therefore, $F_3 = F + F_2$ is a free abelian subgroup of G. Moreover, F_3 contains the dense subgroup $F_1 \subseteq H = \ker q$ and $q(F_3) = q(F_2) = F_1$ is a dense subgroup of G/H. Therefore, F_3 is a dense subgroup of G by Lemma 2.1. $\square$

The next claim is used essentially in the proof of Proposition 2.4, which solves one of the implications of Theorem A.

Lemma 2.3. *Let G be an abelian group, K a subgroup of G of infinite rank $r(K)$ and let F be a finitely generated subgroup of G. If $s \in \mathbb{N}_+$ and H is an s-generated subgroup of G, then there exists a free abelian subgroup F_1 of rank s of G, such that $F_1 \cap F = 0$ and $H \subseteq K + F_1$.*

Proof. Since $F + H$ is a finitely generated subgroup of G, the intersection $N = K \cap (F + H)$ is a finitely generated subgroup of K. Therefore, the rank $r(K/N)$ is still infinite. In particular, there exists a free abelian subgroup F_2 of rank s of K, such that $F_2 \cap N = 0$. Then also

$$F_2 \cap (F + H) = 0. \tag{2}$$

Let $x_1, \dots, x_s$ be the generators of H and let $t_1, \dots, t_s \in K$ be the free generators of F_2. Let $z_i = x_i + t_i$ for $i = 1, \dots, s$ and $F_1 = \langle z_1, \dots, z_s\rangle$. Then, obviously $H \subseteq K + F_1$ as $x_i = t_i - z_i \in K + F_1$.

The subgroup F_1 is free since any linear combination $k_1z_1 + \ldots + k_sz_s = 0$ produces an equality $k_1x_1 + \ldots + k_sx_s = -(k_1t_1 + \ldots + k_st_s) \in F_2$. Since $k_1x_1 + \ldots + k_sx_s \in H$, from (2) one can deduce that $k_1t_1 + \ldots + k_st_s = 0$. Since $t_1, \dots, t_s$ are independent, this gives $k_1 = \ldots = k_s = 0$. This concludes the proof that F_1 is free.

It remains to prove that $F_1 \cap F = 0$. So let $x \in F_1 \cap F$ and let us verify that necessarily $x = 0$. Since $x \in F_1$, there exist some integers $a_1, \dots, a_s$ such that $x = a_1z_1 + \ldots + a_sz_s = (a_1x_1 + \ldots + a_sx_s) + (a_1t_1 + \ldots + a_st_s)$. Then $a_1t_1 + \ldots + a_st_s \in F_2 \cap (F + H)$ and this intersection is trivial by (2). By the independence of $t_1, \ldots, t_s$ we have $a_1 = \ldots = a_s = 0$ and hence $x = 0$ as desired. $\square$

PROPOSITION 2.4. *Let G be a topological abelian group and K a closed subgroup of G with $r(K) \geq \omega$. If K contains a dense free abelian subgroup F of rank $m \in \mathbb{N}_+$ and G/K is topologically s-generated for some $s \in \mathbb{N}_+$, then G admits a dense free abelian subgroup E of rank $m+s$ (i.e., $G \in \mathcal{F}$ and $r_t(G) \leq m+s$).*

Proof. Let $q : G \to G/K$ be the canonical projection and consider G/K endowed with the quotient topology of the topology of G. Let Y be a subset of size s of G/K generating a dense subgroup of G/K. Pick a subset X of size s of G such that $q(X) = Y$ and let $H = \langle X \rangle$. By Lemma 2.3 there exists a free abelian subgroup F_1 of rank s of G such that $F_1 \cap F = 0$ and $H \subseteq K + F_1$. Hence, $q(F_1)$ is a dense subgroup of G/K as it contains $q(H) = \langle Y \rangle$. Let $E = F + F_1$. Then E is a free abelian subgroup of G of rank $m + s$. Moreover, E contains a dense subgroup of $K = \ker q$ and $q(E) = q(F_1)$ is dense in G/K. Therefore, E is dense in G by virtue of Lemma 2.1. $\square$

COROLLARY 2.5. *If G is a topologically finitely generated abelian group with infinite rank $r(G)$, then $G \in \mathcal{F}_{\text{fin}}$.*

Proof. Let H be the finitely generated dense subgroup of G. Then $H = L \times F$, where L is a free abelian subgroup of finite rank and F is a finite subgroup. Then the closure K of L in G is a closed finite index subgroup of G, so K is open too. Since K has finite index in G, its rank $r(K)$ is infinite. By Proposition 2.4, G contains a dense free abelian subgroup F of finite rank. $\square$

In particular, topologically finitely generated compact abelian groups belong to $\mathcal{F}$. The same holds relaxing compactness to pseudocompactness since non torsion pseudocompact abelian groups have rank at least $\mathfrak{c}$ as proved in [6].

