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A boundary value problem on the half-line for superlinear differential equations with changing sign weight[1]

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Dedicated to the 60th birthday of Professor Fabio Zanolin

Abstract. *The existence of positive solutions x for a superlinear differential equation with p-Laplacian is here studied, satisfying the boundary conditions $x(0) = x(\infty) = 0$. Under the assumption that the weight changes its sign from nonpositive to nonnegative, necessary and sufficient conditions for the existence are derived by combining Kneser-type properties for solutions of an associated boundary value problem on a compact set, a-priori bounds for solutions of suitable boundary value problems on noncompact intervals, and continuity arguments.*

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1. Introduction

In this paper we study the existence of solutions for the second order nonlinear differential equation with p-Laplacian

$$\big(r(t)\Phi(x')\big)' = q(t)f(x), \tag{1}$$

satisfying the boundary conditions

$$x(0) = 0, \quad \lim_{t\to\infty} x(t) = 0, \quad x(t) > 0 \quad \text{for } t > 0. \tag{2}$$

We will assume the following conditions:

H1. $\Phi(u) = |u|^p \operatorname{sgn} u$, for $u \in \mathbb{R}$ and $p > 0$;

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H2. f is a continuous function on $\mathbb{R}$ such that $uf(u) > 0$ for $u \neq 0$, and

$$\text{(a)} \quad \lim_{u\to 0^+} \frac{f(u)}{\Phi(u)} = 0, \qquad \text{(b)} \quad \lim_{u\to\infty} \frac{f(u)}{\Phi(u)} = \infty; \tag{3}$$

H3. r, q are continuous functions for $t \geq 0$, $r(t) > 0$ for $t \geq 0$, and q satisfies the sign condition

$$q(t) \leq 0,\ q(t) \not\equiv 0, \text{ for } t \in [0,1],$$
$$q(t) \geq 0 \text{ for } t > 1,\ q(t) \not\equiv 0 \text{ for large } t.$$

Boundary values problems (BVPs) associated to (1) on infinite intervals have been considered in many papers. For instance, in [14, 18, 20] some asymptotic problems for second-order equations with the Sturm-Liouville operator, possibly singular, are studied and BVPs, concerning equations with p-Laplacian, are considered, e.g., in [9, 11, 17]. For other contributions we refer to the monograph [1] and references therein.

As usual, by a solution of (1), we mean a continuously differentiable function x such that $r(t)\Phi(x')$ has a continuous derivative satisfying (1). For any solution x of (1), denote its quasiderivative as

$$x^{[1]}(t) = r(t)\Phi(x').$$

Let

$$R(t) = \int_0^t r^{-\frac{1}{p}}(s)\, ds.$$

The limit $\lim_{t\to\infty} R(t)$ will be denoted by $R(\infty)$; both the cases $R(\infty) < \infty$ and $R(\infty) = \infty$ will be considered. If $R(\infty) < \infty$, we put

$$\rho(t) = \int_t^\infty r^{-\frac{1}{p}}(s)\, ds.$$

The sign condition on q is motivated by the following. When q has constant sign on the whole half-line, and $q \not\equiv 0$, we can distinguish three cases: i_1) $q(t) \geq 0$ for $t \geq 0$, i_2) $q(t) \leq 0$ for $t \geq 0$ and $R(\infty) = \infty$, i_3) $q(t) \leq 0$ for $t \geq 0$ and $R(\infty) < \infty$. In cases i_1) or i_2), the problem (1)-(2) is not solvable. To see this, if i_1) holds, consider the function $G(t) = r(t)\Phi(x')x$, where x is a solution of (1)-(2). Since $G'(t) = q(t)f(x)x + r(t)|x'|^{p+1}$, then G is nondecreasing, and, as $G(0) = 0$, we obtain $G(t) \geq 0$ for $t > 0$. Thus, the positivity of x yields the existence of a point $t_0 > 0$ such that $G(t_0) > 0$. Since G is nondecreasing, x' is eventually positive, which contradicts the asymptotic condition in (2). In case i_2), for any solution x of (1)-(2) the quasiderivative $x^{[1]}$ is nonincreasing. If $\lim_{t\to\infty} x^{[1]}(t) = k \geq 0$, we immediately get a contradiction with the boundary conditions (2), since x should be eventually nondecreasing. Therefore $\lim_{t\to\infty} x^{[1]}(t) = -k < 0$, which implies $x^{[1]}(t) < -k/2$ for large t. Integrating the inequality $x'(t) < -r(t)^{-1/p}(k/2)^{1/p}$ on $[T, t]$, with T sufficiently large, we get

$$x(t) - x(T) < -\left(\frac{k}{2}\right)^{\frac{1}{p}} \int_T^t r^{-\frac{1}{p}}(s)ds,$$

which contradicts as $t \to \infty$ the positivity of x.

