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A Stabilizing Distributed Receding Horizon Control Scheme for Cooperative Linear and Nonlinear Systems.

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Introduction

By definition, cooperation is the act of working in compliance with others to obtain a mutual benefit.

Consider an engineering system composed by several subunits, which can be either dynamically coupled or decoupled. Suppose we want to control such system: if all the subunits are able to interact and cooperate in order to pursue and attain the control objective of the overall system, we may gain great advantages. Let's look at some typical examples.

A large data network may be composed by several routing units that are responsible for the data traffic organization: depending on the terminals requests and the channels communication capacity, an optimal data exchange rate should be chosen for the different links. The overall objective of the routers is to allow a maximal overall flow of information, avoiding bottleneck links, data bouncing and overexploited (or unused) paths.

Similarly we can consider a flow control system, composed by an interconnection of pipes serving different users (e.g. different tanks of a large plant, hydraulic basins ...) and having different capacities. The distribution system could be composed by different units responsible for the flow to be assigned to each pipe: this should respect the users requests, the pipes capacities and the sources availability.

A final and very popular example is a group of robots that have to explore an environment of interest and reach a certain formation, which is their ultimate goal: each of them needs information on important features of the environment, such as obstacles, and on the other robots positions.

For the three cases we shortly presented, we can immediately draw some important structural analogies. The set of routers, the flow assignment units and the robots are Decision Makers (DMs) or agents (we will utilize these words as synonyms), nodes where the information regarding the overall system is utilized and the control strategy is actuated. The data rates at the different nodes, the flow rates and the environmental features are the information based on which the control action has to be chosen. It is reasonably intuitive, that the more the DMs are, the faster and better they may know the system configuration and fulfill their goal. But this is true only if they implement the right cooperation strategy.

A centralized supervisor constantly gathering the information collected by every DM and accordingly distributing instructions would perfectly do the job, finding the optimal solution to the overall problem. But often, such as in the network routing case, a model for the DMs does not exist, or it's too expensive to obtain. Moreover a centralized controller would generate an unsustainable fragility in the system, which would critically fail if this unique supervisor was put down. At the other extreme, we could allow each DM to communicate with all the others (all-to-all connection), and provide it with the knowledge of all the models of the other DMs and the data they gathered. In this other case, it could be practically difficult or impossible to gather all the models; the computational power needed by each unit could become enormous; there would be further issues regarding the problem solution numerically attained at each node. Furthermore, if N is the number of agents, this would introduce a number of links of the order of N^2.

In between those two critical solutions, there's the vast land of possible engineering architectures helping the agents to efficiently exchange data and strategic information and reach the desired system state.

To distribute means to divide a certain quantity in different parts, that can be assigned each to a certain individual. A distributed system is therefore a set of coherently autonomous units, that can achieve the overall solution of an engineering problem by individually solving its parts [49]. Going back to our example: can we divide the DMs tasks in many autonomous subtasks to be assigned to different members the agents set? Doing so, can we maintain the overall cooperation? This way we would not need a central calculator, nor an all-to-all communication connectivity.

The design and synthesis of a distributed cooperative system needs two crucial technological aspects to coexist: the first is the availability of suitable hardware, in the form of embedded distributed computational units or DMs, sensor and actuator networks; the second is the existence of efficient software, allowing to break down the computational burden among the DMs or route the control instructions to the suitable actuation unit. In the recent years great attention has been given to both hardware and software dedicated to this category of systems, which are becoming more and more common. It is enough to think of more examples such as site control and surveillance, that require distributed monitoring of large areas and coordinated actions; environmental applications, where several sensors can be displaced to collect and fuse different types of information over a large territory, to make weather predictions, or screen pollution levels; automated military missions, where small exploration or foraging units need to cooperate to perform their task with precision and efficacy.

Our interest is devoted to the theoretical side of cooperative systems design: we will study a class of control algorithms that can be distributed, guarantee cooperation and stability properties to a general multi agent system.

A special case of cooperation is the dynamic coupling arising in large-scale systems: classical references are among others the work by D. D. Siljak [68], the seminal paper [76], the adaptive techniques for decentralized systems developed in [28]. The structural properties of decentralized controlled large-scale systems are considered in the work of R. D'Andrea and co-workers (see, for instance, [11, 40]), which can be used in several applications, such as flight formations and distributed sensors. Studies on topology independent control have also been explored [9].

This work is though dedicated to the study of cooperation as coupling arising among independent systems through their control actions, where the classical (but not exclusive) case study is represented by Uninhabited Autonomous/Air Vehicles (see, among others, [6, 66, 77]). We are moreover interested in a certain category of distributed cooperative control algorithms: a new version of distributed Receding Horizon (RH) control will be proposed and analyzed, in particular from the point of view of its stabilizing properties. RH control is a framework that arises from Model Predictive Control (MPC): [4, 56, 25, 23, 24]; [2]. Its basic principle relies on the knowledge of a model for the system, the computation of an optimal control sequence minimizing a user defined cost function over a finite future time window and the application of the first element of such regulation sequence; this procedure has to be repeated at each time. When referring to UAV types of applications, such MPC cost function can take into account several issues, such as collision avoidance and formation constraints, and may reward the tracking of a certain path. The RH control techniques are nowadays mature, allow to handle linear and nonlinear systems, constraints on the state and the control actions; the theory applies to both continuous and discrete time cases. The choice of exploring its applicability to cooperative systems is therefore well justified [47, 15, 45, 52, 51].

In [43], [44] and [45], the authors consider a two-degrees of freedom team of UAVs assigned to visit a certain number of points. The team of UAVs is controlled in a *centralized* RH framework and by exploiting global potential functions, the authors prove certain stationarity properties of the generated trajectories in the case of two agents searching for multiple targets. The main drawback is here the centralized approach, that limits the scalability of the analysis; the recent publication [46] the authors develop a distributed cooperative controller which does not require a vehicle to maintain perfect information on the entire team and whose computational cost is scalable and lower than the centralized case; its effectiveness is illustrated through simulation-based comparisons with the centralized case.

One of the first works proposing a distributed MPC framework for large-scale linear systems was proposed in [29, 5], where a one-step delayed communication was assumed. Another early study on the subject is [16]. The PhD work by W. Dunbar focuses instead specifically on the coordination of a large group of cooperating nonlinear vehicles ([15] and related works), where a centralized RH problem is decomposed and solved locally. Each vehicle knows the model of its neighbors (other vehicles with which the communication is allowed) and receives their predicted control action (in practice, their strategy), together with their current state: based on such information, it can find its optimal strategy and broadcast it. Convergence to the formation equilibrium point is assured by guaranteeing frequent updates and a bounded error between the assumed and the predicted trajectories, which every agent computes for itself and it's neighbors in the model predictive control process. Such assumption can be seen as a *constraint on the strategy*, and can be beneficial in terms of final performance, though detrimental as far as computational burden and velocity. This approach is very close to the one we will propose in the present work, and we will often draw comparisons between the two.

A RH control scheme has also been proposed in [32], [33], where the centralized problem is decomposed into local computations and the feasibility issues are thoroughly examined; stability is obtained in [32] by exploiting a hierarchical decomposition of the team in suitable subgraphs with assigned priorities. Recently, a distributed RH control architecture for spatially invariant systems was proposed in [57], while [58] is dedicated to spatially distributed systems with arbitrary connection topology.

The case of autonomous aerial vehicles was considered by J. P. How and co-authors, in works such as [1, 38], where Mixed-Integer Linear Programming (MILP) is exploited to solve a RH control problem for trajectory planning.

A theoretical framework for distributed MPC with guaranteed nominal stability and performance properties is offered in [75]: distributed regulators work iteratively and cooperatively towards achieving a team control objective, guaranteeing feasibility and stabilization at each iteration step.

It is worth noting that RH control is not the only choice we have in cooperation problems. Some important theoretical results on the stability of swarms are [48, 22]; string stability is a concept proposed and studied in [71] and [72], where adaptive control algorithms are applied. Coordination of a large group of cooperating nonlinear vehicles is considered in [67]: a centralized cost function is decomposed and locally minimized; stability is assured by exploiting graph theory assumptions and potential functions. Robotic networks are considered in [54, 55], coverage control algorithms for mobile sensor networks are presented in [10]. Decentralized algorithms for motion coordination of a group of autonomous vehicles, aimed at minimizing the expected waiting time to service stochastically-generated targets are offered in [21].

We need at this point to mention that towards a broad analysis of the structural properties of cooperative systems, an Input-to-State Stability (ISS) analysis has recently been proposed by several authors. In [73], [74] the concept of Leader to Formation Stability is developed. A discussion of some of the issues arising in the study of non-holonomic vehicles using ISS can be found in [8]. ISS tools have been successfully applied to the specific case of networked systems with serial communication, where Nesic and Teel propose a new unified framework for modelling and analyzing networked control systems [60], [61]; previous work on parameterized interconnected system is given by [39]. Finally, in [47, 51] connections are made between ISS and MPC.

In this work, we consider a team of discrete time, dynamically uncoupled DMs: each has a twofold *local control objective*: (i) a control objective that depends only on the agent's own state variables and (ii) a control objective that depends on the information exchanged with neighboring/cooperating agents, which aims to achieve a desired global cooperation behavior for the team of agents. Such objectives are embedded into a finite horizon, *local*, user-defined cost function, which is to be minimized; such cost function clearly depends on the state of the minimizing agent and on the states of the neighbors, gathered with a certain delay. The obtained control sequence is applied in a RH fashion, and allows the presence of constraints. Given this general structure, and with respect to the presented state of the literature, the main contributions of this work can be summarized as follows:

- We propose a stabilizing and totally distributed RH framework for decoupled cooperative systems, with no constraints on the strategy, considering both cases of a team of linear and nonlinear agents. In the linear unconstrained case, the control law becomes particularly simple, being explicit and linear.
- This distributed cooperative control problem explicitly and thoroughly handles state and control constraints, where the formal proofs are carried out utilizing ISS concepts.
- Each agent does not need to know the models of its neighbors nor the overall team structure.
- The presence of delays is taken into account, that can be different among the different agents, even though they are assumed to be known and deterministic.

The fundamental features of this control architecture will be described in Chapter 1 with particular attention to the general formulation of a RH control problem and the expression of the cost function to be minimized.

In the subsequent chapters, two case studies will be presented: the first dealing with a team of linear agents, in Chapter 2; the second regarding the study of a team of agents with nonlinear dynamics, in Chapters 4 and 5. In all these subcases, the same series of analysis steps will be made:

- The specific RH control problem is formulated; a certain cost function is introduced, which embeds individual agent and team goals. Assumptions on the dynamics, control requirements and constraints are introduced, where necessary.

- Each agent is first viewed as an entity separated from the team: the delayed information gathered from the neighbors is treated as an external reference to be tracked in the prediction fame of the MPC problem. Sufficient conditions are stated to guarantee stability of the single agent. Where constraints are present, local stability is achieved.
- The agents are gathered together and analyzed as a single dynamical system in a closed loop, where the delayed information is viewed as a delay block into the loop. Asymptotic stability is proved utilizing small gain results.

When the agents dynamics are linear, the RH control problem without state or input constraints can be easily solved in analytical form, by utilizing the so-called Fake Algebraic Riccati Equation [3]. The information from the neighboring agents can be seen as an external input to be tracked: therefore the control law is in this case linear with respect to both the state of the agent which is the subject of the optimization, and the states of the neighbors. The total dynamics of the team can be thus modeled through an overall linear dynamical system, whose stability can be guaranteed by a suitable choice of the local cost function. In particular, as will be shown in Section 2.4.2, the matrices weighting the information incoming from the neighbors have to respect certain bounds in order to ensure asymptotic stability of the group. This property will be proved to hold utilizing the discrete time small gain theorem.

A more complicated case is that of a team made up of nonlinear dynamical agents. Despite the general control framework remains practically unvaried, we chose to carry on the stability analysis through the concept of Input-to-State Stability (ISS): Chapter 3 is dedicated to an introduction to the main definitions regarding this field. Some basic results on regional ISS will be also presented in Section 3.2, which are useful for the analysis on constrained cooperative control in Chapter 5. In fact, the case of nonlinear agents will be split in two sub-cases, namely one where the RH control problem is solved without constraints (Chapter 4) and another were we introduce constraints both on the state and on the input (Chapter 5). The latter problem is by far the most interesting and complete, and represents a very powerful result also from an implementation point of view. While the stability proofs for both the unconstrained and the constrained case are solved by utilizing the small gain theorem, the necessary preliminary tools to show stability at the level of the single agent differ in several points. In particular, the constrained case needs a slightly different problem formulation (that follows [51]), and the utilization of regional ISS stability concepts.

Application examples will be offered in Chapters 2, 4 and 5: the algorithms will be tested using simulated teams of UAV's in Matlab.

Finally, an overall summary with remarks and hints for future work will be discussed in the thesis conclusions.

Notation

This section provides the main notations, definitions and abbreviations that will be extensively utilized throughout this work. Further definitions regarding Input to State Stability, that require more formal effort, are provided in Chapter 3.

$\mathbb{R}$, $\mathbb{R}_{\geq 0}$	Set of real numbers and non-negative reals respectively.
$\mathbb{Z}, \mathbb{Z}_{\geq 0}$	Set of integer numbers and non-negative integers respectively.
$P > 0$, $P \geq 0$	Positive definite and positive semidefinite matrix respectively, $P \in \mathbb{R}^{n \times n}$.
ϕ_k	kth value of the discrete time sequence $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$.
$\{\phi_k, ..., \phi_{k+N}\}$	Time window from time k to time $k+N$ of the discrete time sequence ϕ.
$\phi_{k-\Delta}$	kth value of the discrete time sequence ϕ, subject to a time delay Δ.
$\lvert\cdot\rvert$, $\lvert\cdot\rvert^2$	Euclidean norm and squared Euclidean norm respectively.
$\lvert\cdot\rvert_P$	Weighted Euclidean norm, where P is a positive definite matrix.
$\lvert\cdot\rvert_\infty$	Infinity norm.
$\lVert\phi\rVert$	$\sup_{k\geq 0}\{\lvert\phi_k\rvert\}$, for the discrete time sequence ϕ.

$\|\phi\|_T$	$\sup_{0\leq k\leq T}\{\|\phi_k\|\}$, for the discrete time sequence ϕ.
$\mathcal{M}_\Phi$	Set of discrete–time sequences ϕ taking values in some subset $\Phi \subset \mathbb{R}^m$.
id	Identity function from $\mathbb{R}$ to $\mathbb{R}$.
$\gamma_1 \circ \gamma_2$	Composition of two functions γ_1 and γ_2 from $\mathbb{R}$ to $\mathbb{R}$.
∂A	Boundary of a closed set $A \subset \mathbb{R}^n$.
$A\backslash B$	Difference between two given sets $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$, with $B \subseteq A$, i.e. $A\backslash B = \{x : x \in A, x \notin B\}$.
$d(\zeta, A)$	Point-to-set distance from $\zeta \in \mathbb{R}^n$ to A, $A \subseteq \mathbb{R}^n$, i.e. $d(\zeta, A) = \inf\{\|\eta - \zeta\|, \eta \in A\}$.
$\mathcal{A}$	Set of cooperative agents operating in the same environment.
$\mathcal{A}^i$	Agent belonging to the set $\mathcal{A}$, characterized by the index $i \in \{1, ..., M\}$, where M is the total number of agents.
$\mathcal{G}^i$	Set of agents (neighbors) exchanging information with agent $\mathcal{A}^i$.
G^i	Set of indexes characterizing the agents belonging to $\mathcal{G}^i$.
x_t^i	Discrete time state vector of the ith agent $\mathcal{A}^i$, at time t.
u_t^i	Discrete time control vector of the ith agent $\mathcal{A}^i$, at time t.
$\bar{w}_t^i$	Discrete time information vector received at time t by the ith agent $\mathcal{A}^i$, coming from the neighboring agents.
Δ_{ji}	Communication delay between agent $\mathcal{A}^j$ and agent $\mathcal{A}^i$.
$\bar{x}^{ij}$	Delayed discrete time state vector communicated from agent $\mathcal{A}^j$ to agent $\mathcal{A}^i$ at time t (the time index is dropped for simplicity). Alternative notation: $x_{t-\Delta_{ji}}^{ij}$.