We recall the following result stated in [11] giving, for an infinite compact abelian group K, several equivalent conditions characterizing the density character $d(K)$ of K.

PROPOSITION 2.6. [11, Exercise 3.8.25] *Let K be an infinite compact abelian group and κ an infinite cardinal. Then the following conditions are equivalent:*

(a) $d(K) \leq \kappa$;

(b) there exists a homomorphism $f : \bigoplus_\kappa \mathbb{Z} \to K$ *with dense image;*

(c) there exists an injective homomorphism $\widehat{K} \to \mathbb{T}^\kappa$;

(d) $|\widehat{K}| \leq 2^\kappa$;

(e) $w(K) \leq 2^\kappa$*;*

(f) there exists a continuous surjective homomorphism $(M \times \widehat{\mathbb{Q}})^{2^\kappa} \to K$*.*

Proof. (a)⇒(b) Since $d(K) \leq \kappa$ and it is infinite, then there exists a dense subgroup D of G with $|D| \leq \kappa$, so in particular there exists a homomorphism $f : \bigoplus_\kappa \mathbb{Z} \to D$, which has dense image in K.

(b)⇒(a) is obvious and (b)⇔(c) is an easy application of Pontryagin duality.

(c)⇔(d) follows from the fact that $\mathbb{T}^\kappa$ is divisible with $r(\mathbb{T}^\kappa) = r_p(\mathbb{T}^\kappa) = 2^\kappa$ for every prime p.

(d)⇔(e) follows from the fact that $w(K) = |\widehat{K}|$.

(c)⇔(f) follows from the fact that as a discrete abelian group $\mathbb{T}^\kappa$ is isomorphic to $\bigoplus_{2^\kappa} \mathbb{Q} \oplus \bigoplus_{2^\kappa} \mathbb{Z}(p^\infty) = (\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})^{2^\kappa}$. □

Remark 2.7. *As a by-product of this proposition we show an easy argument for the well known equality* $d(K) = \log w(K)$ *for a compact group* K *in case* K *is* abelian *(the argument in the non-abelian case makes use of the highly non-trivial fact of the dyadicity of the compact groups). The inequality* $\log w(K) \leq d(K)$ *follows from the well known fact that* $w(K) \leq 2^{d(K)}$*. Since* $w(K) \leq 2^{\log w(K)}$ *obviously holds by the definition of* $\log$*, the equivalence of (a) and (e) from above proposition, applied to* $\kappa = \log w(K)$ *gives the desired inequality* $d(K) \leq \log w(K)$*.*

The equivalent conditions of Proposition 2.6 appear to be weaker than those of Theorem B (see also Lemma 4.2). On the other hand, these conditions become equivalent to those of Theorem B assuming the infinite compact abelian group K to be in $\mathcal{F}$; indeed, the point is that $K \in \mathcal{F}$ is equivalent to $d(K) \leq r(K)$ by Corollary 1 in the Introduction.

3. Compact abelian groups with dense free subgroups of finite rank

In the next lemma we give a computation of the value of the invariant ρ_p of a compact abelian group K in terms of the p-rank of the discrete dual group $\widehat{K}$.

Lemma 3.1. *For a prime* p *and a compact abelian group* K*, we have that*

$$\rho_p(K) = \begin{cases} r_p(\widehat{K}) & \text{if } \rho_p(K) \text{ is finite,} \\ 2^{r_p(\widehat{K})} & \text{if } \rho_p(K) \text{ is infinite.} \end{cases}$$

Proof. Let $G = \widehat{K}$. Then $K/pK \cong \widehat{G[p]}$. If K/pK is finite then $K/pK \cong G[p]$ and so $r_p(K/pK) = r_p(G[p])$. Assume now that K/pK is infinite; therefore $G[p]$ is infinite as well. So $G[p] \cong \bigoplus_{r_p(G[p])} \mathbb{Z}(p)$, consequently $K/pK \cong \mathbb{Z}(p)^{r_p(G[p])}$ and hence $r_p(K/pK) = 2^{r_p(G[p])}$. □

Since pK contains the connected component $c(K)$, which is divisible, K/pK is a quotient of $K/c(K)$ and it is worth to compare their p-ranks. Note that in general $\rho_p(K) = r_p(K/pK)$ does not coincide with $r_p(K/c(K))$ for a compact abelian group K; indeed, take for example $K = \mathbb{J}_p$. On the other hand, one can easily prove the following properties of the invariant ρ_p for compact abelian groups.