Finally, if the case i_3) holds, the change of variable

$$\tau(t) = R(t)$$

transforms (1) into

$$\frac{d}{d\tau}\big(\Phi(\dot{x})\big) = q(t(\tau))f(x(t(\tau))),$$

where $\dot{} = d/d\tau$, and $t(\tau)$ is the inverse function of $\tau(t)$. Since τ is an increasing bounded function, the problem (1)-(2) is transformed into a boundary value problem, possibly singular, on a bounded interval, and a very wide literature is devoted to this kinds of problems.

Therefore, the most interesting case for the solvability of (1)-(2) is that the function q changes its sign at least once.

Let

$$J =: \lim_{T\to\infty} \int_1^T \left(r^{-1}(t)\int_t^T q(s)\,ds\right)^{1/p} dt.$$

The main result of this paper is the following.

THEOREM 1.1. *Assume either $R(\infty) = \infty$ and $J = \infty$, or $R(\infty) < \infty$. Then the BVP (1)-(2) has a solution. Further, in the remaining case $J < \infty$ and $R(\infty) = \infty$, the BVP (1)-(2) has no solution.*

The tools used for proving Theorem 1.1 are a combination of a shooting method in a compact interval, following some ideas by Gaudenzi, Habets and Zanolin [12], a study of some topological properties of positive solutions of (1) in the half-line $[1, \infty)$, and some arguments in the phase space.

More in detail, we will consider two auxiliary BVPs, the first one on the compact interval $[0, 1]$, where q is nonpositive, and the second one on the half-line $[1, \infty)$, where q is nonnegative. The existence of solutions for (1), emanating from zero, positive in the interval $(0, 1)$, and satisfying additional assumptions at $t = 1$, is considered in the first problem, namely

$$\begin{cases} \big(r(t)\Phi(x')\big)' = q(t)f(x), \quad t \in [0, 1], \\ x(0) = 0, \quad x(t) > 0 \text{ for } t \in (0, 1), \\ \gamma x(1) + \delta x'(1) = 0, \end{cases} \tag{4}$$

where $\gamma + \delta > 0$, $\delta\gamma = 0$. The boundary conditions in (4) are a particular case of the well known Sturm-Liouville conditions. A wide literature has been devoted to the existence and the multiplicity of solutions of second order linear and nonlinear equations with Sturm-Liouville boundary conditions, see for instance [2, 15, 16] and the references therein. On the half-line $[1, \infty)$, we analyze the existence of positive decreasing solutions for (1), starting from a given positive value, and approaching zero as $t \to \infty$, namely the BVP

$$\begin{cases} \bigl(r(t)\Phi(x')\bigr)' = q(t)f(x), \quad t \in [1, \infty) \\ x(1) = x_0, \lim\limits_{t\to\infty} x(t) = 0, \ x(t) > 0, \ x'(t) < 0. \end{cases} \tag{5}$$

The existence of a solution of (1)-(2) is obtained, roughly speaking, as the intersection of two connected sets in the space $\mathbb{R}^2$, the first set representing the final values of the solutions (x, x') of (4), and the other set representing the initial values of solutions for (5).

Our method is based on a Kneser type property, concerning solutions emanating from a continuum set of initial data; moreover, principal solutions of suitable associated half-linear equations play a crucial role for obtaining suitable upper and lower bounds.

The paper is organized as follows. In Section 2 we recall the notion of principal solutions in the half-linear case and some properties which will be used in the following. In Section 3 the BVPs (4) and (5) are solved and some additional properties of solutions are proved. The proof of Theorem 1.1 is given in Section 4. Finally, some comments and suggestions for future researches complete the paper.

2. Preliminary results

As claimed, a key role will be played by the so-called *principal solutions* of some half-linear equations associated to (1).

The notion of principal solution, introduced by Leighton and Morse for second-order linear nonoscillatory differential equations, see, e.g., [13, Ch. 11], has been extended to the half-linear equation

$$\bigl(r(t)\Phi(x')\bigr)' = q(t)\Phi(x) \qquad (t \geq 1) \tag{6}$$

in [10] (see also [19, Ch. 4.15]) by using the Riccati equation approach, and reads as follows.

Definition 2.1. *A nontrivial solution z of (6) is said to be* principal solution of *(6) if for every nontrivial solution x of (6), such that $x \neq \lambda z$, $\lambda \in \mathbb{R}$, it holds*

$$\frac{z'(t)}{z(t)} < \frac{x'(t)}{x(t)} \qquad \text{as } t \to \infty. \tag{7}$$

Observe that, in view of the sign assumptions on q, the equation (6) is nonoscillatory. The set of principal solutions of (6) is nonempty ([10, 19]) and for any $\mu \neq 0$ there exists a unique principal solution z such that $z(1) = \mu$, i.e. principal solutions are determined up to a constant factor.