γ_{RH}, γ_{RH^o}	Receding Horizon control law, and optimal RH control law respectively.
γ_{FH}, γ_{FH^o}	Finite Horizon control law and optimal FH control law respectively.
ISS	Input to State Stable (also Input to State Stability).
AS, $0 - AS$	Asymptotically Stable and zero-Asymptotically Stable.
GAS, $0 - GAS$	Globally Asymptotically Stable and zero-Globally Asymptotically Stable.
$LISS$	Locally Input to State Stable.
AG	Asymptotic Gain property.
L_f	Lipschitz constant (global or local as specified in the text) for function f from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Chapter 1

Formulation of the RH control problem

This chapter will introduce the general problem of cooperative and distributed RH control, giving more details on the structure of the system type that will be considered. As anticipated in the Introduction, we are interested in systems that can overall be seen as a set of separated subunits that cooperate through their control action in order to fulfill an *a priori* determined control objective. Therefore the generic system we will look at is composed by a set of dynamically decoupled subsystems, each computing a *local* control law trying to accomplish both global-level and agent-level requirements. Such architecture can be referred to as *team*. Throughout this work we will consider only discrete time settings, where the agents are discrete time linear or nonlinear systems.

The control laws are obtained in a RH fashion: at each DM, the basic algorithm proceeds by minimizing a suitably defined FH cost function with respect to a finite sequence of controllers, the first element of such sequence is actuated and the algorithm is then repeated at each successive discrete time. Section 1.2 offers an overview of the RH strategy: in the subsequent chapters such basic control structure will be added several specifications that depend on the linearity/nonlinearity of the DMs, on the control objective, requirements and constraints.

1.1 Structure of the team of agents

A *distributed dynamic system* will be considered, made of a set of M agents denoted as $\mathcal{A} \triangleq \{\mathcal{A}^i \; : \; i = 1, ..., M\}$. Each DM $\mathcal{A}^i$ is described by the generic time-invariant state equation:

$$x^i_{t+1} = g^i(x^i_t, u^i_t), \; t \geq 0, \; x^i_0 = \bar{x}^i \tag{1.1}$$

where, for each $i = 1, ..., M$, $x^i_t \in \mathbb{R}^{n^i}$ denotes the local state vector and $u^i_t \in \mathbb{R}^{m^i}$ denotes the local control vector of agent $\mathcal{A}^i$ at time t. We assume that $g(\cdot, \cdot)$ for now is just a generic function describing the system dynamics, either linear or nonlinear, and that $g^i(0,0) = 0$, $i = 1, ..., M$. We also suppose that the dynamics of all M agents evolve on the same discrete-time space (that is, the DMs are synchronized).

In open–loop mode, each agent is dynamically decoupled from the remaining DMs and the dynamics of the other agents are not assumed to be known. The coupling between agents arises due to the fact that they operate in the same environment and due to the "cooperative" objective imposed on each DM by a cost function that will be defined later on.

To achieve some degree of cooperation, each agent $\mathcal{A}^i$ exchanges an information vector w^i_t with a given set of neighboring agents $\mathcal{G}^i \triangleq \{\mathcal{A}^j \; : \; j \in G^i\}$, where G^i denotes the set of indexes identifying the DMs belonging to the set $\mathcal{G}^i$. More precisely, the information exchange pattern is defined as follows. Let us consider a generic time–instant t; then for each $i = 1, ..., M$, the agent $\mathcal{A}^i$ receives from each neighboring cooperating DM $\mathcal{A}^j \in \mathcal{G}^i$ the value of its local state vector with a delay of Δ_{ij} time steps, that is, agent $\mathcal{A}^i$ receives the vector $x^j_{t-\Delta_{ij}}$ from agent $\mathcal{A}^j \in \mathcal{G}^i$. To gain some more insight into the information exchange pattern, refer to Fig. 1.1, where a simple three–agent example is shown pictorially. In this specific example, each DM receives information from all remaining DMs. At each time–instant t, we group all inputs to agent $\mathcal{A}^i$ into a vector $\bar{w}^i_t$ defined as $\bar{w}^i_t \triangleq \mathrm{col}\,(x^j_{t-\Delta_{ij}}, \; j \in G^i)$. The size of vector $\bar{w}^i$ is equal to $n^i_w = \sum_{j \in G^i} n^j$ and clearly

$$\bar{w}^i_t \in W^i \tag{1.2}$$

where W^i denotes the cartesian product of all sets $X^j, j \in G^i$, that is,

$$W^i \triangleq \prod_{j \in G^i} X^j .$$

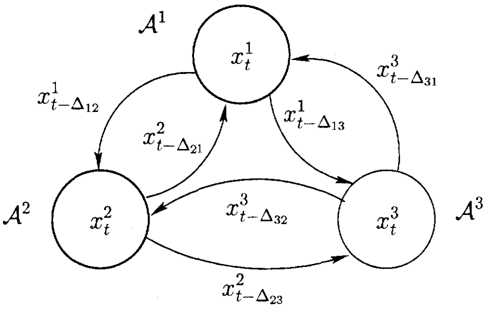

Figure 1.1: Three agents exchanging delayed state information.

It is worth noting that the above setting allows the investigation of quite a large class of distributed cooperating dynamic systems like teams of mobile vehicles, cooperating robotic arms, routing nodes in communications and/or transportation networks where agents cooperate to minimize the total traffic delay, networks of reservoirs in water-distribution networks.

For instance, let us introduce an example that will be then utilized in Chapter 5 to illustrate the effectiveness of our framework. Such example is drawn from [31] and is given by a set of M hovercrafts that we want to coordinate to autonomously achieve a certain spatial configuration. Each toy vehicle can be described by the following continuous time equations:

$$\begin{aligned} m\ddot{x}^i &= -\mu_1\dot{x}^i + (u_R^i + u_L^i)\cos(\theta^i), \\ m\ddot{y}^i &= -\mu_1\dot{y}^i + (u_R^i + u_L^i)\sin(\theta^i), \\ J\ddot{\theta}^i &= -\mu_2\dot{\theta}^i + (u_R^i - u_L^i)r_v, \end{aligned} \tag{1.3}$$

which can be trivially discretized; the corresponding discrete time variables will be indicated as x_t^i, y_t^i and u_t^i. A graphical sketch of the vehicle is given in Figure 1.2

Suppose that we are able to embed a controller in each of the hovercrafts, and we can endow them with wireless sensors that allow them to sense or directly communicate their state, according to a certain topology of the communication links. So the ith vehicle has access to the state of a subset of

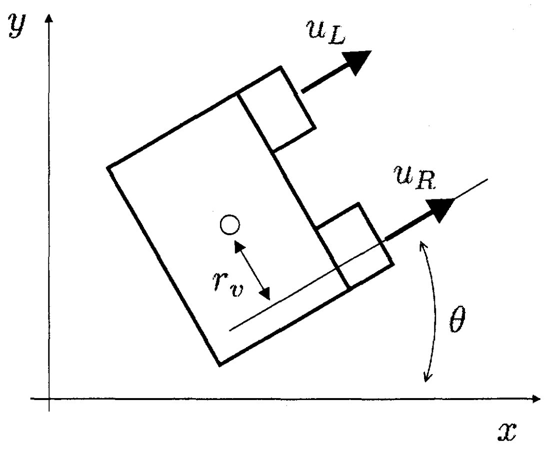

Figure 1.2: Scheme of the hovercraft example

other vehicles, which are to be considered its neighbors. Now, this ith DM must seek to reach a certain position in its environment, that corresponds to an overall configuration that was assigned to the group by *apriori* design: this is exemplified in Figure 1.3. This has to be accomplished without an overall optimization, and with the partial information about the state of the neighbors only. This can be done by minimizing a *local* cost function (that will be introduced formally with (1.4)) that is able to account for the *cooperative* final task including the information - previously denoted as $\bar{w}_t^i$ - incoming from the neighbors: depending on such information, a controller that minimizes such cost will be found at each time instant, as we will detail in the next section.

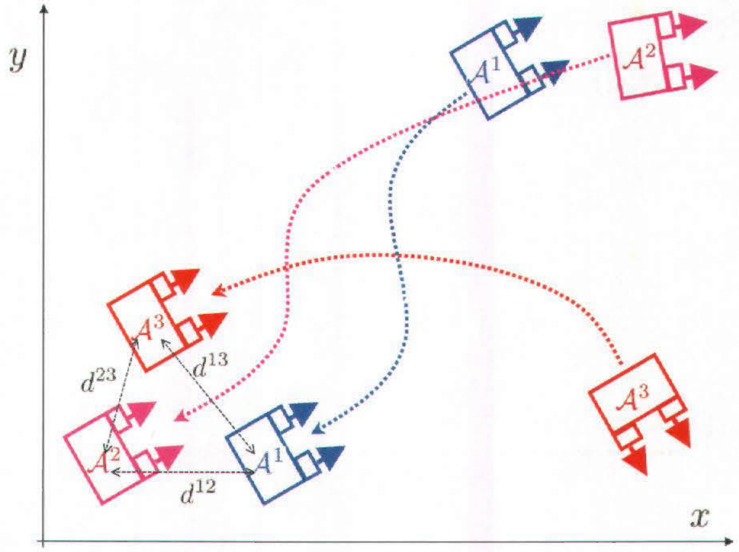

Figure 1.3: Example of configuration of a group of hovercrafts: collision avoidance is clearly required.

1.2 The RH framework

Throughout this work we assume that each agent locally computes a Receding Horizon control law, to achieve both cooperative and local objectives. As anticipated in the previous section, such controller is the one that minimizes a cost function whose parameters are chosen depending on the designer objectives: such objectives are indeed cooperative, but must also be aimed at guaranteeing stability.

For each member of the team, for given values of the state vector x_t^i and of the information vector $\bar{w}_t^i$ at time–instant t, we introduce the following finite–horizon (FH) cost function (in general, nonquadratic):

$$J^i_{FH} = \sum_{k=t}^{t+N^i-1} \left[h^i(x^i_k, u^i_k, d^{h^i}_k) + q^i(x^i_k, \bar{w}^i_t, d^{q^i}_k) \right] + h^i_f(x^i_{t+N^i}, d^{h^i}_{t+N^i}) \,. \quad (1.4)$$

Such function depends therefore on the specific initial state x^i_t and on the information vector, but also on other factors. Indeed the prediction window length N^i, the cost–to–go elements h^i, q^i, the final cost h^i_f and the parameters $d^{h^i}_k$, $d^{q^i}_k$ are all deliberately chosen according to the cooperation and the own agent's objectives. Some of them also play a crucial role in guaranteeing stability, as will be thoroughly detailed in the following chapters. The control variables u^i_k, $k = t, \dots, t + N^i - 1$ are the argument of a suitable optimization problem, where one seeks to minimize (1.4).

Recalling the hovercraft example, the states are positions and velocities of the vehicles, while the inputs are the lateral fans' thrust to be applied to attain the desired trajectory.

We want to immediately point out that (1.4) is composed of two terms: a partial cost term given by $\sum_{k=t}^{t+N^i-1} h^i(x^i_k, u^i_k, d^{h^i}_k) + h^i_f(x^i_{t+N^i}, d^{h^i}_{t+N^i})$, and a "cooperation" cost term given by $\sum_{k=t}^{t+N^i-1} q^i(x^i_k, \bar{w}^i_k, d^{q^i}_k)$. The quantities $d^{h^i}_k$, $d^{q^i}_k$, $i = 1, \dots, M$ denote some given vectors of appropriate dimensions and in general $d^{h^i}_k$ are useful to specify a desired reference value for some or all components of the local state variables, whereas the vectors $d^{q^i}_k$ can be used to parametrize the cooperation between the DMs. For example, if the agents are our hovercrafts, then vectors $d^{h^i}_k, d^{q^i}_k$ could be defined so as to specify given trajectories to be followed by each agent or also set the desired "formation structures" for the DMs.

At every time–instant t, vector $\bar{w}^i_t$ can be considered as known external input in the cost function: its value can remain constant within the prediction horizon (information is exchanged prior to the evaluation and minimization of (1.4)) or be paired with a *forward-forgetting factor* that will decrease the "importance" of the information vector in the FH cost function along the time window. This latter possibility will be exploited in Chapter 4, where its mathematical implementation and advantages will be described and discussed.

Going back to our toy example, the information vector $\bar{w}_t^i$ is nothing but a vector where the states of the neighbors (that can be known with a certain delay) are stacked and utilized within (1.4) to determine the best control sequence.

The local control law is designed according to a RH strategy: in the literature several different problem formulations can be found depending on the particular setting (see also the well–known survey paper [56]). At this level, our aim is to introduce the reader to the general formulation of a RH control problem in a cooperative context: such formulation will be furtherly detailed in Chapters 2 and 4 according to the characteristics of the considered team of agents and to the design objectives.

Problem 1.2.1 (FH Optimal Control Problem) *At every time instant* $t \geq 0$, *for every agent* $\mathcal{A}^i$, $i = 1, \ldots, M$ *described by (1.1), and for given values* x_t^i *and* $\bar{w}_t^i$ *of the state and the information vectors, find the optimal FH control sequence* $\{u_t^{iFH^\circ}, \ldots, u_{t+N^i-1}^{iFH^\circ}\}$ *that minimizes the chosen cost (1.4).* □

The above control problem can be extended to include constraints: we could require for instance that the states and the control sequences belong to specific predefined sets, or impose constraints on the terminal state. This case is, in general, more realistic but represents a harder minimization problem to solve. This type of formulation will be discussed in Chapter 4.

Now, the RH procedure can be described in the usual way as follows. When the controlled agent $\mathcal{A}^i$ is in the state x_t^i at stage t, the FH optimal control Problem 1.2.1 is solved, thus obtaining the sequence of optimal control vectors, $\{u_t^{iFH^\circ}, \ldots, u_{t+N^i-1}^{iFH^\circ}\}$. The first control action of this sequence becomes the control action $u_t^{iRH^\circ}$ generated by the RH local controller at time–instant t (i.e., $u_t^{iRH^\circ} \triangleq u_t^{iFH^\circ}$). This procedure is repeated stage after stage and a *feedback–feedforward* control law $\gamma_{RH^\circ}^i(x_t^i, \bar{w}_t^i)$ is obtained, as the control vector $u_t^{iFH^\circ}$ depends on the local current state x_t^i and on the vector of delayed states $\bar{w}_t^i$ communicated to the agent $\mathcal{A}^i$ by the cooperating agents $\mathcal{G}^i =$

$\{\mathcal{A}^j,\, j \in G^i\}$. The system (1.1) under the action of the *RH* optimal control law can thus be rewritten as

$$x_{t+1}^i = f^i(x_t^i, \bar{w}_t^i) \triangleq g^i(x_t^i, \gamma_{RH^\circ}^i(x_t^i, \bar{w}_t^i)),\ t \geq 0,\ x_0^i = \bar{x}^i. \tag{1.5}$$

It is worth noting that, from well-known results on RH control (see, for instance, [56] and the references cited therein), we have $\gamma_{RH^\circ}^i(0,0) = 0$ and hence $f^i(0,0) = 0$, that is, the origin is an equilibrium state for agent $\mathcal{A}^i$ when $\bar{w}_t^i = 0,\, t \geq 0$.