LEMMA 3.2. *Let p be a prime and K be a compact abelian group. Then:*

(a) $\rho_p(K) \geq \rho_p(K_1)$ if K_1 is a quotient of K;

(b) $\rho_p(K) = \rho_p(K/c(K))$;

(c) $\rho_p(K^n) = n\rho_p(K)$.

The next lemma proves in particular the equivalence of conditions (b), (c) and (d) in Theorem A of the Introduction.

LEMMA 3.3. *Let K be an infinite compact abelian group and $n \in \mathbb{N}_+$. Then the following conditions are equivalent:*

(a) $w(K) \leq \mathfrak{c}$ and $K/c(K)$ is a quotient of M^n;

(b) $w(K) \leq \mathfrak{c}$ and $\rho(K) \leq n$ (i.e., $\rho_p(K) \leq n$ for every prime p);

(c) K is a quotient of $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times M^n$.

Proof. (a)⇒(b) By Lemma 3.2 we have $\rho_p(K) = \rho_p(K/c(K)) \leq \rho_p(M^n) = n\rho_p(M) = n$ for every prime p.

(b)⇒(c) Let $G = \widehat{K}$ be the discrete dual of K and denote by $D(G)$ the divisible hull of G. Since $|G| = w(K) \leq \mathfrak{c}$ and $\widehat{c(K)} \cong G/t(G)$, in particular $r(D(G)) = r(G) = r(G/t(G)) \leq \mathfrak{c}$. On the other hand, for every prime p, the p-rank of $D(G)$ is $r_p(D(G)) = r_p(G) = r(G[p]) \leq n$ by Lemma 3.1. Then $D(G)$ is contained in $\bigoplus_{\mathfrak{c}} \mathbb{Q} \times (\mathbb{Q}/\mathbb{Z})^n$, hence by Pontryagin duality we have the condition in (c).

(c)⇒(a) Since the weight is monotone under taking quotients, we have $w(K) \leq \mathfrak{c}$. By hypothesis we have that K is a quotient of $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times (\widehat{\mathbb{Q}/\mathbb{Z}})^n$, so by Pontryagin duality G admits an injective homomorphism in $\bigoplus_{\mathfrak{c}} \mathbb{Q} \oplus (\mathbb{Q}/\mathbb{Z})^n$ and in particular $t(G)$ is contained in the subgroup $(\mathbb{Q}/\mathbb{Z})^n$. Applying again Pontryagin duality we conclude that $K/c(K) \cong \widehat{t(G)}$ is a quotient of $(\widehat{\mathbb{Q}/\mathbb{Z}})^n \cong M^n$. □

The following result on monothetic groups is known, we give it here as a consequence of the previous lemma noting that an infinite quotient of a monothetic group is monothetic as well.

PROPOSITION 3.4. *Let K be an infinite compact abelian group. If $w(K) \leq \mathfrak{c}$ and $\rho_p(K) \leq 1$ for every prime p, then K is monothetic.*

Proof. By Lemma 3.3 the group K is a quotient of $\widehat{\mathbb{Q}}^{\mathfrak{c}} \times M$, which is monothetic. □

We are now in position to prove the next characterization of compact abelian groups admitting a dense free abelian subgroup of finite rank, i.e., with finite topological free rank. Along with Lemma 3.3, this concludes the proof of Theorem A of the Introduction.

THEOREM 3.5. *Let K be an infinite compact abelian group and $n \in \mathbb{N}_+$. Then the following conditions are equivalent:*

(a) $K \in \mathcal{F}$ and $r_t(K) \leq n$;

(b) $w(K) \leq \mathfrak{c}$ and $\rho(K) \leq n$ (i.e., $\rho_p(K) \leq n$ for every prime p);

Proof. (a)⇒(b) By hypothesis in particular $d(K) \leq \omega$, so $w(K) \leq \mathfrak{c}$ as $d(K) = \log w(K)$ (see Remark 2.7).

Let p be a prime. Let $q : K \to K/pK$ be the canonical projection and let F be a dense free abelian subgroup of K of finite rank $r(F) \leq n$. Since $pK \cap F \supseteq pF$ and $\ker q = pK$, we have that $q(F) \cong F/\ker q$ is finite, being a quotient of the finite group F/pF. Now the density of F in K implies the density of the finite subgroup $q(F)$ in K/pK, which has exponent p. Therefore $K/pK = q(F)$ has at most n many generators, in other words $r_p(K/pK) \leq n$. This proves $\rho_p(K) \leq n$.