The characteristic properties of principal solutions for (6), when q is positive for $t \geq 1$, are investigated in [4]. In particular, it is shown that, roughly speaking, principal solutions of (6) are the smallest solutions in a neighborhood of infinity. Here we summarize further properties which will be useful in the sequel. Observe that these properties continue to hold also when $q(t) \geq 0$ for $t > 1$, $q(t) \not\equiv 0$ for large t.

Proposition 2.2 ([4, Theorem 3.1, Corollary 1]). *Assume either $R(\infty) = \infty$ and $J = \infty$ or $R(\infty) < \infty$. Then any principal solution z of (6) satisfies $z(t)z'(t) < 0$ on $[1, \infty)$ and $\lim_{t\to\infty} z(t) = 0$.*

A comparison between principal solutions of a suitable half-linear equation, and the solutions of (5) is needed for proving our main result, and is given in the following. The argument is similar to the one given in [3, Theorem 5].

Lemma 2.3. *Let $c > 0$ be a fixed constant, and assume that $M > 0$ (depending on c) exists, such that*

$$f(u) \leq Mu^p \text{ on } [0, c]. \tag{8}$$

Further, assume either $R(\infty) = \infty$ and $J = \infty$, or $R(\infty) < \infty$. Let z_γ be the principal solution of the half-linear equation

$$\big(r(t)\Phi(z')\big)' = Mq(t)\Phi(z)$$

with $z_\gamma(1) = \gamma$, $0 < \gamma \leq c$. Then for any solution x of (5) with $x_0 = c$ we have

$$x(t) \geq z_\gamma(t), \qquad t \geq 1, \tag{9}$$

$$x'(1) \geq \frac{c}{\gamma}\, z'_\gamma(1). \tag{10}$$

Moreover, if $R(\infty) < \infty$, then

$$x(t) \leq \frac{c}{\rho(1)}\rho(t). \tag{11}$$

Proof. Set $g(t) = x(t) - z_\gamma(t)$. Since $g(1) \geq 0$, and, in view of Proposition 2.3, it holds $\lim_{t\to\infty} g(t) = 0$, for proving (9) it is sufficient to show that g does not have negative minima. By contradiction, let $T > 1$ be a point of negative minimum for g. Hence $g(T) < 0$, $g'(T) = 0$. Moreover, there exists $t_0 > T$

such that $g'(t_0) > 0$ and $g(t) < 0$ on $[T, t_0]$. Thus

$$r(t_0)\left(\Phi(x'(t_0)) - \Phi(z'_\gamma(t_0))\right) = \int_T^{t_0} q(s)\left(f(x(s)) - M\Phi(z_\gamma(s))\right) ds$$
$$\le M \int_T^{t_0} q(s)\left(\Phi(x(s)) - \Phi(z_\gamma(s))\right) ds.$$

Since $g(t) < 0$ on $[T, t_0]$, we obtain $\Phi(x'(t_0)) - \Phi(z'_\gamma(t_0)) \le 0$, which contradicts $g'(t_0) > 0$.

Now let us show that (10) holds. Consider $g_c(t) = x(t) - z_c(t)$. Using the same argument as above, since $g_c(1) = 0$, we obtain $x'(1) \ge z'_c(1)$. Since principal solutions of a half-linear equation are uniquely determined up to a constant factor, and being z_c and z_γ two principal solutions of the same half-linear equation, we have for any $t \ge 1$

$$z_c(t) = \frac{c}{\gamma} z_\gamma(t),$$

from which (10) follows.

Finally, considering the function

$$h(t) = x(t) - \frac{c}{\rho(1)}\rho(t),$$

the inequality (11) follows by observing that $h(1) = 0 = \lim_{t\to\infty} h(t)$ and observing that the function $c\rho(t)/\rho(1)$ is the principal solution of $(r(t)\Phi(z'))' = 0$, $z(1) = c$. $\square$

We close this section with a result which describes a general asymptotic property of solutions for (1), depending on the behavior of the nonlinear term f in a neighborhood of zero.

Lemma 2.4. *Assume that f satisfies*

$$\limsup_{u\to 0^+} \frac{f(u)}{\Phi(u)} < \infty. \tag{12}$$

Then any nontrivial solution x of (1), defined on $[1,\infty)$, satisfies

$$\sup_{t\in[\tau,\infty)} |x(t)| > 0 \quad \text{for any } \tau \ge 1,$$

that is, x is not eventually zero.