Chapter 2

The case of a team of discrete time linear agents

If we consider a set of agents where each member is described by linear dynamics, the application of the general RH framework introduced in Chapter 1 allows for a straightforward stability analysis. In particular, at the level of each agent we will derive a linear RH control law that allows to solve Problem 1.2.1 when the considered cost function is quadratic and no constraint is imposed. Such RH regulator has a feedback and a feedforward component: the latter is linear in the delayed states of the neighboring agents. At Section 2.3.1 we will show how each DM under the action of the derived RH regulator is stable.

The introduction of cooperation and of the feedforward control terms is equivalent to generating interconnections subject to delays within the team. Therefore the overall system can be modeled as a linear system, where the delays are taken into account, and under suitable restrictions on the interconnections size we can prove the team stability utilizing small gain theorem reasonings in Section 2.3. The bounds on the size of the interconnections will be discussed in detail in Section 2.4, reaching specifications on the overall control design. In fact these bounds will be specifically linked with the cooperation terms in the local cost functions to be minimized by each DM.

2.1 Problem formulation

The distributed dynamic system made of a set of M agents denoted as $\mathcal{A} \triangleq \{\mathcal{A}^i,\ i = 1, ..., M\}$, is now composed by DMs described by a LTI state equation:

$$x^i_{t+1} = A^i x^i_t + B^i u^i_t, \quad i = 1, ..., M \tag{2.1}$$

where, for each $i = 1, ..., M$, $x^i_t \in \mathbb{R}^{n^i}$ denotes the state vector and $u^i_t \in \mathbb{R}^{m^i}$ denotes the control vector. The characteristics of the team in terms of dynamic coupling and information exchange remain the same as in Section 1.1. The following assumption is added:

- The pair $[A^i,\ B^i]$ is stabilizable, for each $i = 1, ..., M$.

We will now specify the form of the cost function 1.4, which is to be minimized at the level of the single DM. In this chapter, we will drop the notation $\bar{w}^i(t)$ denoting a column vector composed by all the incoming state information from the DMs that cooperate with agent $\mathcal{A}^i$. For each $i = 1, ..., M$ and for a given value of the state vector x^i_t at time–instant t, we introduce the following finite–horizon (FH) cost function, where the norm considered is the euclidean norm:

$$\begin{aligned} J^i_{FH} \;=\; & \sum_{k=0}^{N^i-1} \left(|x^i_{t+k}|^2_{P^i} + |u^i_{t+k}|^2_{R^i} \right) + |x^i_{t+N^i}|^2_{P^i_{N^i}} + \\ & + \sum_{k=0}^{N^i-1} \sum_{j \in G^i} |x^i_{t+k} - Z^{ji}_{\Delta} x^j_{t-\Delta_{ji}} + d^{ij}_k|^2_{S^{ij}} + \\ & + \sum_{j \in G^i} |x^i_{t+N^i} - Z^{ji}_{\Delta} x^j_{t-\Delta_{ji}} + d^{ij}_{N^i}|^2_{S^{ij}_{N^i}} \end{aligned} \tag{2.2}$$

where $N^i, i = 1, \ldots, M$ denote the lengths of the control horizons for each DM and $d^{ij}_k \in \mathbb{R}^{n^i}$, $k = 0, \ldots, N^i - 1, i = 1, \ldots, M, j \in G^i$ are given constant vectors representing the desired "distance" between the state variables of co-operating agents. Matrices $Z^{ji}_{\Delta} \in \mathbb{R}^{n^i \times n^j}$ "select" the components of vector $x^j_{t-\Delta_{ji}}$ that should be considered in the cooperation part of the cost function. Clearly, if the dimension of the state variables are the same for all DMs, the cost (2.2) can be rewritten in the simpler form

$$\begin{aligned} J^i_{FH} &= \sum_{k=0}^{N^i-1} \left(|x^i_{t+k}|^2_{P^i} + |u^i_{t+k}|^2_{R^i}\right) + |x^i_{t+N^i}|^2_{P^i_{N^i}} + \\ &+ \sum_{k=0}^{N^i-1} \sum_{j\,\in G^i} |x^i_{t+k} - x^j_{t-\Delta_{ji}} + d^{ij}_k|^2_{S^{ij}} + \sum_{j\,\in G^i} |x^i_{t+N^i} - x^j_{t-\Delta_{ji}} + d^{ij}_{N^i}|^2_{S^{ij}_{N^i}} \end{aligned}$$

where we set $n^i = n$. The symmetric state–weighting matrices satisfy $P^i \geq 0, P^i_{N^i} \geq 0, S^{ij} \geq 0, S^{ij}_{N^i} \geq 0$ and the symmetric control weighting matrices satisfy $R^i > 0$, and the corresponding quadratic forms replace the functions h^i, q^i and h^i_f introduced at Section 1.2.

It is worth noting that:

1. The use of the quadratic cost function (2.2) is quite common in the recent literature (see, for instance [26, 14]) even though it is rather restrictive. On the other hand, as will be seen later on, this choice allows the formulation of the "linear–quadratic" framework thus making possible the off–line analytical determination of local control laws (see the next section). The use of the past values $x^j_{t-\Delta_{ji}}$ instead of some prediction $\hat{x}^j_{t+k}$ of x^j_{t+k} as is done, for instance, in [13], can be motivated in the same way: since $x^j_{t-\Delta_{ji}}$ does not depend on the index k, it will be possible again to determine analytically off line the local control laws, without assuming that each agent knows the dynamics of the neigboring ones. This significant computational simplification may clearly result in some performance degradation.

2. Several interesting control objectives in the framework of cooperative DMs can be embedded into a cost function of the form (2.2). For example, in the context of cooperating UAVs, it is well known that collision avoidance constraints can be handled through the so–called "potential fields" [35], [37], [7]. It is possible to show that this approach can be reformulated in terms of the optimization of quadratic cost functions like (2.2), for a suitable choice of the terms d^{ij} and of the matrix weights.

The local control strategy is based on the RH framework which has been described at Section 1.2. The RH procedure is repeated stage after stage and a *feedback–feedforward* control law is obtained, as the control vector $u_t^{FH^{io}}$ depends on the local current state x_t^i and on the delayed states $x_{t-\Delta_{ji}}^j$, $j \in G^i$ communicated to the agent $\mathcal{A}^i$ by the cooperating agents belonging to $\mathcal{G}^i$. We can state the following problem, which details Problem 1.2.1 for the considered case.

Problem 2.1.1 *At every time instant* $t \geq 0$ *and for every agent* $\mathcal{A}^i$, $i = 1, \ldots, M$, *find the RH optimal control law* $u_t^{RH^{io}} = \gamma_{RH}^{io}(x_t^i; x_{t-\Delta_{ji}}^j, j \in G^i)$ *where* $u_t^{RH^{io}}$ *is the first vector of the control sequence* $u_t^{FH^{io}}, \ldots, u_{t+N^i-1}^{FH^{io}}$ *that minimizes the cost* (2.2) *for the local state* x_t^i *and the delayed states* $x_{t-\Delta_{ji}}^j$, $j \in G^i$.

As previously mentioned in Section 1.2, the structure of the control objective is made of two parts: a "local" control objective aiming at the minimization of the partial cost given by the terms $\sum_{k=0}^{N^i-1}(|x_{t+k}^i|_{P^i}^2 + |u_{t+k}^i|_{R^i}^2) + |x_{t+N^i}^i|_{P_{N^i}^i}^2$ and a "cooperation" control objective aiming at the minimization of the partial cost given by the remaining terms $\sum_{k=0}^{N^i-1} \sum_{j \in G^i} |x_{t+k}^i - x_{t-\Delta_{ji}}^j + d_k^{ij}|_{S^{ij}}^2 + \sum_{j \in G^i} |x_{t+N^i}^i - x_{t-\Delta_{ji}}^j + d_{N^i}^{ij}|_{S_{N^i}^{ij}}^2$. Then, as the control law for each DM $\mathcal{A}^i$, $i = 1, \ldots, M$ takes on the form $\gamma_{RH}^{io}(x_t^i; x_{t-\Delta_{ji}}^j, j \in G^i)$, then the dynamic behaviors of the agents are coupled and depend on the values of $P^i, P_{N^i}^i, S^{ij}, S_{N^i}^{ij}, R^i$ and d_k^{ij} .

The cooperation stemming from the use of local control laws solving Problem 2.1.1 clearly introduces a dynamic coupling between the cooperating DMs, as already pointed out. This cost–function–induced coupling is well known in the literature and is particularly well suited for distributed frameworks where cooperation between decision makers is pursued. A classical example is the case where a group of decision makers have access to different information and cooperate on the accomplishment of a common goal, the organization becomes a *team* and the related optimization problems are named *team optimal control problems* (see, for instance the classical works [27, 53] and the very recent works [45], [32], [15], [26]).

2.2 The local RH control law

Problem 2.1.1 is an unconstrained minimization problem in the unknowns $u_t^i, \ldots, u_{t+N^i-1}^i$. The analytical solution can be obtained by resorting to results available in the literature (see, for instance, [42]).

For simplicity, this change of notation is now introduced for the remaining part of the chapter: $\bar{x}^{ij} \triangleq x_{t-\Delta_{ji}}^j$, $j \in G^i$. For $k = N^i - 1, N^i - 2, \ldots, 0$, we obtain the backwards difference equations:

$$q_k^i = P^i + \sum_{j \in \mathcal{G}^i} S^{ij} + A^{i^\top} [q_{k+1}^i - q_{k+1}^i B^i \times (B^{i^\top} q_{k+1}^i B^i + R^i)^{-1} B^{i^\top} q_{k+1}^i] A^i \tag{2.3}$$

$$v_{t+k}^i = [A^{i^\top} - A^{i^\top} q_{k+1}^i B^i (B^{i^\top} q_{k+1}^i B^i + R^i)^{-1} \times B^{i^\top}] v_{k+1}^i + \sum_{j \in G^i} S^{ij} (\bar{x}^{ij} - d_k^{ij}) \tag{2.4}$$

with boundary conditions

$$q_{N^i}^i = P_{N^i}^i + \sum_{j \in \mathcal{G}^i} S_{N^i}^{ij}$$

$$v_{t+N^i}^i = \sum_{j \in \mathcal{G}^i} S_{N^i}^{ij} (\bar{x}^{ij} - d_{N^i}^{ij})$$

From now on, without loss of generality and again for the sake of notational simplicity we let $S_{N^i}^{ij} = S^{ij}$ and $d_{N^i}^{ij} = d_k^{ij} = d^{ij}$, $k = 0, \ldots, N^i - 1$. It is useful to rewrite (2.4) defining the following matrix depending only on the i-th DM local parameters (and thus *computable off line*):

$$\Phi_k^i \triangleq A^{i^\top} - A^{i^\top} q_k^i B^i (B^{i^\top} q_k^i B^i + R^i)^{-1} B^{i^\top} \tag{2.5}$$

Inside the FH time window $\{t, \ldots, t + N^i\}$, the quantities $(\bar{x}^{ij} - d^{ij})$ are constant. Hence, (2.4) becomes:

$$v_{t+k}^i = [I + \Phi_{k+1}^i + \Phi_{k+2}^i \Phi_{k+1}^i + \ldots + \Phi_{N^i}^i \cdots \Phi_{k+2}^i \Phi_{k+1}^i] \sum_{j \in G^i} S^{ij} (\bar{x}^{ij} - d^{ij})$$

and letting

$$\tilde{\Phi}^i_k \triangleq [I + \Phi^i_{k+1} + \Phi^i_{k+2}\Phi^i_{k+1} \cdots + \Phi^i_{N^i} \cdots \Phi^i_{k+2}\Phi^i_{k+1}],$$

we have

$$v^i_{t+k} = \tilde{\Phi}^i_k \sum_{j \in G^i} S^{ij}(\bar{x}^{ij} - d^{ij}) \tag{2.6}$$

Therefore, for $k = 0, \ldots, N^i - 1$, matrix gains are obtained: $K^{x^i}_k = (B^{i^\top} q^i_{k+1} B^i + R^i)^{-1}(B^{i^\top} q^i_{k+1} A^i)$ and $K^{v^i}_k = (B^{i^\top} q^i_{k+1} B^i + R^i)^{-1} B^{i^\top}$. For each k, the FH control action is then given by

$$u^i_{t+k} = -K^{x^i}_k x^i_{t+k} + K^{v^i}_k v^i_{t+k+1} \tag{2.7}$$

where v^i_{t+k+1} is given by (2.6).

Finally, the RH control law solving Problem 2.1.1 is obtained from (2.7) by setting $k = 0$, that is

$$u^{RH^{io}}_t = -K^{x^i}_0 x^i_t + K^{v^i}_0 \tilde{\Phi}^i_1 \sum_{j \in G^i} S^{ij}(\bar{x}^{ij} - d^{ij}) \tag{2.8}$$

It is worth noting the particular structure of the RH control law (2.8). Specifically, the control law is made of two parts: a *feedback* part and a *feedforward* part where the latter depends on the information exchange pattern and is therefore related to the cooperation between DMs.

2.3 Stability of the team of cooperating agents

The stability analysis will be carried out in two main steps. First, we shall address the stability properties of a single agent $\mathcal{A}^i$ when controlled by the local RH control law (2.8) and without considering the coupling effects due to the information exchange with the other DMs. Subsequently, the coupling effects will be taken into account. References for this chapter are [18],[19] and [20].

2.3.1 Stability of the local system

Stability conditions for agent $\mathcal{A}^i$ can be proved in two steps (see [3]). First we have to recast the FH optimal control problem considered in Problem 2.1.1 into a linear quadratic regulation problem and consider its extension to an infinite–horizon (IH) regulation problem with an associated Algebraic Riccati Equation (ARE). Then, we will interpret the RH problem as an approximation of the IH problem and we'll be able to state stability conditions using known results involving the Fake Riccati Equation (FARE).

Let us assume that all the DMs have the same input dimension, and also the same finite optimization horizon, without loss of generality. More precisely, we assume that, for each $i = 1, \dots, M$, we have $m^i = m$, and $N^i = N$. Moreover, we'll suppose that all the DMs in the set $\mathcal{A}$ cooperate to minimize their local costs, that is $G^i = \{1, \dots, M\} \setminus \{i\}$, $i = 1, \dots, M$.

To carry out the first step, at time t and for each $i = 1, ..., M$ we introduce an *auxiliary* dynamic system described by the state equation

$$x^{i,h}_{t+k+1} = A^{i,h} x^{i,h}_{t+k} , \quad k = 0, \dots, N-1 \tag{2.9}$$

where $x^{i,h}_{t+k} \in \mathbb{R}^{n(M-1)}$ and the initial condition is given by

$$x^{i,h}_t \triangleq [(\bar{x}^{i1} - d^{i1})^\top \cdots (\bar{x}^{i,i-1} - d^{i,i-1})^\top \qquad (\bar{x}^{i,i+1} - d^{i,i+1})^\top \cdots (\bar{x}^{iM} - d^{iM})^\top]^\top .$$

Matrix $A^{i,h}$ in system (2.9) is an identity matrix of appropriate dimensions. Hence, the following augmented systems is obtained[1]:

$$x^{i,r}_{t+k+1} = A^{i,r} x^{i,r}_{t+k} + B^{i,r} u^{i,r}_{t+k} , \quad k = 0, ..., N-1 \tag{2.10}$$

where $x^{i,r}_{t+k} \triangleq \mathrm{col}\,[x^i_{t+k}, x^{i,h}_{t+k}]$, $u^{i,r}_{t+k} \triangleq u^i_{t+k}$,

$$A^{i,r} \triangleq \begin{bmatrix} A^i & \emptyset \\ \emptyset & A^{i,h} \end{bmatrix} , \quad B^{i,r} \triangleq [B^i \ \emptyset \cdots \emptyset]^\top$$

Clearly, $A^{i,r} \in \mathbb{R}^{nM \times nM}$, $B^{i,r} \in \mathbb{R}^{nM \times m}$ and $\emptyset$ will denote from now on zero–matrices of appropriate dimensions.