(b)⇒(a) By Lemma 3.3 we know that $K/c(K)$ is a quotient of $M^n \cong \prod_p \mathbb{J}_p^n$. In particular this implies that $K/c(K)$ is a product of at most n monothetic subgroups. Indeed, for a prime p, a quotient of $\mathbb{J}_p^n$ is always of the form $C_{p,1} \times \ldots \times C_{p,n}$ where $C_{p,i}$ is either $\mathbb{J}_p$ or a cyclic p-group for every $i = 1, \ldots, n$, or 0. Therefore, one can write $K/c(K) = M_1 \times \ldots \times M_n$, letting $M_i = \prod_p C_{p,i}$ for every $i = 1, \ldots, n$; observe that M_i is monothetic for every $i = 1, \ldots, n$.

Let now $q : K \to K/c(K)$ be the canonical projection and $K_1 = q^{-1}(M_1)$. Then $c(K_1) = c(K)$ and $K_1/c(K) \cong M_1$ is monothetic, so K_1 is monothetic as well by Proposition 3.4. Since $K/K_1 \cong M_2 \times \ldots \times M_n$ is topologically $(n-1)$-generated, Proposition 2.4 implies that K contains a dense free abelian subgroup of rank $\leq n$. □

One can easily see that if K is a totally disconnected compact abelian group with a dense finitely generated subgroup, then $K = M_1 \times \ldots \times M_n$, where M_i are compact monothetic groups (see the proof of the implication (b)⇒(a) in Theorem 3.5). We do not know whether this factorization in direct product of compact monothetic groups remains true without the additional restraint of total disconnectedness.

4. Compact abelian groups with infinite topological free rank

We recall the following known result in terms of the divisible weight.

THEOREM 4.1. [14, Theorem 2.6] *Let κ be a cardinal. A discrete abelian group G admits a dense embedding into $\mathbb{T}^\kappa$ if and only if $|G| \le 2^\kappa$ and $\log\kappa \le w_d(G)$.*

Recall that the bimorphisms (i.e., monomorphisms that are also epimorphisms) in the category $\mathcal{L}$ of LCA groups are precisely the continuous injective homomorphisms with dense image. Therefore, applying the Pontryagin duality functor $\widehat{\ }\colon \mathcal{L} \to \mathcal{L}$, we deduce that for a cardinal κ the following conditions are equivalent:

(a) there exists a bimorphism $\bigoplus_\kappa \mathbb{Z} \to K$;

(b) there exists a bimorphism $\widehat{K} \to \mathbb{T}^\kappa$.

In equivalent terms:

LEMMA 4.2. *Let K be a compact abelian group and let κ be a cardinal. Then the following conditions are equivalent:*

(a) K admits a dense free abelian subgroup of rank κ (in particular, $K \in \mathcal{F}$);

(b) $\widehat{K}$ admits a dense embedding in $\mathbb{T}^\kappa$.

The following easy relation between the divisible weight of a compact abelian group and that of its discrete Pontryagin dual group was already observed in [9].

LEMMA 4.3. *Let K be a compact abelian group. Then $w_d(K) = w_d(\widehat{K})$.*

Consequently, K is w-divisible if and only if $\widehat{K}$ is w-divisible.

Proof. Let $G = \widehat{K}$. Since $nG \cong \widehat{nK}$ for every $n \in \mathbb{N}_+$, one has $|nG| = w(nK)$, hence the conclusion follows. □

We recall now a fundamental relation given in [9] between the divisible weight and the rank of a compact abelian group. It is worth to note that the rank is a purely algebraic invariant, while the divisible weight is a topological one.

THEOREM 4.4. [9, Corollary 3.9] *Let K be a compact abelian group. Then $r(K) = 2^{w_d(K)}$.*

Applying these observations we can give now the proof of Theorem B. Note that in the proof of the implication (c)⇒(b) we apply both Theorem 4.4 and Theorem 4.1.

Proof of Theorem B. We have to prove that if K is an infinite compact abelian group and κ is an infinite cardinal, then the following conditions are equivalent:

(a) $K \in \mathcal{F}$ and $r_t(K) \leq \kappa$;

(b) $\widehat{K}$ admits a dense embedding in $\mathbb{T}^\lambda$ for some $\lambda \leq \kappa$;

(c) $d(K) \leq r(K)$ and $d(K) \leq \kappa$.

The equivalence (a)$\Leftrightarrow$(b) is contained in Lemma 4.2 and (a)$\Rightarrow$(c) is clear.

(c)$\Rightarrow$(b) Let $G = \widehat{K}$. Put $\lambda = d(K)$. Since $d(K) = \log w(K)$ and $w(K) = |G|$, we have that $\lambda = \log |G|$, and so $2^\lambda \geq |G|$. On the other hand,

$$\log \lambda = \log \log |G| = \log d(K) \leq \log r(K) \leq w_d(K),$$

where the last inequality follows from Theorem 4.4. So $\log \lambda \leq w_d(G)$ by Lemma 4.3, and Theorem 4.1 guarantees that G admits a dense embedding into the power $\mathbb{T}^\lambda$. $\square$

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