Proof. The assertion follows, from instance, from [19, Theorem 1.2 and Remark 1.1] with minor changes. For sake of completeness, we give here another simple alternative proof. By contradiction, let $x(t) = 0$ for $t \ge T > 1$. Since the function $G(t) = r(t)\Phi(x'(t))x(t)$ is not decreasing and $G(T) = 0$, we have $x(t)x'(t) \le 0$ on $[1, T]$. Without loss of generality, suppose $x(1) = x_0 > 0$. In view of (12), there exists $M > 0$ such that

$$f(u) \le Mu^p \text{ on } [0, x_0]. \tag{13}$$

By integration of (1), taking into account (13) and that x is positive nonincreasing on $[1, T)$, we get

$$\begin{aligned} x(t) &= \int_t^T \left(\frac{1}{r(s)} \int_s^T q(\sigma) f(x(\sigma))\, d\sigma \right)^{\frac{1}{p}} ds \\ &\le M^{\frac{1}{p}} x(t) \int_t^T \left(\frac{1}{r(s)} \int_s^T q(\sigma)\, d\sigma \right)^{\frac{1}{p}} ds, \end{aligned}$$

that is

$$1 - M^{\frac{1}{p}} \int_t^T \left(\frac{1}{r(s)} \int_s^T q(\sigma)\, d\sigma \right)^{\frac{1}{p}} ds \le 0$$

for all $t \in [1, T)$, which is a contradiction as $t \to T$. □

Remark 2.5. *The assumption (12) plays a crucial role in Lemma 2.4. Indeed, if the estimation (13) does not hold, then (1) can have solutions x such that $x(t) \equiv 0$ for large t, the so-called singular solutions, see, e.g., [6].*

3. Some Auxiliary Boundary Value Problems

In this section we study the existence of positive solutions for the problems (4) and (5).

The existence of solutions for (4) follows from a classical result by Wang [22], which makes use of the Krasnoselskii fixed point theorem on cone compressions or expansions. Here, by means of a change of variable, we show how it is possible to apply that result, overcoming the problems due to the lack of concavity of the positive solutions of (1), due to the presence of the coefficient r.

Theorem 3.1. *If f satisfies (3), then the BVP (4) has at least one positive solution.*

Proof. Let

$$\tau(t) = \frac{R(t)}{R(1)}.$$

Since r is a positive continuous function on $[0, 1]$, it follows that τ is a positive C^1-function, with $\tau' > 0$ on the whole interval, and $\tau(0) = 0$, $\tau(1) = 1$. It therefore defines a change of the independent variable $\tau = \tau(t)$. Consider the function $y(\tau) = x(t(\tau))$, where $t = t(\tau)$ is the inverse function of τ. Simple calculations show that x is a solution of (4) if and only if y is a solution of the problem

$$\begin{cases} \dfrac{d}{d\tau}\big(\Phi(\dot{y})\big) = \hat{q}(\tau) f(y), \quad \tau \in [0,1], \\ y(0) = 0, \quad y(\tau) > 0 \text{ for } \tau \in (0,1), \\ \gamma y(1) + \hat{\delta} y'(1) = 0, \end{cases} \tag{14}$$

where $\dot{} = d/d\tau$, $\hat{q}(\tau) = (R(1))^{p+1} \left(r(t(\tau))\right)^{1/p} q(t(\tau))$, and $\hat{\delta} = \delta \left(r(1)\right)^{-1/p} (R(1))^{-1}$. Clearly, $\hat{q}(\tau) \le 0$, $\hat{q}(\tau) \not\equiv 0$ in $[0,1]$, and $\gamma + \hat{\delta} > 0$, $\gamma\hat{\delta} = 0$.

Problem (14) is a particular case of the BVPs studied in [22]. The assumption

$$0 < \int_0^{1/2} \left(\int_s^{1/2} q(t)\, dt \right)^{\frac{1}{p}} ds + \int_{1/2}^{1} \left(\int_{1/2}^{s} q(t)\, dt \right)^{\frac{1}{p}} ds < \infty,$$

which plays a key role in [22], is satisfied in our setting, since here $\hat{q}$ is continuous in $[0,1]$, and at least an interval $(\tau_1, \tau_2) \subseteq (0,1)$ exists, such that $\hat{q}(\tau) < 0$ for $\tau \in (\tau_1, \tau_2)$. Therefore Theorem 3 in [22] can be applied to (14), leading to the existence of at least a solution $\bar{y}$. Then $\bar{x}(t) = \bar{y}(\tau(t))$ is a solution of (4). □

Now, we study the properties of the solutions of the BVP on the half-line (5). The solvability of (5) is proved in the subsequent theorem, which easily follows from a well-known result of Chanturia.

THEOREM 3.2. *Assume (3)-(a). Then (5) is solvable for any $x_0 > 0$ if either $R(\infty) = \infty$ and $J = \infty$, or $R(\infty) < \infty$.*

Proof. Using [7, Theorem 1], we obtain the existence of a solution x of (1) on $[1, \infty)$ such that

$$x(1) = x_0, \ x(t) \ge 0, \ x'(t) \le 0, \tag{15}$$

for any $x_0 > 0$. The positivity of x follows from Lemma 2.4. Let us show that $\lim_{t\to\infty} x(t) = 0$. We consider separately the case $R(\infty) = \infty$ and $R(\infty) < \infty$.