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[1] The original dynamics of agent $\mathcal{A}^i$ are not affected by the augmentation with the auxiliary system.

Now, to obtain the equivalent optimal regulator problem, we also need to suitably modify the weighting matrices in the original FH cost function (2.2). We first consider matrices S^{ij}: as we assumed for simplicity that all DMs have the same state dimension, the transformation can be done by defining matrix $C^{i,j} \in \mathbb{R}^{n\times n(M-1)}$ as $C^{i,j} \triangleq [\ \emptyset \ \cdots \ \emptyset \ I \ \emptyset \ \cdots \ \emptyset\ ]$, where the $n \times n$ identity matrix I is placed inside matrix $C^{i,j}$ according to the definition of vector $x_t^{i,h}$. Then, we let

$$S^{ij,r} \triangleq \begin{bmatrix} I \\ -(C^{i,j})^\top \end{bmatrix} S^{ij} \begin{bmatrix} I & -C^{i,j} \end{bmatrix}$$

where I again denotes an $n \times n$ identity matrix. Matrices $S^{ij,r}$ are thus belonging to $\mathbb{R}^{r\times r}$ and have the following structure:

$$S^{ij,r} = \begin{bmatrix} S^{ij} & \emptyset & \cdots & \emptyset & -S^{ij} & \emptyset & \cdots & \emptyset \\ \emptyset & \emptyset & \cdots & \emptyset & \emptyset & \emptyset & \cdots & \emptyset \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ -S^{ij} & \emptyset & \cdots & \emptyset & S^{ij} & \emptyset & \cdots & \emptyset \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \emptyset & \emptyset & \cdots & \emptyset & \emptyset & \emptyset & \cdots & \emptyset \end{bmatrix}$$

The block elements S^{ij} are suitably positioned according to the j-th element of vector $x_t^{i,h}$, right or down shifted of n positions. Analogously, we introduce the new matrix $P^{i,r} \in \mathbb{R}^{r\times r}$ as

$$P^{i,r} \triangleq \left[\begin{array}{c|ccc} P^i & \emptyset & \cdots & \emptyset \\ \hline \emptyset & \emptyset & \cdots & \emptyset \\ \emptyset & \emptyset & \cdots & \emptyset \end{array}\right] + \sum_{j\in G^i} S^{ij,r}$$

and a similar definition stands for $P_N^{i,r}$. Finally, we let $R^{i,r} \triangleq R^i$. The FH cost function (2.2) can be rewritten as

$$J^{i,r} = \sum_{k=0}^{N-1} \left(|x_{t+k}^{i,r}|^2_{P^{i,r}} + |u_{t+k}^{i,r}|^2_{R^{i,r}} \right) + |x_{t+N}^{i,r}|^2_{P_N^{i,r}} \tag{2.11}$$

Therefore, Problem 2.1.1 can be stated in the following equivalent form.

Problem 2.3.1 *At every time instant* $t \geq 0$ *and for every agent* $\mathcal{A}^i$, $i = 1, \dots, M$, *find the RH optimal control law* $u_t^{RH^{i,r\circ}} = \gamma_{RH}^{i,r\circ}(x_t^{i,r})$ *where* $u_t^{RH^{i,r\circ}}$ *is the first vector of the control sequence* $u_t^{FH^{i,r\circ}}, \dots, u_{t+N-1}^{FH^{i,r\circ}}$ *that minimizes the cost* (2.11) *for the local augmented state* $x_t^{i,r}$.

Let us now consider the FARE associated to Problem 2.3.1:

$$\begin{aligned} q_k^{i,r} &= A^{i,r^\top}[q_k^{i,r} - q_k^{i,r} B^{i,r}(B^{i,r^\top} q_{k+1}^{i,r} B^{i,r} + R^{i,r})^{-1} B^{i,r^\top} q_k^{i,r}] A^{i,r} + \bar{P}_k^{i,r} \\ q_N^{i,r} &= P_N^{i,r} \end{aligned} \tag{2.12}$$

for $k = N-1, N-2, \dots, 0$, where

$$\bar{P}_k^{i,r} \triangleq P^{i,r} - (q_k^{i,r} - q_{k+1}^{i,r}) \tag{2.13}$$

The asymptotic stability of the local closed loop system, when the RH control law solving Problem 2.3.1 is applied, is related to the eigenvalues of the closed loop matrix $\bar{A}_{k=0}^{i,r}$ of system (2.10), where

$$\bar{A}_k^{i,r} \triangleq A^{i,r} - B^{i,r}(B^{i,r^\top} q_{k+1}^{i,r} B^{i,r} + R^{i,r})^{-1} \times B^{i,r^\top} q_{k+1}^{i,r} A^{i,r} \tag{2.14}$$

The conditions for which the eigenvalues of matrix $\bar{A}_0^{i,r}$ are all strictly inside the unit circle can be found in the following result [3].

Theorem 2.3.1 *Consider the FARE* (2.12) *and definition* (2.13); *if the following assumptions hold:*

- *a)* $R^{i,r}$ *is positive definite;*
- *b)* $[A^{i,r}, B^{i,r}]$ *is stabilizable;*
- *c)* $[A^{i,r}, (\bar{P}_0^{i,r})^{\frac{1}{2}}]$ *is detectable;*
- *d)* $\bar{P}_0^{i,r}$ *is positive semidefinite.*

Then matrix $q_1^{i,r}$ *is stabilizable, that is, the eigenvalues of matrix* $\bar{A}_0^{i,r}$ *are all strictly within the unit circle.*

It is worth noting that assumptions made in Section 2.1 imply that hypotheses a) and b) of Theorem 2.3.1 are fulfilled by construction. As to Assumption c), the following alternative result holds true (the proof of this Corollary follows immediately from Lemma 4.1 in [3]).

Corollary 2.3.1 *Let us take into consideration definition* (2.13)*; if the following assumptions hold:*

a) $[A^{i,r}, B^{i,r}]$ *is stabilizable and* $R^{i,r}$ *is positive definite;*

b) $[A^{i,r}, (P^{i,r})^{\frac{1}{2}}]$ *is detectable;*

c) $\bar{P}_0^{i,r} \geq P^{i,r}$*;*

then the eigenvalues of matrix $\bar{A}_0^{i,r}$ *are all strictly within the unit circle.*

To sum up, the above results state that a suitable choice of the weighting matrices in the cost function (2.11) (and then the original cost function (2.2)) allows us to guarantee the local stability of the controlled DM $\mathcal{A}^i$. In particular, choosing a suitably large final weighting matrix $P_N^{i,r}$ always allows to satisfy Assumption d) in Theorem 2.3.1. It is also worth noting that to guarantee $\bar{P}_0^{i,r} \geq P^{i,r}$ for any choice of $P^{i,r}$, then a suitably large value of N has to be chosen. The reader is referred to [3] and the references cited therein for more details.

2.3.2 Stability of the team of cooperating agents

Under the application of the RH optimal control law (2.8), the closed loop dynamics of the i-th DM can be described as

$$x_{t+1}^i = A^i x_t^i - B^i K_0^{x^i} x_t^i + B^i K_0^{v^i} \tilde{\Phi}_1^i \sum_{j \in G^i} S^{ij} (\bar{x}^{ij} - d^{ij}) \tag{2.15}$$

Letting $\tilde{A}^i \triangleq A^i - B^i K_0^{x^i}$ and $F^{ij} \triangleq B^i K_0^{v^i} \tilde{\Phi}_1^i S^{ij}$, it follows that

$$x_{t+1}^i = \tilde{A}^i x_t^i + \sum_{j \in G^i} F^{ij} (\bar{x}^{ij} - d^{ij}) \tag{2.16}$$

A global state equation,which describes the dynamics of the whole group of cooperating agents, can be now written . Let us introduce the maximum delay $\Delta \triangleq \max\limits_{i,j;\, i \neq j} \Delta_{ji}$. To account for the delay, we introduce $M\Delta$ further state equations $\rho_{t+1}^{i,1} = x_t^i$, $\rho_{t+1}^{i,2} = \rho_t^{i,1}$, ..., $\rho_{t+1}^{i,\Delta} = \rho_t^{i,\Delta-1}$, $i = 1, \ldots, M$. Accordingly, we define an augmented state vector $x_t^a \in \mathbb{R}^{nM(1+\Delta)}$ in the form $x_t^a \triangleq [x_t^{1\top} \ldots x_t^{M\top} \; \rho_t^{1,1\top} \cdots \rho_t^{M,1\top} \ldots \rho_t^{1,\Delta\top} \ldots \rho_t^{M,\Delta\top}]^\top$. Then, the global system dynamics can be described as

$$x_{t+1}^a = A^a x_t^a + \sum_{i=1}^{M} \mathcal{F}^i d^i . \tag{2.17}$$

First of all, A^a is defined as

$$A^a \triangleq \left[\begin{array}{c|cccc|c} \tilde{A} & \bar{F}^1 & \bar{F}^2 & \cdots & & \bar{F}^\Delta \\ \hline \multicolumn{5}{c|}{I_{nM\Delta}} & \emptyset \end{array} \right] , \tag{2.18}$$

where $\tilde{A} \in \mathbb{R}^{nM \times nM}$, $\tilde{A} \triangleq blkdiag(\tilde{A}^1, ..., \tilde{A}^M)$, $I_{nM\Delta}$ is an $nM\Delta \times nM\Delta$ identity matrix, and $\emptyset$ denotes a zero rectangular matrix of dimension $nM\Delta \times nM$. Matrices $\bar{F}^k \in \mathbb{R}^{nM \times nM}$, $k = 1, ..., \Delta$ are block matrices whose structure and values depend on the specific delays Δ_{ji}. More precisely, for $k = 1, \ldots, M$, we let

$$\bar{F}^k \triangleq \begin{bmatrix} \emptyset & \cdots & \delta^{1M}(k) F^{1M} \\ \delta^{21}(k) F^{21} & \cdots & \delta^{2M}(k) F^{2M} \\ \vdots & \ddots & \vdots \\ \delta^{M1}(k) F^{M1} & \cdots & \emptyset \end{bmatrix} ,$$

where $\delta^{ij}(k) = 1$ if agent $\mathcal{A}^i$ receives the information about the state of DM $\mathcal{A}^j$ with a delay of k time–steps and $\delta^{ij}(k) = 0$ otherwise. The matrices $\mathcal{F}^i$

are defined as

$$\mathcal{F}^i \triangleq \begin{bmatrix} F^i & \emptyset \\ \emptyset & \emptyset \end{bmatrix}$$

where

$$F^i \triangleq \begin{bmatrix} \emptyset & \emptyset & \emptyset & \emptyset & \cdots & \emptyset \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ F^{i1} & F^{i2} & F^{i3} & F^{i4} & \cdots & F^{iM} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \emptyset & \emptyset & \emptyset & \emptyset & \cdots & \emptyset \end{bmatrix} .$$

$\emptyset$ denote zero–matrices of appropriate dimensions $F^{ii} = \emptyset$ and $F^{ij} = \emptyset$ if there is no cooperation between agents $\mathcal{A}^i$ and $\mathcal{A}^j$. Finally vectors d^i are defined as: $d^i \triangleq -[d^{i1\top} \quad d^{i2\top} \ldots d^{iM\top} \,|\, \emptyset]^\top$, where $d^{ii} = \emptyset$; if there is no cooperation between DMs $\mathcal{A}^i$ and $\mathcal{A}^j$, then the value of d^{ij} is irrelevant can can be conventionally set to zero. Consider the following partition of matrix A^a:

$$A^a \triangleq \begin{bmatrix} \tilde{A} & \mathcal{F}^\Delta \\ \mathcal{I}^\Delta & \mathcal{H}^\Delta \end{bmatrix} ,$$

where $\mathcal{F}^\Delta \in \mathbb{R}^{nM \times nM\Delta}$, $\mathcal{I}^\Delta \in \mathbb{R}^{nM\Delta \times nM}$, and $\mathcal{H}^\Delta \in \mathbb{R}^{nM\Delta \times nM\Delta}$ are appropriately defined on the basis of A^a given in (2.18)

Let us first assume that the assumptions stated in Theorem 2.3.1 are verified. Then, matrix $\tilde{A}$ is asymptotically stable and obviously we have that $|(zI - \mathcal{H}^\Delta)^{-1}\mathcal{I}^\Delta(zI - \tilde{A})^{-1}|_\infty < \infty$. Specifically, there exists a scalar $\alpha > 0$ such that

$$|(zI - \mathcal{H}^\Delta)^{-1}\mathcal{I}^\Delta(zI - \tilde{A})^{-1}|_\infty < \alpha$$

Now, it can be immediately shown that $\forall\, \varepsilon > 0$, $\exists\, \beta > 0$ such that $|S^{ij,r}|_\infty < \beta$, $\forall\, i = 1, \ldots, M$, $\forall\, j \in G^i$ implies that $|\mathcal{F}^\Delta|_\infty < \varepsilon$. Hence, let us choose a suitable $\bar{\beta} > 0$ such that $|\mathcal{F}^\Delta|_\infty < 1/\alpha$. Then, using the discrete–time small–gain theorem, it follows immediately that matrix A^a is asymptotically stable and the asymptotic stability of A^a is guaranteed for all matrices $\mathcal{F}^\Delta$ such that $|\mathcal{F}^\Delta|_\infty < \dfrac{1}{\alpha}$.

Moreover, assume now that the delays Δ_{ji} are affected by some uncertainty. It is trivial to show that such uncertainty results in a suitable perturbation $\Delta\tilde{F}$ to the block matrix $\left[\bar{F}^1\ \bar{F}^2\ \cdots\ \bar{F}^\Delta\right]$ appearing in (2.18), that is matrix A^a becomes

$$A^a = \left[\begin{array}{c|ccc|c} \tilde{A} & \bar{F}^1 & \bar{F}^2 & \cdots & \bar{F}^\Delta \\ \hline \multicolumn{4}{c|}{I_{nM\Delta}} & \emptyset \end{array}\right] + \left[\begin{array}{c|c} \emptyset & \Delta\tilde{F} \\ \hline \emptyset & \emptyset \end{array}\right], \tag{2.19}$$

Therefore, if the uncertainty on the communication delays is sufficiently small, that is, if there exists a sufficiently small scalar $\gamma > 0,\, \gamma < 1/\alpha$ such that $|\Delta\tilde{F}|_\infty < \gamma$, then the condition $|\mathcal{F}^\Delta|_\infty < \dfrac{1}{\alpha} - \gamma$ guarantees again the asymptotic stability of matrix A^a.

Summing up, we proved the following:

Proposition 2.3.1 *Consider the global coupled system* (2.17). *Moreover, for each agent $\mathcal{A}^i$, $i = 1, \ldots, M$, consider the FARE* (2.12) *and definition* (2.13) *and suppose that:*

a) $R^{i,r}$ is positive definite;

b) $[A^{i,r}, B^{i,r}]$ is stabilizable;

c) $[A^{i,r}, (\bar{P}_0^{i,r})^{\frac{1}{2}}]$ is detectable;

d) $\bar{P}_0^{i,r}$ is positive semidefinite.