Case I). Assume $R(\infty) = \infty, J = \infty$. Since $x^{[1]}$ is nondecreasing and $x^{[1]}(t) \le 0$, the limit $\lim_{t\to\infty} x^{[1]}(t)$ is finite. If $\lim_{t\to\infty} x^{[1]}(t) = x^{[1]}(\infty) < 0$, from $x^{[1]}(t) \le x^{[1]}(\infty)$ we obtain

$$x(t) \le x(1) + \Phi^* \left(x^{[1]}(\infty) \right) \int_1^t r^{-1/p}(s) ds,$$

where Φ^* is the inverse function of Φ. Letting $t \to \infty$, we get a contradiction with the positivity of x. Thus $\lim_{t\to\infty} x^{[1]}(t) = 0$. Now suppose $\lim_{t\to\infty} x(t) =$

$x(\infty) > 0$ and set $k = \min_{x(\infty)\le u\le x_0} f(u)$. Hence $k > 0$. Integrating (1) we have

$$x(t) \le x(1) - k^{1/p} \int_1^t \left(r^{-1}(s) \int_s^\infty q(\sigma)d\sigma \right)^{1/p} ds,$$

which gives again a contradiction as $t \to \infty$.

Case II). Assume $R(\infty) < \infty$. The assertion follows reasoning as in the proof of [9, Theorem 1.1], with minor changes.

Finally, let us prove that $x'(t) < 0$ on $[1, \infty)$. Assume, by contradiction, that $\bar{t} \ge 1$ exists, such that $x'(\bar{t}) = 0$. Let $G(t) = r(t)\Phi(x')x$. Since $G'(t) = q(t)f(x) + r(t)|x'|^{p+1} \ge 0$, then G is nondecreasing, with $G(\bar{t}) = 0$. Assuming that $G(t) = 0$ for every $t \ge \bar{t}$, we immediately get a contradiction, since the positivity of r yields $x' \equiv 0$ on $[\bar{t}, \infty)$, i.e. x is eventually constant and positive. Then $t_1 > \bar{t}$ exists, such that $G(t) > 0$ for every $t > t_1$. Thus, $x'(t) > 0$ for every $t > t_1$, which is again a contradiction. □

Remark 3.3. *When $R(\infty) = \infty$, condition $J = \infty$ is necessary for the existence of solutions of the BVP (5). Indeed, if $J < \infty$, then any bounded solution x of (1) satisfies $\lim_{t\to\infty} |x(t)| = |x(\infty)| > 0$, see, e.g., [3, Th. 6] with minor changes. When $R(\infty) < \infty$ and $J < \infty$, this fact does not occur, because in this case (1) can have positive (bounded) solutions both approaching zero and a non-zero limit when t tends to infinity, as the Emden-Fowler equation*

$$(r(t)\Phi(x'))' = q(t)|x|^\beta \, sgn\, x, \qquad p < \beta,$$

illustrates, see, e.g. [5, Theorem 3].

Remark 3.4. *If (3)-(a) holds and f is increasing for $u > 0$, then (5) is uniquely solvable for any $x_0 > 0$. This property is a consequence of the fact that, in this case, two positive solutions of (1) defined for $t \ge 1$, can cross at most in one point, including $t = \infty$. We refer the reader to a classical result by Mambriani (see, e.g., [21, Cap. XII, Section 5]), in which the same property is proved for a generalized Thomas-Fermi equation.*

Finally, the following "continuity" result holds for solutions of (5).

Theorem 3.5. *Assume (3)-(a) and either $R(\infty) = \infty$ and $J = \infty$, or $R(\infty) < \infty$. Then the set*

$$S = \left\{ (x(1), x^{[1]}(1)) \right\},$$

where x is a solution of (5) for some $x_0 > 0$, contains a connected subset S_1 such that $P(S_1) = (0, \infty)$, where P is the projection $P(u, v) = u$. Moreover, if $(c_n, d_n) \in S_1$ and $\lim_n c_n = 0$, then $\lim_n d_n = 0$, and S_1 is contained in the set $\pi = \{(u, v) : u > 0, v < 0\}$.

Proof. Let $c > 0$ be fixed. In virtue of Theorem 3.2, the boundary value problem

$$\begin{cases} \big(r(t)\Phi(x')\big)' = q(t)f(x), \quad t \in [1,\infty) \\ x(1) = c - n^{-1}, \ \lim\limits_{t\to\infty} x(t) = 0, \\ x(t) > 0, \ x'(t) < 0, \end{cases} \tag{16}$$

is solvable for any positive integer n. Let $\{x_n\}$ be a solution of (16). Fixed $\gamma < c$, choose n large so that $\gamma \le c - n^{-1}$. In view of (3)-(a), the inequality (8) holds, and so, from Lemma 2.3, taking into account that x_n is nonincreasing, we obtain for $t \ge 1$

$$z_\gamma(t) \le x_n(t) \le c,$$

i.e. $\{x_n\}$ is equibounded on $C[1,\infty)$. Moreover, in view of Proposition 2.2, $z'_\gamma(1) < 0$, and again from Lemma 2.3 we have

$$x'_n(1) \ge \frac{c - n^{-1}}{\gamma}\, z'_\gamma(1) \ge \frac{c}{\gamma}\, z'_\gamma(1),$$

and so $0 \ge x_n^{[1]}(1) \ge c z_\gamma^{[1]}(1)/\gamma$, i.e. $\left\{x_n^{[1]}(1)\right\}$ is bounded on $\mathbb{R}$. Integrating (1), we get

$$x_n^{[1]}(t) = x_n^{[1]}(1) + \int_1^t q(s)f(x_n(s))ds. \tag{17}$$

Thus, since $\{x_n\}$ is equibounded and $\left\{x_n^{[1]}(1)\right\}$ is bounded in $\mathbb{R}$, also $\left\{x_n^{[1]}\right\}$ is equibounded on $C[1,\infty)$, i.e. $\{x_n\}$ is compact on $C[1,T]$ for every $T > 1$. Fixed $T > 1$, without loss of generality, suppose $\lim_n x_n(t) = x(t)$ for $t \in [0,T]$ and $\lim_n x_n^{[1]}(1) = d$. Thus, from (17) the sequence $\left\{x_n^{[1]}\right\}$ uniformly converges on $[1,T]$ and

$$\lim_n x_n^{[1]}(t) = x^{[1]}(t).$$

Hence from

$$\begin{aligned} x_n(t) &= \left(c - \frac{1}{n}\right) + \int_1^t \left(\frac{1}{a(s)}\left(x_n^{[1]}(1) + \int_1^s q(\sigma)f(x_n(\sigma))d\sigma\right)\right)^{1/p} ds = \\ &= \left(c - \frac{1}{n}\right) + \int_1^t \left(\frac{x_n^{[1]}(s)}{a(s)}\right)^{1/p} ds, \end{aligned}$$

we obtain for $t \in [1,T]$

$$x(t) = c + \int_1^t \left(\frac{x^{[1]}(s)}{a(s)}\right)^{1/p} ds,$$

that is x is solution of (1).

Now, let us prove that $\lim_{t\to\infty} x(t) = 0$. If $R(\infty) = \infty$, $J = \infty$, since x is bounded, this property can be proved using the same argument to that given in the proof of Theorem 3.2, case I). If $R(\infty) < \infty$, being x_n a solution of (16), from Lemma 2.3 we get

$$x_n(t) \leq \frac{c - n^{-1}}{\rho(1)}\rho(t) \leq \frac{c}{\rho(1)}\rho(t).$$

Since the sequence $\{x_n\}$ uniformly converges to x on every compact interval in $[1, \infty)$ and it is dominated by a zero-convergent function, again we have $\lim_{t\to\infty} x(t) = 0$. Clearly $x'(t) \leq 0$. The argument for proving that $x'(t) < 0$ is analogous to the one in the final part of the proof of Theorem 3.2. Thus, there exists at most a solution x of (5) such that

$$\lim_n x_n^{[1]}(1) = x^{[1]}(1).$$

This means that S contains a connected subset S_1, contained in π, and, in view of the arbitrariness of c, $P(S_1) = (0, \infty)$.

Finally, let $(c_n, d_n) \in S_1$, with $c_n \to 0$, and let x_n be the solution of (5) with initial data (c_n, d_n). Then, from Lemma 2.3, we obtain $0 > x_n'(1) = d_n \geq z_{c_n}'(1) = c_n z_1'(1)$, and letting $n \to \infty$ we get the assertion. □

Remark 3.6. *Theorem 3.5 can be view also as a "selection" theorem and extends to (5) a property of principal solutions of linear equations stated by Hartman and Wintner, see [13, Corollary 6.6]. Indeed, from the proof of Theorem 3.5, if $\{c_n\}$ is a real positive sequence converging to $c > 0$, the sequence $\{x_n\}$ of solutions of (5) starting at $x_0 = c_n$ admits a subsequence which uniformly converges, on every closed interval of $[1, \infty)$, to a solution of (5) starting at $x_0 = c$. Observe that the selection is unnecessary if (5) has a unique solution, see Remark 3.4.*

4. Proof of Theorem 1.

The following generalization of the well known Kneser's theorem, see for instance [8, Section 1.3], plays a key role in the proof of Theorem 1.1.

Proposition 4.1 ([8]). *Consider the system*

$$z' = F(t, z), \quad (t, z) \in [a, b] \times \mathbb{R}^n$$

where F is continuous, and let K_0 be a continuum (i.e., compact and connected) subset of $\{(t, z) : t = a\}$ and $\mathcal{Z}(K_0)$ the family of all the solutions emanating from K_0. If any solution $z \in \mathcal{Z}(K_0)$ is defined on the interval $[a, b]$, then the cross-section $\mathcal{Z}(b; K_0) = \{z(b) : z \in \mathcal{Z}(K_0)\}$ is a continuum in $\mathbb{R}^n$.