Then, there exists a bounded $\alpha > 0$ such that $|(zI - \mathcal{H}^\Delta)^{-1}\mathcal{I}^\Delta(zI - \tilde{A})^{-1}|_\infty < \alpha$. Moreover, referring to (2.19), *if there exists a sufficiently small scalar $\gamma > 0,\, \gamma < 1/\alpha$ such that $|\Delta\tilde{F}|_\infty < \gamma$, and if the cooperation weighting matrices are chosen in such a way that $|\mathcal{F}^\Delta|_\infty < \dfrac{1}{\alpha} - \gamma$, then the team of DMs is asymptotically stable (all the eigenvalues of matrix A^a lie strictly inside the unit circle).*

Some observations about the stability result reported in Proposition 2.3.1 are now in place.

1. Proposition 2.3.1 is a sufficient condition for the asymptotic stability of the team of cooperating DMs and may be conservative in practice.

2. The stability of the team of cooperating DMs should be associated with the global equilibrium state which in general is not the origin of the state space but depends on the specific values of the terms d^{ij} that act as "reference variables" in the cooperative control scheme (recall that these scalars enter the feedforward terms in the local control laws – see Eqs. (2.8) and (2.17)). As an example, if the cooperative control problem refers to a formation trajectory tracking for autonomous vehicles, the desired formation is associated with the equilibrium state and the stability as to be associated with the "steady–state formation".

3. There is an inherent compromise between the overall stability of the team of agents and the amount of cooperation between the DMs themselves. This is not surprising and in several works (see, for instance, the recent paper by Dunbar [13]) the fact that, loosely speaking, the amount of cooperation between DMs and the stability of the team are conflicting objectives is emphasized.

4. The global system (2.17) can be seen as a closed–loop one where the communication delays are included in the feedback interconnection. Clearly, in qualitative terms, increasing the amount of cooperation means to increase the relative weight of the exchanged delayed information between the agents. On the other hand, as for all feedback systems, it is not surprising that these delays have a strong influence on the closed–loop stability. In this connection, in the next section, a constructive sufficient condition is given (see Proposition 2.4.1) showing that an upper bound on the relative weight of the cooperation terms should be given to guarantee the stability of the global cooperative control scheme.

2.4 Stability of the team and choice of the local cost functions

2.4.1 Bounding expressions for the linear control law

Following [19], in order to relate the stability of the team with the coupling matrices chosen in the local cost functions, some bounds on matrices F^{ij} in (2.15) will be given as functions of matrices S^{ij}, exploiting some of the results reported in [36] and [41].

Let us recall equation (2.4), which is the backwards recursive expression needed to compute the local feedforward input of each agent:

$$v^i_{t+k} = [I + \Phi^i_{k+1} + \Phi^i_{k+2}\Phi^i_{k+1} + \ldots + \Phi^i_{N^i} \cdots \Phi^i_{k+2}\Phi^i_{k+1}] \times \sum_{j \in G^i} S^{ij}(\bar{x}^{ij} - d^{ij}),$$

where we also recall the definition of each matrix Φ^i_k :

$$\Phi^i_k \triangleq A^{i^{\top}} - A^{i^{\top}} q^i_k B^i (B^{i^{\top}} q^i_k B^i + R^i)^{-1} B^{i^{\top}}.$$

In section 2.2 we finally expressed in equation (2.6) the feedforward input v^i_{t+k} as:

$$v^i_{t+k} = \tilde{\Phi}^i_k \sum_{j \in G^i} S^{ij}(\bar{x}^{ij} - d^{ij}).$$

Matrix $\tilde{\Phi}^i_k$ takes on the form

$$\tilde{\Phi}^i_k \triangleq [I + \Phi^i_{k+1} + \Phi^i_{k+2}\Phi^i_{k+1} \cdots + \Phi^i_{N^i} \cdots \Phi^i_{k+2}\Phi^i_{k+1}]. \qquad (2.20)$$

In the following, a bound on $\tilde{\Phi}^i_k$ will be determined as a function of the cost matrices S^{ij}. After this step, we will be able to relate also matrices F^{ij} with the cost function cooperation parameters, and finally rephrase in these terms also the sufficient conditions given in Theorem 2.3.1.

First of all, let us consider the Riccati equation (2.3). Recall that, for a suitably large choice of

$$q^i_{N^i} = P^i_{N^i} + \sum_{j \in \mathcal{G}^i} S^{ij}_{N^i},$$

it can be shown that $q^i_k \leq q^i_{k+1}$, for $k = 0, ..., N^i - 1$. Letting

$$\bar{P}^i_{N^i} \triangleq P^i_{N^i} + \sum_{j \in \mathcal{G}^i} S^{ij}_{N^i}$$

we obtain the inequality [2]

$$\Phi^i_k \le {A^i}^\top \left[I - q^i_k B^i ({B^i}^\top \bar{P}^i_{N^i} B^i + R^i)^{-1} {B^i}^\top \right], \quad k = 0, \dots, N^i - 1 .$$

Hence, defining

$$\Theta^i_{\bar{P}^i_{N^i}} \triangleq B^i ({B^i}^\top \bar{P}^i_{N^i} B^i + R^i)^{-1} {B^i}^\top \ge 0$$

it follows that

$$\Phi^i_k \le {A^i}^\top \left[I - q^i_k \, \Theta^i_{\bar{P}^i_{N^i}} \right], \quad k = 0, \dots, N^i - 1.$$

Let us now determine a lower bound on q^i_k as a function of $\sum_{j \in \mathcal{G}^i} S^{ij}_{N^i}$. To this end, we assume that weight matrices R^i are identity matrices of appropriate dimension. Then, applying the matrix inversion lemma we can write

$$\begin{aligned} q^i_k &= {A^i}^\top \left[q^i_{k+1} - q^i_{k+1} B^i ({B^i}^\top q^i_{k+1} B^i + I)^{-1} {B^i}^\top q^i_{k+1} \right] A^i + P^i + \sum_{j \in \mathcal{G}^i} S^{ij} \\ &= {A^i}^\top \left[(q^i_{k+1})^{-1} + B^i \, {B^i}^\top \right]^{-1} A^i + P^i + \sum_{j \in \mathcal{G}^i} S^{ij}, \\ &\qquad k = N^i - 1, N^i - 2, \dots, 0. \end{aligned}$$

Since $q^i_{k+1} \ge 0$, the first term on the right–hand side of the above equality is positive; since $P^i \ge 0$ and $S^{ij} \ge 0$ by assumption, we have

$$q^i_k \ge \underline{P}^i \triangleq P^i + \sum_{j \in \mathcal{G}^i} S^{ij}, \quad k = N^i - 1, N^i - 2, \dots, 0.$$

and then the upper bound on Φ^i_k takes on the form

$$\Phi^i_k \le {A^i}^\top \left[I - \underline{P}^i \, \Theta^i_{\bar{P}^i_{N^i}} \right], \quad k = 0, \dots, N^i - 1.$$

---

[2] We remark that, for the sake of simplicity, we are dealing with the case of cooperating DMs with the same dimension of the state and control vectors, without losing the general validity of the results.

Now, define the matrix

$$\Psi^i_{S^{ij}} \triangleq \left[I - \underline{P}^i \, \Theta^i_{\bar{P}^i_{N^i}} \right], \tag{2.21}$$

which depends on the choice of matrices S^{ij}. By substitution in (2.20), we draw the following upper bound for $\tilde{\Phi}^i_1$:

$$\tilde{\Phi}^i_1 \leq [I + {A^i}^\top \Psi^i_{S^{ij}} + ({A^i}^\top \Psi^i_{S^{ij}})^2 + \cdots + ({A^i}^\top \Psi^i_{S^{ij}})^{N^i}].$$

As a result, the following upper bound on matrices F^{ij} in (2.15) has been derived:

$$F^{ij} = B^i K_0^{v^i} \tilde{\Phi}^i_1 S^{ij} \leq \mathcal{E}^i_{S^{ij}} \, S^{ij}, \tag{2.22}$$

where

$$\mathcal{E}^i_{S^{ij}} \triangleq B^i K_0^{v^i} \left[I + {A^i}^\top \Psi^i_{S^{ij}} + ({A^i}^\top \Psi^i_{S^{ij}})^2 + +({A^i}^\top \Psi^i_{S^{ij}})^{N^i} \right].$$

It is worth noting that the subscript S^{ij} in matrices $\Psi^i_{S^{ij}}$ and $\mathcal{E}^i_{S^{ij}}$ emphasizes the fact that these matrices depend on the choice of sum of the cooperation weighting matrices S^{ij} for each DM $\mathcal{A}^i$.

2.4.2 Choice of the stabilizing cost function

Now that a bound on matrices F^{ij} has been derived, we can proceed to finding a choice of matrices S^{ij} in (2.2), thus guaranteing the stability of the team of cooperating agents.

In Section 2.3, the overall system dynamics was described by

$$x^a_{t+1} = A^a x^a_t + \sum_{i=1}^{M} \mathcal{F}^i d^i,$$

where matrix A^a was partitioned as follows:

$$A^a \triangleq \begin{bmatrix} \tilde{A} & \mathcal{F}^\Delta \\ \mathcal{I}^\Delta & \mathcal{H}^\Delta \end{bmatrix}.$$

In Proposition 2.3.1, the stability condition was given in terms of the norm

$|\mathcal{F}^{\Delta}|_{\infty}$. In the following, by exploiting the bounds (2.22) on matrices F^{ij}, we shall analyze how the cooperation weighting matrices S^{ij} in the cost function alterate the norm $|\mathcal{F}^{\Delta}|$ thus influencing the stability property of the system. More specifically, once it has been verified that $|(zI-\mathcal{H}^{\Delta})^{-1}\mathcal{I}^{\Delta}(zI-\tilde{A})^{-1}|_{\infty} < \alpha$ for some positive scalar α, let us find some explicit condition on the coupling matrices to ensure that $|\mathcal{F}^{\Delta}| < 1/\alpha$, thus fulfilling Proposition 2.3.1.

Let us first recall the structure of matrix $\mathcal{F}^{\Delta}$:

$$\mathcal{F}^{\Delta} = \left[\bar{F}^{1} \quad \bar{F}^{2} \ldots \bar{F}^{\Delta}\right]$$

Assuming again that all the DMs have the same state dimension, and if the communication between the DMs is not replicated over the time frame Δ, it follows that

$$|\mathcal{F}^{\Delta}|_{\infty} = \max_{i=1,..,M} |\sum_{j\in\mathcal{G}^i} F^{ij}|_{\infty} .$$

Supposing that all the coupling matrices S^{ij} are diagonal, from (2.22), we obtain immediately

$$\sum_{j\in\mathcal{G}^i} F^{ij} \leq \mathcal{E}^{i}_{S^{ij}} \sum_{j\in\mathcal{G}^i} S^{ij}$$

and thus

$$\begin{aligned} |\mathcal{F}^{\Delta}|_{\infty} &= \max_{i=1,..,M} |\sum_{j\in\mathcal{G}^i} F^{ij}|_{\infty} \leq \max_{i=1,...,M} |\mathcal{E}^{i}_{S^{ij}} \sum_{j\in\mathcal{G}^i} S^{ij}|_{\infty} \\ &\leq \max_{i=1,..,M} |\mathcal{E}^{i}_{S^{ij}}|_{\infty}| \sum_{j\in\mathcal{G}^i} S^{ij}|_{\infty} . \end{aligned} \tag{2.23}$$

Summing up, we get the following result:

Proposition 2.4.1 *Let us take into account the global coupled system* (2.17). *Moreover, for each agent* $\mathcal{A}^i$, $i = 1,\ldots,M$, *consider the FARE* (2.12) *and definition* (2.13) *and suppose that:*

a) $R^{i,r}$ *is positive definite;*

b) $[A^{i,r}, B^{i,r}]$ *is stabilizable;*

c) $[A^{i,r}, (\bar{P}_0^{i,r})^{\frac{1}{2}}]$ *is detectable;*

d) $\bar{P}_0^{i,r}$ *is positive semidefinite.*

Then, there exists a bounded $\alpha > 0$ *such that* $|(zI - \mathcal{H}^{\Delta})^{-1}\mathcal{I}^{\Delta}(zI - \tilde{A})^{-1}|_{\infty} < \alpha$. *Moreover, if the cooperation weighting matrices* S^{ij} *are chosen in such a way that*

$$\max_{i=1,..,M} |\mathcal{E}^i_{S^{ij}}|_{\infty} \, |\sum_{j \in \mathcal{G}^i} S^{ij}|_{\infty} < \frac{1}{\alpha}$$

then the team of agents is asymptotically stable. □

It is worth noting that, once the other parameters of the local system and the cost function are set, and asymptotic stability of matrix $\tilde{A}$ is ensured, inequality (2.23) can be tested adapting suitably $\sum_{j \in \mathcal{G}^i} S^{ij}$; the choice of the single matrices S^{ij} can thus be interpreted as a tuning parameter in the design of the cooperative control scheme.

Proposition 2.4.1 has been stated and proved for the case where no uncertainties affect the communication delays Δ_{ji} between the DMs. The extension to the case of uncertain communication delays can be done in a straightforward way along the same lines as for Proposition 2.3.1.

2.5 Illustrative example

We took as an example a set of UAVs moving in $\mathbb{R}^3$, using simplified and linear models. The objective of the team (and therefore of the cooperative controller) is to reach a certain "formation" around the origin, on the plane $z = 0$, maintaining as much as possible the formation through the whole trajectory. The state of the vehicles in such simplified model is given by the space position and velocity components in the three space directions, which are assumed to be independent from each other. The discretized state equations for each UAV take on the linear structure

$$x^i_{t+1} = A^i x^i_t + B^i u^i_t, \tag{2.24}$$

where matrices A^i and B^i were obtained from the discretization of linear damped double integrator equations, with sampling time $T = 0.1s$.

The physical parameters have been set identical for all the DMs: the mass is $m = 0.75\,Kg$ and the viscosity parameter is set to $\mu = 0.15\,Kg/m\,s$. The delay occurring in the information received by agent i from agent j, which has been indicated as Δ^{ij}, is set as $\Delta^{ij} = 3T$ for all the DMs, while $\Delta^{ii} = 0$.

The desired formation is to spread the vehicles at an even distance of $\sqrt{2 \cdot (1.5)^2}\,m$ along a line of 45° crossing the origin, reaching this position on the plane $z = 0$.

For the local cost function (2.2), we chose the prediction horizon $N^i = 9$, and the following weighting matrices: $R^i = blkdiag(1,1,1)$, $P^i = blkdiag(10,1,10,1,10,1)$ and $P_N^i = 10 * P^i$ for every $i = 1, ..., M$. As to the cooperation weight matrices S^{ij}, several values have been considered, in order to test the efficacy of the algorithm and of the proposed bounds on the $\sum_{j \in \mathcal{G}^i} S^{ij}$, for each i . The initial condition is a set of matrices that causes instablity of the team, with all the matrices S^{ij} equal to $S_0^{ij} = 100 * blkdiag(10,1,10,1,10,1)$, $S_0^{ii} = 0$.

Two different ways to tune the cooperation matrices were tried.

First of all we trivially ensured stability by imposing that the eigenvalues of the system matrix in (2.18) be inside the unit circle: starting from a destabilizing value, the cooperation matrices were gradually decreased (they are diagonal, so we simply multiplied by a scaling factor) until such condition was satisfied. This first algorithm guarantees stability of the team, and a very good cooperative behavior; clearly this technique is computationally demanding if the team is composed by several agents, since it is related to an eigenvalue problem.

For a set of 5 agents, we found that stability was assured by setting all the coupling matrices as $S_b^{ij} = blkdiag(42.4, 4.24, 42.4, 4.24, 42.4, 4.24)$, for each i, j, (which is the the bound on the sum multiplied for a factor $1/M$), the performance respects the desired formation (see Figures 2.1 and Figure 2.2). The dashed colored lines always represent the desired trajectories, while the solid lines render the actual behavior of the DMs. The colored circles represent the positions of the vehicles taken each second, while the black circles are the desired position at the same time instants. The distances among the agents reach the desired value of $\sqrt{2 \cdot (1.5)^2}\,m$ and vehicles never hit each other (see Figure 2.2). The objective distances are here shown with respect to agent 1, the red agent, and are represented by the dark dashed lines. The dark and colored circles show respectively the desired and actual positions of the DMs every two seconds. No input constraints have been imposed, which explains the ideally fast behavior of the agents.

On the other hand, finding a bound on the $\sum_{j \in \mathcal{G}^i} S^{ij}$ through small gain reasoning has shown poorer performances on the considered system; tuning the bound according to the sufficient condition (2.23) leads to set $S_b^{ij} = blkdiag(6.36, 0.636, 6.36, 0.636, 6.36, 0.636)$, which forces the system to lower the cooperative behavior . In Figure 2.3 the team trajectories are reported in the three dimensional space; the projection on the plane $z = 0$ is shown in Figure 2.4. Figure 2.5 represents the performance of the members in terms of desired distances.

The simulations on this example showed a strong dependence of the control performances (like, for example, reaching of desired final configurations) on the choice of the initial set of cooperation matrices and on the FH length N; in particular, the length of the control horizon and the weight on the final state greatly affects the stability of the system. The geometry of the weighting matrices naturally leads the control to affect more strongly either the position state variables or the velocities, depending on the higher chosen weights; the simulations proposed for the presented example were targeted for position tracking.

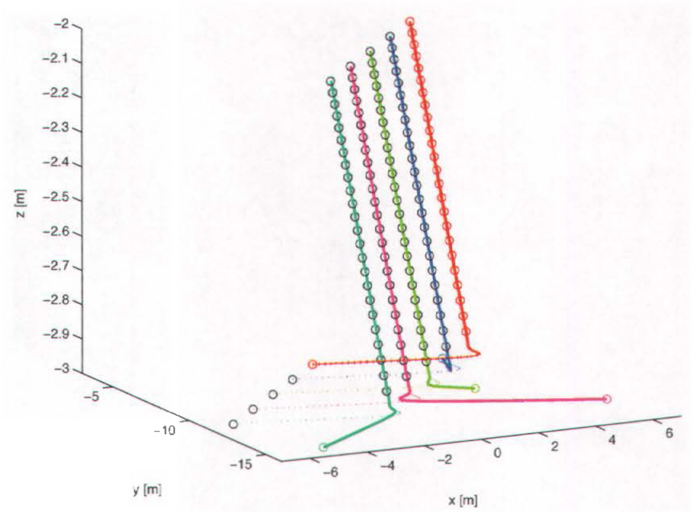

Figure 2.1: Trajectories of the UAVs with tighter cooperation: red–agent 1, blue–agent 2, green–agent 3, magenta–agent 4, cyan–agent 5

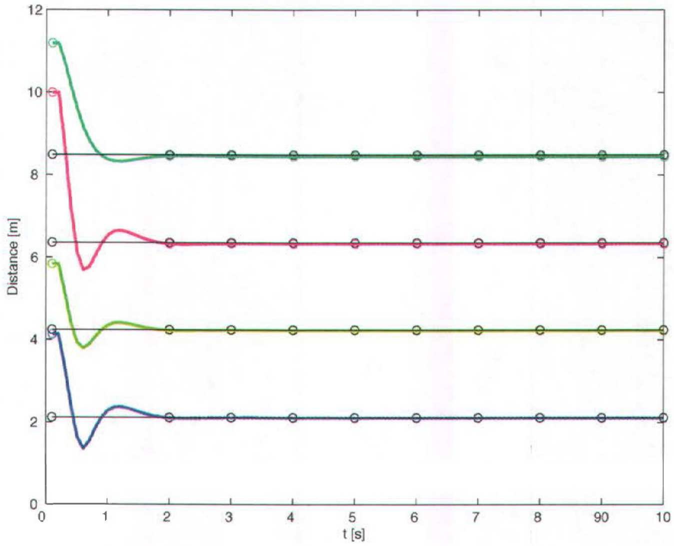

Figure 2.2: Distances among the UAV's and the red agent, with tighter cooperation; in black line the desired distance

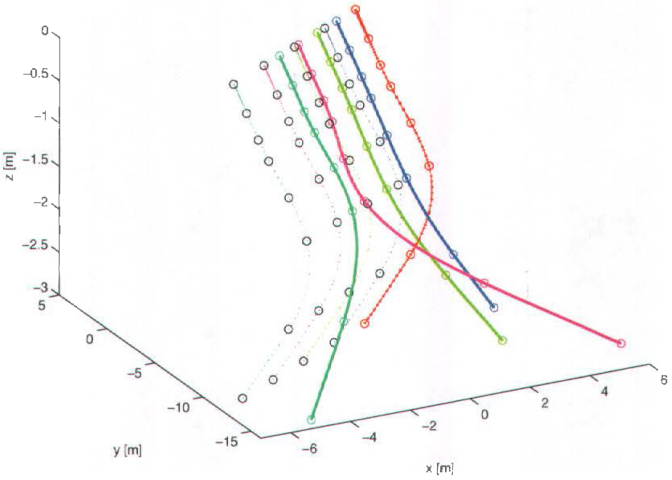

Figure 2.3: Trajectories of the UAVs: red–agent 1, blue–agent 2, green–agent 3, magenta–agent 4, cyan–agent 5

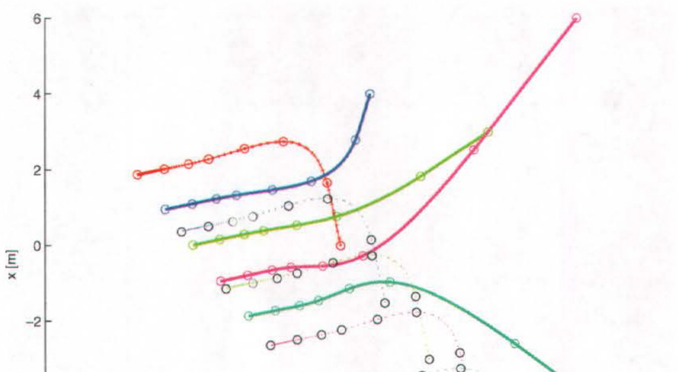

Figure 2.4: Trajectories of the UAV's, projection on the plane $z = 0$: red–agent 1, blue–agent 2, green–agent 3. magenta–agent 4, cyan–agent 5

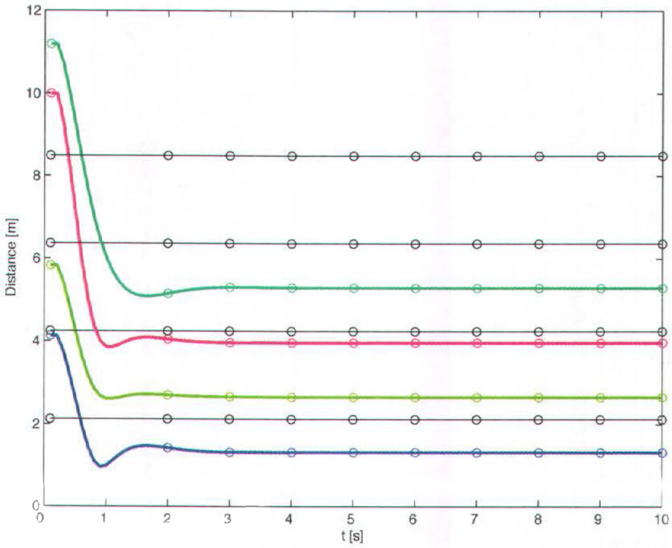

Figure 2.5: Distances among the UAV's and the red agent; in black line the desired distance

Chapter 3

Input to State Stability

Several studies on cooperative distributed systems have recently taken advantage of Input to State Stability (ISS) and related concepts, which can be utilized in order to prove structural properties of certain cooperation strategies [73], [74]. It is also necessary to mention that a unified framework to analyze networked control systems was proposed in [60], [61]. Connections between MPC and ISS can be found in [47, 51].

Input to State Stability, [70, 69, 30] among others, is a concept that answers an important question: how much does the external input magnitude affect the system state time profile? And if we substitute the words external input with *external disturbance*, it is even more evident how crucial such answer could be. Intuitively, if we recall from the previous chapters how we compared the information vector received by each agent to an external input or external tracking signal, we can immediately see that ISS could be extremely useful if it could help us classifying the behavior of our team in the presence of all the cooperation interconnections.

Therefore, ISS is the main tool that will be utilized in the following chapters when investigating the stability properties of a team of nonlinear agents. In particular, the previously described RH control framework will be proved to guarantee ISS and stability of the team. It is therefore necessary to dedicate a chapter to this subject, recalling the main definitions and properties related to ISS. In Section 3.2 some results on regional ISS properties will be reported: they will be crucial basis for the stability analysis in Chapter 5.

3.1 Definitions

The notations and definitions introduced in this chapter are fairly standard in the literature (see, for instance, [30]) and are briefly reported for the reader convenience.

Definition 3.1.1 ($\mathcal{K}$-function) *A function* $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ *is of class* $\mathcal{K}$ *(or a "$\mathcal{K}$-function") if it is continuous, positive definite and strictly increasing.* □

Definition 3.1.2 ($\mathcal{K}_\infty$-function) *A function* $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ *is of class* $\mathcal{K}_\infty$ *if it is a* $\mathcal{K}$*-function and* $\gamma(s) \rightarrow +\infty$ *as* $s \rightarrow +\infty$*.* □

Definition 3.1.3 ($\mathcal{KL}$-function) *A function* $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ *is of class* $\mathcal{KL}$ *if, for each fixed* $t \geq 0$, $\beta(\cdot, t)$ *is of class* $\mathcal{K}$*, for each fixed* $s \geq 0$, $\beta(s, \cdot)$ *is decreasing and* $\beta(s,t) \rightarrow 0$ *as* $t \rightarrow \infty$*.* □

Definition 3.1.4 (Upper limit) *Given a bounded sequence* $s : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$*, the upper limit is defined as*

$$\overline{\lim_{t \rightarrow \infty}} s_t \triangleq \inf_{t \geq 0} \sup_{\tau \geq t} s_\tau$$

□

Consider the following nonlinear discrete-time dynamic system

$$x_{t+1} = \tilde{f}(x_t, w_t), \; t \geq 0, \; x_0 = \bar{x}, \tag{3.1}$$

where $\tilde{f}(0,0) = 0$, and where $x_t \in \mathbb{R}^n$ and $w_t \in W \subset \mathbb{R}^r$ are the state and the bounded input of the system, respectively. The discrete–time state trajectory of the system (3.1) with initial state $\bar{x}$ and input sequence $w = \{w_t, \, t \geq 0\}$ is denoted by $x(t, \bar{x}, w)$, $t \geq 0$. We have the following definitions.

Definition 3.1.5 (ISS-Lyapunov function) *A function* V*:* $\mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}_{\geq 0}$ *is an ISS-Lyapunov function for system (3.1), if:*

1. *There exist two functions* $\underline{\alpha}(\cdot)$, $\bar{\alpha}(\cdot)$ *of class* $\mathcal{K}_\infty$ *such that:*

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|), \quad \forall x \in \mathbb{R}^n. \tag{3.2}$$

2. *There exist a function* $\alpha(\cdot)$ *of class* $\mathcal{K}_\infty$*, and a function* $\sigma(\cdot)$ *of class* $\mathcal{K}$ *such that defining* $\Delta V_w \triangleq V(f(x_t, w_t) - V(x_t)$ *one has:*

$$\Delta V_w \leq -\alpha(\|x_t\|) + \sigma(\|w_t\|). \tag{3.3}$$

Definition 3.1.6 (Robust positively invariant set) *A set* $\Xi \subset \mathbb{R}^n$ *is a robust positively invariant set for system (3.1) if* $\tilde{f}(x,w) \in \Xi$, $\forall x \in \Xi$ *and* $\forall w \in W$*.* □

Definition 3.1.7 (0-*AS* in Ξ) *Given a compact set* $\Xi \subset \mathbb{R}^n$ *including the origin as an interior point, the system (3.1) with* $w_t = 0$, $\forall t \geq 0$ *is said to be 0-AS (zero-asymptotically stable) in* Ξ*, if* Ξ *is robust positively invariant for (3.1) and if there exists a* $\mathcal{KL}$*-function* β *such that*

$$|x(t, \bar{x}, 0)| \leq \beta(|\bar{x}|, t), \ \forall t \geq 0, \ \forall \bar{x} \in \Xi. \tag{3.4}$$

□

Definition 3.1.8 (Regional ISS in Ξ) *Given a compact set* $\Xi \subset \mathbb{R}^n$ *including the origin as an interior point, the system (3.1) with* $w \in \mathcal{M}_W$*, is said to be regionally ISS in* Ξ*, if* Ξ *is robust positively invariant for (3.1) and if there exist a* $\mathcal{KL}$*-function* β *and a* $\mathcal{K}$*-function* γ *such that*

$$|x(t, \bar{x}, w)| \leq \beta(|\bar{x}|, t) + \gamma(\|w\|), \ \forall t \geq 0, \ \forall \bar{x} \in \Xi. \tag{3.5}$$

□

Definition 3.1.9 (*AG* in Ξ) *Given a compact set* $\Xi \subset \mathbb{R}^n$ *including the origin as an interior point, the system (3.1) is said to have the Asymptotic Gain (AG) property in* Ξ*, if* Ξ *is robust positively invariant for (3.1) and if there* *exists a* $\mathcal{K}_\infty$ *function* γ_{AG} *such that, for all initial state vectors* $\bar{x} \in \Xi$ *and all input sequences* $w \in \mathcal{M}_W$*, we have*

$$\overline{\lim_{t\to\infty}} |x(t,\bar{x},w)| \le \gamma_{AG}(\|w\|)\,.$$

□

Definition 3.1.10 (LISS) *The system (3.1) is locally input to state stable (LISS) if there exist a positive scalar* ρ*, a* $\mathcal{K}$*-function* γ*, and a* $\mathcal{KL}$*-function* β *such that*

$$|x(t,\bar{x},w)| \le \beta(|\bar{x}|,t) + \gamma(\|w\|)$$

for all initial states $\bar{x}$ *such that* $|\bar{x}| \le \rho$*, for all input sequences* $w \in \mathcal{M}_W$ *such that* $\|w\| \le \rho$*, and for all* $t \ge 0$. □

3.2 Regional ISS results

Some basic results concerning the regional input–to–state stability properties of general discrete–time systems of the form (3.1) will be stated and proved, following the approach presented in [51]. The regional ISS stability analysis will now be associated to the existence of a suitable Lyapunov function (in general, a-priori non smooth) defined as follows.