Proof of Theorem 1.1. Consider the Cauchy problem

$$\begin{cases} \big(r(t)\Phi(x')\big)' = q(t)f(x_+), & t \in [0,1] \\ x(0) = 0, \ x'(0) = A > 0 \end{cases}, \tag{18}$$

where $x_+ = \max\{x, 0\}$. Clearly, every nonnegative solution of (18) is also solution of (1) in $[0,1]$. Vice versa, if x is a solution of (1), with $x(0) = 0$, and $x > 0$ in $(0,1)$, then x is also solution of (18). Indeed, since $r(t)\Phi(x')$ is nonincreasing, assuming by contradiction $x'(0) = 0$, it follows that $x'(t) \le 0$ for $t \in [0,1]$, which, together with the condition $x(0) = 0$, contradicts the positivity of x in $(0,1)$.

Now, we show that all solutions of (18) are persistent, i.e., are defined for all $t \in [0,1]$. To see this, first of all notice that all the solutions of (18) have an upper bound, since from $x^{[1]}(t) \le x^{[1]}(0)$ we get

$$x(t) \le A \, r^{\frac{1}{p}}(0) \; R(t).$$

Moreover, if x is a solution of (18) such that $x(t) > 0$ in $(0, t_1)$ and $x(t_1) = 0$, $0 < t_1 \le 1$, then $x'(t_1) < 0$. Indeed, integrating the equation in (18) over $[0, t_1]$ we obtain

$$0 = x(t_1) - x(0) = \int_0^{t_1} \left(\frac{1}{r(s)}\right)^{\frac{1}{p}} \Phi^* \left(x^{[1]}(0) + \int_0^s q(r)f(x(r))\,dr\right) ds.$$

Since $x^{[1]}$ is nonincreasing, $x^{[1]}(0) > 0$, and $q(t) \le 0$ in $[0,1]$, the quasiderivative

$$x^{[1]}(t) = x^{[1]}(0) + \int_0^t q(r)f(x(r))\,dr$$

has to assume a negative value for $s = t_1$, and so $x'(t_1) < 0$. Hence, if $t_1 < 1$, x is negative in a right neighborhood (t_1, t_2) of t_1, and satisfies $(x^{[1]}(t))' = 0$ in (t_1, t_2), i.e., $x^{[1]}(t) = x^{[1]}(t_1) < 0$, which yields $x(t) < 0$ on $(t_1, 1]$. By integration we obtain for $t > t_1$:

$$x(t) = x^{[1]}(t_1) \int_{t_1}^t \left(\frac{1}{r(s)}\right)^{\frac{1}{p}} ds,$$

that is, x is also bounded from below.

Notice that, by the above argument, we get the following property, that will be used several times in the remaining part of the proof.

(P) *If x is a solution of (18), with $x(t_0) \le 0$, $0 < t_0 \le 1$, then $x'(t_0) < 0$.*

By Theorem 3.1, equation (1) have solutions y and w, which are positive in $(0,1)$ and satisfy $y(0) = 0$, $y'(1) = 0$ and $w(0) = w(1) = 0$, respectively. Let

$A_1 = y'(0)$, $A_2 = w'(0)$. Then, from the first part of the proof, $A_1, A_2 > 0$ and y, w are also solutions of (18) for $A = A_1$ and $A = A_2$, respectively. Assume, without restriction, $A_2 < A_1$ and let

$$T = \{(x(1), x'(1)) : x \text{ sol. of (18) s.t. } x'(0) = A \in [A_2, A_1]\}$$

Since all the solutions of (18) are defined on $[0, 1]$, Proposition 4.1 assures that T is a continuum in $\mathbb{R}^2$, containing the points $(y(1), 0)$ and $(0, w'(1))$. Notice that, from property (P), it results $y(1) > 0$ and $w'(1) < 0$. Further, T does not contain any point $(0, c)$ with $c \geq 0$. It follows that a continuum $T_1 \subseteq T$ exists, such that T_1 is contained in $\overline{\pi} = \{(u, v) : u \geq 0, v \leq 0\}$, $(0, 0) \notin T_1$, and there exist $R, M > 0$ such that $(R, 0) \in T_1$, $(0, -M) \in T_1$, see Figure 1.