Definition 3.2.1 (ISS-Lyapunov function in Ξ) *A function* $V\colon \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}_{\ge 0}$ *is an ISS-Lyapunov function in* Ξ *for system (3.1), if:*

1. Ξ *is a compact robust positively invariant set including the origin as an interior point;*

2. *There exist a compact set* $\Omega \subseteq \Xi$ *(including the origin as an interior point) and suitable* $\mathcal{K}_\infty$*-functions* α_1, α_2, σ_1 *such that:*

$$V(x,w) \ge \alpha_1(|x|),\ \forall x \in \Xi,\ \forall w \in W \tag{3.6}$$

$$V(x,w) \le \alpha_2(|x|) + \sigma_1(|w|),\ \forall x \in \Omega,\ \forall w \in W. \tag{3.7}$$

3. *There exists a suitable $\mathcal{K}_\infty$-functions α_3 and some $\mathcal{K}$-functions σ_2, σ_3 such that:*

$$\begin{aligned} V(\tilde{f}(x,w_1),w_2)-V(x,w_1) &\leq -\alpha_3(|x|)+\sigma_2(|w_1|)+\sigma_3(|w_2|), \\ &\quad \forall x\in\Xi,\ \forall w_1,w_2\in W. \end{aligned} \tag{3.8}$$

4. *There exist some suitable $\mathcal{K}_\infty$-functions ε and ρ (ρ should be such that $(id-\rho)$ is a $\mathcal{K}_\infty$-function, too) such that the following compact set $D\subset\Omega$ (including the origin as an interior point) can be defined for some constant $c>0$:*

$$D \triangleq \ \{x : d(x,\partial\Omega)>c,\ V(x,w)\leq b(\hat{w}),\ \forall w\in W\}\subset\Omega\,, \tag{3.9}$$

where $b\triangleq\alpha_4^{-1}\circ\rho^{-1}\circ\sigma_4$, *with* $\alpha_4\triangleq\underline{\alpha}_3\circ\overline{\alpha}_2^{-1}, \underline{\alpha}_3(s)\triangleq\min(\alpha_3(s/2),\varepsilon(s/2))$, $\overline{\alpha}_2(s)\triangleq\alpha_2(s)+\sigma_1(s)$, $\sigma_4(s)\triangleq\varepsilon(s)+\sigma_2(s)+\sigma_3(s)$, *and* $\hat{w}\triangleq\max_w\{|w| : w\in W\}$.

□

Furthermore, the following regularity assumption is needed.

Assumption 1 *For every $t\geq 0$, the state trajectories $x(t,\bar{x},w)$ of the system (3.1) are continuous in $\bar{x}=0$ and $w=0$ with respect to the initial condition $\bar{x}$ and the input sequence w.* □

A sufficient condition for regional ISS of system (3.1) can now be stated.

Theorem 3.2.1 *Suppose that Assumption 1 holds. If the system (3.1) admits an ISS-Lyapunov function in Ξ, then it is ISS in Ξ and $\lim_{t\to\infty} d(x(t,\bar{x},w),D)=0$.*
□

The proof here is omitted: we address the interested reader to [17]

Chapter 4

Stability analysis of a team of nonlinear agents

If each member of the team is described by a nonlinear dynamic equation, the application of the distributed RH framework in the form described at Section 1.2 remains unvaried: the stability analysis at the level of the single agent and of the team becomes slightly more complicated though.

An approach based on ISS will be adopted, maintaining the general structure of Chapter 2: first we will show that every DM is ISS under the action of a RH control policy that minimizes the usual cooperation oriented cost function. Then we will lift our perspective to the overall team of DMs where we now find all the interconnections due to the application of the cooperative controller, which are affected by delays.

The *closed loop* team will be divided in two main dynamic subsystems: one describing the single agent dynamics and the other accounting for the occurring delays. Then we will exploit ISS tools which have been introduced at Chapter 3 and small gain reasonings - already utilized at Section 2.3.2 we will be able to prove the global asymptotic stability of the team.

4.1 Problem formulation

The set of M agents $\mathcal{A} \triangleq \{\mathcal{A}^i : i = 1, ..., M\}$ composing our team are now each described by a nonlinear time-invariant state equation:

$$x^i_{t+1} = f^i(x^i_t, u^i_t), \qquad t = 0,\, 1,\, 2,\, \ldots \tag{4.1}$$

where $x^i_t \in \mathbb{R}^{n^i}$ and $u^i_t \in \mathbb{R}^{m^i}$ are the state and control vector at time t, for each $i = 1, ..., M$. We assume again that all the M agents are synchronized and dynamically decoupled, but as specified in Section 1.1 there is a delayed information exchange among them. The coupling arises through the minimization of a cost function aimed at satisfying individual and cooperative objectives, as previously detailed in Section 1.2. In particular, for each $i = 1, ..., M$ and for a given value of the state vector x^i_t at time–instant t, we recast (1.4) into the following finite–horizon (FH) cost function to be minimized:

$$\begin{aligned} & J^i_{FH}[x^i_t, \bar{w}^i_t, u^i_{t,t+N^i-1}, N^i, h^i_F(\cdot)] = \\ & \sum_{k=t}^{t+N^i-1} \left[h^i(x^i_k, u^i_k) + q^i(x^i_k, \bar{w}^i_t) \right] + h^i_F(x^i_{t+N^i}) \,, \end{aligned} \tag{4.2}$$

where N^i, $i = 1, \ldots, M$ are positive integers denoting the lengths of the control horizons. Moreover, for each $i = 1, ..., M$, $h^i_F \in \mathcal{C}^1$ (continuously differentiable) is a suitable terminal cost function, with $h^i_F(0) = 0$. In (4.2) and in the following, we define $u^i_{t\tau} \triangleq \mathrm{col}\,(u^i_t, \ldots, u^i_\tau)$ for both finite and infinite values of τ. At time–instant t, the vector $\bar{w}^i_t$ can be considered as a constant external input in the cost function. Finally, let us assume that $f^i, h^i, k^i \in \mathcal{C}^1$, with $f^i(0,0) = 0$, $h^i(0,0) = 0$, and $q^i(0,0) = 0$. The local control strategy is still based on a RH framework, and is obtained by solving the following problem objective:

Problem 4.1.1 . *At every time instant* $t \geq 0$ *and for every agent* $\mathcal{A}^i$, $i = 1, \ldots, M$ *described by* (4.1), *find the RH optimal control law* $u^{iRH^\circ}_t = \gamma^i_{RH^\circ}(x^i_t, \bar{w}^i_t)$ $\in \mathbb{R}^{m^i}$, *where* $u^{iRH^\circ}_t$ *is the first vector of the control sequence* $u^{iFH^\circ}_t, \ldots, u^{iFH^\circ}_{t+N^i-1}$ *(i.e.,* $u^{iRH^\circ}_t \triangleq u^{iFH^\circ}_t$ *), that minimizes cost* (4.2) *for the state* $x^i_t \in \mathbb{R}^{n^i}$ *and the cooperation vector* $\bar{w}^i_t \in \mathbb{R}^{n^{tot}}$.

By solving Problem 4.1.1 and applying the corresponding controller, the dynamic behaviors of the DMs are coupled, depending on the specific choice of the partial cost terms h^i, h^i_F and q^i.

4.2 Stability properties of the agents

Consider a generic DM $\mathcal{A}^i$ whose dynamics are described by (4.1). We will show that for each $\mathcal{A}^i$, $i = 1, 2, ..., M$, the origin as an equilibrium state of the controlled agent, is zero–globally asymptotically stable (0–GAS). Moreover, we will also show that each $\mathcal{A}^i$ is ISS with respect to the inputs represented by the information vectors $\bar{w}^i_t$ received from its neighbors at each time–step t. Clearly we are now considering each DM as a "separate" dynamic system in the team: the input vectors $\bar{w}^i_t$ are "external" variables that are assumed not to depend on the behavior of its neighbors (i.e., the coupling between the DMs is not taken into account). Let us now introduce some useful notations and assumptions. In general, denote by $\mathcal{Z}$ the class of compact sets, $\mathcal{S} \subset \mathbb{R}^q$, containing the origin as an *internal point*. This means that $\mathcal{S} \in \mathcal{Z} \Leftrightarrow \exists \lambda \in \mathbb{R}, \lambda > 0$ such that $N(\lambda) \subset \mathcal{S}$, where $N(\lambda) \triangleq \{x \in \mathbb{R}^q : |x| \le \lambda\}$. The following assumptions are introduced for each agent $\mathcal{A}^i$, $i = 1, 2, ..., M$:

(i) The linear system $x^i_{t+1} = A^i x^i_t + B^i u^i_t$, obtained via the linearization of system (4.1) in a neighborhood of the origin, is stabilizable.

(ii) The transition cost functions h^i and k^i are such that there exists a strictly increasing function $\underline{r}^i \in \mathcal{C}[\mathbb{R}^+, \mathbb{R}^+]$, with $\underline{r}^i(0) = 0$, such that[1], letting $\tilde{h}^i(x^i, u^i) \triangleq h^i(x^i, u^i) + k^i(x^i, 0)$, we have $\tilde{h}^i(x^i, u^i) \ge \underline{r}^i(|(x^i, u^i)|)$, $\forall x^i \in \mathbb{R}^{n^i}$, $\forall u^i \in \mathbb{R}^{m^i}$, where $(x^i, u^i) \triangleq \mathrm{col}\,(x^i, u^i)$. Moreover, there exist a strictly increasing function $\bar{r}^i \in \mathcal{C}[\mathbb{R}^+, \mathbb{R}^+]$, with $\bar{r}^i(0) = 0$, such that $\tilde{h}^i(x^i, u^i) \le \bar{r}^i(|(x^i, u^i)|)$, $\forall x^i \in \mathbb{R}^{n^i}$, $\forall u^i \in \mathbb{R}^{m^i}$.

(iii) $h^i_F(\cdot) \in \mathcal{H}(a^i, P^i)$, where $\mathcal{H}(a^i, P^i) \triangleq \{h^i_F(\cdot) : h^i_F(x^i) = a^i x^{i\top} P^i x^i\}$, for some $a \in \mathbb{R}$, $a > 0$, and for some positive-definite symmetric matrix $P^i \in \mathbb{R}^{n^i \times n^i}$.

---

[1] When there will be no risk of confusion, notations will be simplified by dropping some subscript and/or superscript from the variables.

(iv) For every neighborhood $N^i(\lambda^i) \subset \mathbb{R}^{n^i}$ of the origin of the state space, there exists a control horizon $M^i \geq 1$ such that there exists a sequence of control vectors $\left\{u_k^i \in \mathbb{R}^{m^i},\ k = t, \ldots, t + M^i - 1\right\}$ that yield a state trajectory $x_k^i \in \mathbb{R}^{n^i}$, $k = t+1, \ldots, t + M^i$ ending in $N^i(\lambda^i)$ (i.e., $x_{t+M^i} \in N^i(\lambda^i)$) for any initial state $x_t^i \in \mathbb{R}^{n^i}$.

(v) The optimal FH feedback control functions $\gamma^i_{FH^\circ}(x_k^i, \bar{w}_t^i, k)$, $k = t, \ldots, t + N^i - 1$, which minimize cost (4.2), are continuous functions with respect to x_k^i, $\bar{w}_t^i$, for any $x_k^i \in \mathbb{R}^{n^i}$, $\bar{w}_t^i \in \mathbb{R}^{n^{tot}}$ and for any finite integer $N^i \geq 1$.

Denote by $J^i_{FH^\circ}[x_t^i, \bar{w}_t^i, N^i, h_F^i(\cdot)] \triangleq J^i_{FH}[x_t^i, \bar{w}_t^i, u^{i\,\circ}_{t,t+N^i-1}, N^i, h_F^i(\cdot)]$ the cost corresponding to the optimal N^i-stage trajectory starting from x_t^i. The following theorem holds.

Theorem 4.2.1 *Consider agent* $\mathcal{A}^i$, $i : 1 \leq i \leq M$. *If assumptions (i) to (v) are verified, there exist a finite control horizon* $\tilde{N}^i \geq M^i$, *a positive scalar* $\tilde{a}^i$ *and a positive-definite symmetric matrix* $P^i \in \mathbb{R}^{n^i \times n^i}$ *such that, for every terminal cost function* $h_F^i(\cdot) \in \mathcal{H}(a^i, P^i)$, *with* $a^i \in \mathbb{R}$, $a^i \geq \tilde{a}^i$, *the following properties hold:*

(a) the origin as an equilibriun point of system (4.1) *under the action of the RH optimal control law* $\gamma^i_{RH^\circ}$ *is GAS for* $\bar{w}^i \equiv 0$; *namely system* (4.1), *rewritten as* $x_{t+1}^i = \tilde{f}^i(x_t^i, \bar{w}_t^i) \triangleq f^i(x_t^i, \gamma^i_{RH^\circ}(x_t^i, \bar{w}_t^i))$, *is* 0*-GAS with respect to input* $\bar{w}_t^i$;

(b) if we furtherly assume that the function f^i *in* (4.1) *and the optimal RH control law* $\gamma^i_{RH^\circ}$ *are globally Lipschitz functions with respect to their arguments, then system* (4.1) *under the action of the RH optimal control law* $\gamma^i_{RH^\circ}$ *is ISS with respect to input* $\bar{w}_t^i$. □

Part (a) of Theorem 4.2.1 is a generalization to the global stability case of the early results published in [65] (see also the related works [12, 64] and the references cited therein) showing that closed–loop stability properties are guaranteed by a suitable choice of the local FH cost. In Part (b) it is shown that, under some further assumptions, each DM shows some ISS property.

Proof. Let us consider a generic agent $\mathcal{A}^i$.

Part (a). The proof that 0 is an equilibrium state of the closed-loop system when the RH regulator is applied and when $\bar{w}_t^i = 0$ is straightforward and it is therefore omitted. Now, we show that the function

$$V^i(x^i) \triangleq J^i_{FH^\circ}[x^i, 0, N^i, h^i_F(\cdot)], \quad x^i \in \mathbb{R}^{n^i} \tag{4.3}$$

is a Lyapunov function in $\mathbb{R}^{n^i}$ for system (4.1) driven by the RH regulator (for now, N^i and $h^i_F(\cdot)$ are not specified). Assumption (v) and the regularity hypotheses on the dynamic system (4.1) and on cost (4.2) ensure that $V^i(\cdot)$ is continuous with respect to all its arguments. Moreover, the control sequence $\{u_k^{iFH^\circ} = 0,\ k = t, t+1, \ldots, t+N^i-1\}$ minimizes cost (4.2) for $x_t^i = 0$, $\bar{w}_t^i = 0$, thus yielding $J^i_{FH^\circ}[0, 0, N^i, h^i_F(\cdot)] = V^i(0) = 0$. By letting $x_t^{iFH^\circ} = x_t^i$, $\forall\, x_t^i \in \mathbb{R}^{n^i} \setminus \{0\}$, we obtain

$$\begin{aligned} V^i(x_t^i) &\geq h^i(x_t^i, u_t^i) + k^i(x_t^i, 0) = \\ &= \tilde{h}^i(x_t^i, u_t^i) \geq \underline{r}^i\left(\left|(x_t^i, u_t^i)\right|\right) \geq \underline{r}^i\left(\left|x_t^i\right|\right) > 0 \end{aligned} \tag{4.4}$$

Then $V^i(\cdot)$ is positive-definite. Moreover, according to (4.4) and the properties of function $\underline{r}^i(\cdot)$, it turns out that $V^i(\cdot)$ is radially unbounded, that is $\lim_{|x^i| \to \infty} V^i(x^i) = \infty$. We have now to evaluate $\Delta V^i(x_t^i) \triangleq V^i(x_{t+1}^{iRH^\circ}) - V^i(x_t^i)$, for x_t^i and $x_{t+1}^{iRH^\circ}$ belonging to the trajectory generated by the RH regulator and starting from a generic initial state $x_t^i \in \mathbb{R}^{n^i}$. The following identity clearly holds:

$$\begin{aligned} &J^i_{FH^\circ}[x_t^i, 0, N^i+1, h^i_F(\cdot)] = \\ &\tilde{h}^i(x_t^i, u_t^{iRH^\circ}) + J^i_{FH^\circ}[x_{t+1}^{iRH^\circ}, 0, N^i, h^i_F(\cdot)], \end{aligned} \tag{4.5}$$

$\forall\, x_t^i \in \mathbb{R}^{n^i}$, $\forall\, N^i \geq 1$, where $u_t^{iRH^\circ} = \gamma^i_{RH^\circ}(x_t^i, 0) = u_t^{iFH^\circ} = \gamma^i_{FH^\circ}(x_t^i, 0)$. We need now the following lemma (the proof is not reported here due to space limitations).