Now consider equation (1) for $t \geq 1$. By Theorem 3.2, for every $x_0 > 0$, there exists a positive solution x of (1) which is defined on $[1, \infty)$, satisfies $x(1) = x_0$, is decreasing and tends to zero as $t \to \infty$. Further, from Theorem 3.5, the set S of the initial values of solutions of (5), contains a connected set $S_1 \subseteq \pi = \{(u, v) : u > 0, v < 0\}$, whose projection on the first component is the half-line $(0, \infty)$. Therefore it holds

$$T_1 \cap S_1 \neq \emptyset.$$

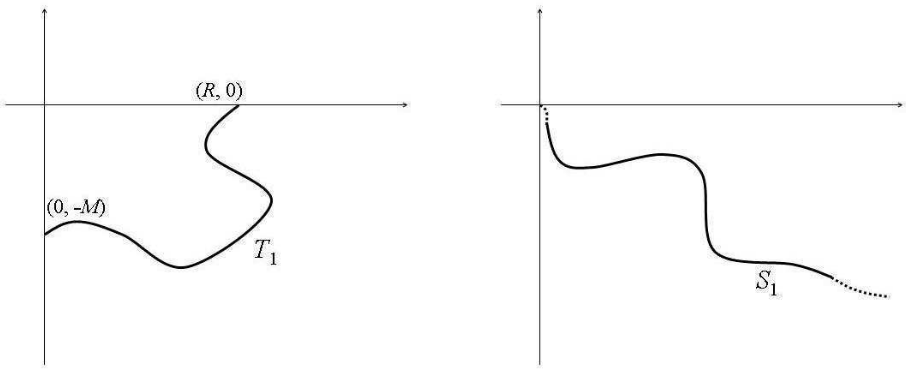

Figure 1: The connected sets T_1 and S_1.

Let us show that to any point $(c_0, c_1) \in T_1 \cap S_1$ corresponds a solution of the BVP (1)-(2). Let $(c_0, c_1) \in T_1 \cap S_1$. Then $c_0 > 0, c_1 < 0$. Moreover, there exists a solution u of (18), for a suitable $A > 0$, such that $u(1) = c_0 > 0$ and

$u'(1) = c_1 < 0$. The condition $u(1) > 0$ implies that u is positive on $(0,1]$, because every solution of (18), which is negative at some point $T \in (0,1)$, is negative also for $t \in [T,1]$, see property (P). Therefore u is solution of (1) in $[0,1]$, with $u(0) = 0$, $u(t) > 0$ for $t \in (0,1]$. Further, as $(c_0, c_1) \in S_1$, a solution v of (5) exists, such that $v(1) = c_0$, $v'(1) = c_1$. Then v is a positive solution of (1) on $[1,\infty)$, and satisfies $\lim_{t\to\infty} v(t) = 0$. Hence the function

$$x(t) = \begin{cases} u(t), & t \in [0,1], \\ v(t), & t > 1. \end{cases}$$

is clearly a solution of the BVP (1)-(2).

Finally, if $J < \infty$ and $R(\infty) = \infty$, the BVP (1)-(2) has no solution, since, in this case, any bounded solution of (1) has a nonzero limit at infinity, see Remark 3.3. □

5. Concluding remarks

1). If the function f satisfies

$$\lim_{u\to 0^+} \frac{f(u)}{\Phi(u)} = l > 0, \quad \lim_{u\to\infty} \frac{f(u)}{\Phi(u)} = L > 0,$$

i.e. (1) is, roughly speaking, close to an half-linear equation near zero and infinity, then all our results concerning the solvability of the second BVP (5) continue to hold. Nevertheless, the solvability of (4) is a more "delicate" problem, and the existence of positive solutions with suitable boundary conditions has been studied by different approaches. A wide literature has been devoted to this topic and we refer to [2, 15, 16] for more details.

If f is sublinear, that is

$$\lim_{u\to 0^+} \frac{f(u)}{\Phi(u)} = \infty, \quad \lim_{u\to\infty} \frac{f(u)}{\Phi(u)} = 0,$$

then the opposite situation occurs. The BVP (4) on $[0,1]$ is now solvable, see [22], but the BVP on the half-line (5) can be not solvable, because in this case the solutions x of (1), obtained via the Chanturia result [7, Theorem 1] and satisfying on $[1,\infty)$ the boundary conditions (15), can be zero for any large t, see [6]. Moreover, under additional assumptions on r and q, the BVP (5) is solvable ([5, Theorem 2]), but not for any small $|x_0|$ and this fact makes inapplicable the crossing method used in the proof of Theorem 1.1.

2). Using an approach similar to that in the proof of Theorem 1.1, we can treat also the existence of solutions x of (1) satisfying any of the following boundary conditions

$$x(0) = 0,\ \lim_{t\to\infty} x(t) = \ell_x, 0 < \ell_x < \infty,\ x(t) > 0 \text{ for } t > 0,$$
$$x(0) = 0,\ \lim_{t\to\infty} x^{[1]}(t) = 0,\ x(t) > 0 \text{ for } t > 0,$$
$$x(0) = 0,\ \lim_{t\to\infty} x^{[1]}(t) = -d_x, 0 < d_x < \infty,\ x(t) > 0 \text{ for } t > 0.$$

In these cases, their solvability on the half-line $[1, \infty)$ requires a different approach, because for obtaining suitable upper and lower bounds, some nontrivial asymptotic properties of nonprincipal solutions of suitable associated half-linear equations are needed. This will be done in a forthcoming paper.

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