Lemma 4.2.1 *There exist a positive-definite symmetric matrix* $P^i \in \mathbb{R}^{n^i \times n^i}$, *a control horizon* $\tilde{N}^i \geq M^i$, *and a positive scalar* $\tilde{a}^i$ *such that*

$$J^i_{FH^\circ}[x_t^i, 0, N^i, h^i_F(\cdot)] \geq J^i_{FH^\circ}[x_t^i, 0, N^i+1, h^i_F(\cdot)], \tag{4.6}$$

$\forall\, x_t^i \in \mathbb{R}^{n^i}$, $\forall\, N^i \geq \tilde{N}^i$, $\forall\, h_F^i(\cdot) \in \mathcal{H}(a^i, P^i)$, *with* $a^i \in \mathbb{R}$, $a^i \geq \tilde{a}^i$. □

It is worth noting that Lemma 4.2.1 specifies N^i and $h_F^i(\cdot)$ introduced in (4.3). From (4.5) and (4.6), it follows that $J_{FH^\circ}^i[x_t^i, 0, N^i, h_F^i(\cdot)] \geq \tilde{h}^i(x_t^i, u_t^{iRH^\circ}) + J_{FH^\circ}^i[x_{t+1}^{iRH^\circ}, 0, N^i, h_F^i(\cdot)]$, $\forall\, x_t^i \in \mathbb{R}^{n^i}$, and then

$$\begin{aligned} \Delta V^i(x_t^i) &= J_{FH^\circ}^i[x_{t+1}^{iRH^\circ}, 0, N^i, h_F^i(\cdot)] - \\ &- J_{FH^\circ}^i[x_t^i, 0, N^i, h_F^i(\cdot)] \leq -\tilde{h}^i(x_t^i, u_t^{iRH^\circ}) \leq \\ &\leq -\underline{r}^i\left(|(x_t^i, u_t^{iRH^\circ})|\right) \leq -\underline{r}^i\left(|x_t^i|\right), \end{aligned} \tag{4.7}$$

$\forall\, x_t^i \in \mathbb{R}^{n^i}$, $x_t^i \neq 0$, with $\Delta V^i(0) = 0$, thus ending the proof of Part (a).

Part (b). We have to prove that the Lyapunov function $V^i(x^i)$, $i = 1, 2, \ldots, M$, is an ISS Lyapunov function, i.e. we have to show that (3.2) and (3.3) are verified.

As to (3.2), we can set $\underline{\alpha}^i \triangleq \underline{r}^i$ (see (4.4)) and by letting

$$\bar{\alpha}^i(|x_t^i|) \triangleq \sum_{k=t}^{t+N^i-1} \bar{r}^i(|(x_k^i, u_k^i)|) + h_F^i(x_{t+N^i}^i)$$

we obtain immediately that $V^i(x^i) \leq \bar{\alpha}^i(|x^i|)$, $\forall\, x^i \in \mathbb{R}^{n^i}$, thus showing that (3.2) is satisfied.

Coming to (3.3), f^i and γ^i being globally Lipschitz by assumption, from the previous definition $\Delta V_{\bar{w}}^i = J_{FH^\circ}^i[f^i(x_t^i, \gamma^i(x_t^i, \bar{w}_t^i)), 0, N^i, h_F^i(\cdot)] - J_{FH^\circ}^i[x_t^i, 0, N^i, h_F^i(\cdot)]$, it follows that

$$\begin{aligned} \Delta V_{\bar{w}}^i &\leq \bar{\alpha}^i(|f^i(x_t^i, \gamma^i(x_t^i, \bar{w}_t^i))|) - \\ &\quad -\underline{\alpha}^i(|x_t^i|) \leq \bar{\alpha}^i(L_f^i\, L_\gamma^i |\bar{w}_t^i|) - \underline{\alpha}^i(|x_t^i|), \\ &\leq \sigma^i(|\bar{w}_t^i|) - \underline{\alpha}^i(|x_t^i|) \end{aligned}$$

where L_f^i and L_γ^i denote the Lipschitz constants associated with f^i and γ^i, respectively. Then, also (3.3) is satisfied and therefore the closed–loop system is ISS with respect to the input $\bar{w}_t^i$. ■

4.3 Stability properties of the team of agents

Let us now consider the agents as a team $\mathcal{A} = \{\mathcal{A}^i,\, i = 1, ..., M\}$ where each $\mathcal{A}^i$ is controlled by the locally–stabilizing RH control law solving Problem 4.1.1. Therefore, we can write:

$$\begin{aligned} x^1_{t+1} &= \tilde{f}^1(x^1_t, \bar{w}^1_t) \triangleq f^1(x^1_t, \gamma^1(x^1_t, \bar{w}^1_t)) \\ &\vdots \\ x^M_{t+1} &= \tilde{f}^M(x^M_t, \bar{w}^M_t) \triangleq f^M(x^M_t, \gamma^M(x^M_t, \bar{w}^M_t)) \end{aligned}$$

where, for the sake of notational simplicity, we keep on denoting by γ^i the RH control law $\gamma^i_{RH^\circ}$.

Then, let us rewrite the team of dynamical systems as a suitable interconnection of two composite systems. To this end, let $X_t \triangleq \mathrm{col}\,(x^1_t, \cdots, x^M_t)$ and $\bar{\mathcal{W}}_t \triangleq \mathrm{col}\,(\bar{w}^1_t, \cdots, \bar{w}^M_t)$. Hence the following state equation can be written, where $\tilde{F}(X_t, \bar{\mathcal{W}}_t) \triangleq \mathrm{col}\,[\tilde{f}^1(x^1_t, \bar{w}^1_t),\, \tilde{f}^2(x^2_t, \bar{w}^2_t),\, ...,\, \tilde{f}^M(x^M_t, \bar{w}^M_t)\,]$,

$$X_{t+1} = \tilde{F}(X_t, \bar{\mathcal{W}}_t) \tag{4.8}$$

Vector $\bar{\mathcal{W}}_t$ can be easily characterized as the output of a system describing the delay dynamics of the information exchange process among the agents. For the sake of simplicity and without loss of generality, we assume that $\dim\,(w^i_t) \geq 1,\, i = 1, \ldots, M$, that is, we assume that each DM receives at least one delayed state information from another neighboring DM. First, we set $\Delta \triangleq \max\{\Delta_{ij},\, i,j = 1, ..., M,\, i \neq j\}$. Then, we introduce the state vector $\mathcal{Z}_t \triangleq \mathrm{col}\,(\rho^1_t, \cdots, \rho^\tau_t, \cdots, \rho^\Delta_t)$, $\mathcal{Z}_t \in \mathbb{R}^{n_\mathcal{Z}}$, where $n_\mathcal{Z} \triangleq \dim\,(\mathcal{Z}_t)$ and where the variables ρ are introduced to store the delayed states; specifically $\rho^1_{t+1} = \mathcal{X}_t$ and $\rho^\tau_{t+1} = \rho^{\tau-1}_t$, $\tau = 2, \ldots, \Delta$. Hence, it follows that

$$\begin{cases} \mathcal{Z}_{t+1} = A\,\mathcal{Z}_t + B\,\mathcal{X}_t \\ \bar{\mathcal{W}}_t = C\,\mathcal{Z}_t \end{cases} \tag{4.9}$$

where

$$A = \begin{bmatrix} \emptyset & \cdots & \cdots & \cdots & \emptyset \\ I^1 & \emptyset & \cdots & \cdots & \emptyset \\ \emptyset & I^2 & \emptyset & \cdots & \emptyset \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \emptyset & \cdots & \cdots & I^{\Delta-1} & \emptyset \end{bmatrix}, B = \begin{bmatrix} I^0 \\ \emptyset \\ \vdots \\ \emptyset \end{bmatrix}, C = \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^M \end{bmatrix}$$

$$C^i = \begin{bmatrix} C^i(1) & \cdots & C^i(\tau) & \cdots & C^i(\Delta) \end{bmatrix}$$

$$C^i(\tau) = \begin{bmatrix} \delta^{i1}(\tau) & \emptyset & \cdots & \cdots & \emptyset \\ \emptyset & \delta^{i2}(\tau) & \emptyset & \cdots & \emptyset \\ \emptyset & \cdots & \delta^{i3}(\tau) & \cdots & \emptyset \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \emptyset & \cdots & \cdots & \emptyset & \delta^{iM}(\tau) \end{bmatrix}$$

All matrices I^τ, for $\tau = 0, ..., \Delta - 1$ are identity matrices of dimension $n^{tot} \times n^{tot}$, where $n^{tot} \triangleq \dim(\mathcal{X}_t)$ and $\delta^{ij}(\tau) \triangleq I$ are identity matrices of dimension n^j, $i, j = 1, ..., M$, $i \neq j$. It is worth noting that DM $\mathcal{A}^i$ does not get replicated information from agent $\mathcal{A}^j$, thus in matrix C the matrix $\delta^{ij}(\tau)$ is equal to the identity for only one value of τ. Summing up, the state equation describing the dynamics of the team of DMs can be written as a feedback interconnection between the dynamic systems (4.8) and (4.9). Let us now prove the following lemma.

Lemma 4.3.1 *Let us suppose that Assumptions in Theorem 4.2.1 are verified. Then dynamic systems* (4.8) *and* (4.9) *are provided with suitable ISS Lyapunov functions* $V(X_t)$ *and* $V^D(\mathcal{Z}_t)$, *respectively.* □

Proof. Consider the Lyapunov function candidate $V(X_t) \triangleq \sum_{i=1}^M V^i(x_t^i)$ for the lumped system (4.8). From (3.2), it follows that

$$\sum_{i=1}^M \underline{\alpha}^i(|x_t^i|) \leq V(X_t) \leq \sum_{i=1}^M \bar{\alpha}^i(|x_t^i|).$$

Clearly $|x_t^i| \leq |X_t|$, and thus

$$V(X_t) \leq \sum_{i=1}^{M} \bar{\alpha}^i(|x_t^i|) \leq \sum_{i=1}^{M} \bar{\alpha}^i(|X_t|) \leq \bar{\alpha}(|X_t|),$$

where we set $\bar{\alpha}(|X_t|) \triangleq \sum_{i=1}^{M} \bar{\alpha}^i(|X_t|)$. Moreover $\sum_{i=1}^{M} |x_t^i| \leq \sum_{i=1}^{M} |X_t| = M|X_t|$. Then $|X_t| \geq \frac{1}{M} \sum_{i=1}^{M} |x_t^i|$ and $|X_t| \leq \sum_{i=1}^{M} |x_t^i|$.

Recall that for any $\mathcal{K}$ function γ it is always true that $\gamma(a+b) \leq \gamma(2a) + \gamma(2b)$ where $a, b > 0$.

Hence it follows that:

$$\underline{\alpha}^i(|X_t|) \leq \underline{\alpha}^i(\sum_{i=1}^{M} |x_t^i|) \leq \sum_{i=1}^{M} \underline{\alpha}^i(M|X_t|)$$

and then $\underline{\alpha}^i(|X_t|/M) \leq \underline{\alpha}^i(\frac{1}{M} \sum_{i=1}^{M} |x_t^i|) \leq \sum_{i=1}^{M} \underline{\alpha}^i(|X_t|)$.

Therefore, letting $\underline{\alpha}(|X_t|) \triangleq \underline{\alpha}^i(|X_t|/M)$ for an arbitrarily chosen index i, we showed that $\underline{\alpha}(|X_t|) \leq V(X_t) \leq \bar{\alpha}(|X_t|)$. Let us now write

$$\Delta V \triangleq \sum_{i=1}^{M} V^i(\tilde{f}^i(x_t^i, \bar{w}_t^i)) - \sum_{i=1}^{M} V^i(x_t^i) \leq$$
$$\leq -\sum_{i=1}^{M} \underline{\alpha}^i(|x_t^i|) + \sum_{i=1}^{M} \sigma^i(|\bar{w}_t^i|)$$

First, we have

$$-\sum_{i=1}^{M} \underline{\alpha}^i(|x_t^i|) \leq -\underline{\alpha}^i\,(|X_t|/M) \text{ and } \sum_{i=1}^{M} \sigma^i(|\bar{w}_t^i|) \leq \sum_{i=1}^{M} \sigma^i(|\bar{\mathcal{W}}_t|).$$

Then, letting $\underline{\alpha}\,(|X_t|) \triangleq \underline{\alpha}^i\,(|X_t|/M)$ and $\sigma(|\bar{\mathcal{W}}_t|) \triangleq \sum_{i=1}^{M} \sigma^i(|\bar{\mathcal{W}}_t|)$, it follows that $\Delta V \leq -\underline{\alpha}\,(|X_t|) + \sigma(|\bar{\mathcal{W}}_t|)$ thus showing that V is an ISS Lyapunov function for the lumped system (4.8).

System (4.9), describing the effects of the time–delays in the information exchange variables, is ISS being an asymptotically stable linear system and a candidate ISS Lyapunov function is $V^D(\mathcal{Z}_t) \triangleq |\mathcal{Z}_t|^2$. It is easy to find two positive constants $\underline{a}^D$ and $\bar{a}^D$ such that $\underline{a}^D|\mathcal{Z}_t|^2 \leq V^D(\mathcal{Z}_t) \leq \bar{a}^D|\mathcal{Z}_t|^2$ and thus the first part of the definition of ISS Lyapunov function holds by defining $\underline{\alpha}^D(|\mathcal{Z}_t|) \triangleq \underline{a}^D|\mathcal{Z}_t|^2$ and $\bar{\alpha}^D(|\mathcal{Z}_t|) \triangleq \bar{a}^D|\mathcal{Z}_t|^2$, which are two $\mathcal{K}$–functions. Moreover

$$\begin{aligned} V^D(\mathcal{Z}_{t+1}) - V^D(\mathcal{Z}_t) = |A\mathcal{Z}_t + BX_t|^2 - |\mathcal{Z}_t|^2 \leq \\ \leq |\mathcal{Z}_t|_Q^2 + |X_t|_{B^\top B}^2 \ . \end{aligned}$$

Then $\Delta V^D \leq -\alpha^D(|\mathcal{Z}_t|) + \sigma^D(|X_t|)$ (the definitions of $\alpha^D(\cdot)$ and $\sigma^D(\cdot)$ are straightforward). ■

Recalling from (4.9) that $\bar{\mathcal{W}}_t = C\,\mathcal{Z}_t$, from the proof of Lemma 4.3.1, it follows immediately that the ISS Lyapunov functions $V(X_t)$ and $V^D(\mathcal{Z}_t)$ satisfy

$$V(X_{t+1}) - V(X_t) \leq -\tilde{\alpha}(V(X_t)) + \tilde{\sigma}(V^D(\mathcal{Z}_t)), \tag{4.10}$$

$$V^D(\mathcal{Z}_{t+1}) - V^D(\mathcal{Z}_t) \leq -\tilde{\alpha}^D(V^D(\mathcal{Z}_t)) + \tilde{\sigma}^D(V(X_t)), \tag{4.11}$$

where $\tilde{\alpha}(\cdot)$ and $\tilde{\alpha}^D(\cdot)$ are $\mathcal{K}_\infty$ functions, and $\tilde{\sigma}(\cdot)$ and $\tilde{\sigma}^D(\cdot)$ are $\mathcal{K}$ functions, respectively. It is easy to show that $\tilde{\alpha} \triangleq \alpha \circ (\bar{\alpha})^{-1}$, $\tilde{\sigma} \triangleq \sigma \circ (\underline{\alpha}^D)^{-1}$, $\tilde{\alpha}^D \triangleq \alpha^D \circ (\bar{\alpha}^D)^{-1}$, and $\tilde{\sigma}^D \triangleq \sigma^D \circ (\underline{\alpha})^{-1}$. Now, the following result about the stability properties of the team of cooperating agents can be immediately proved.

Theorem 4.3.1 *Suppose that Assumptions in Theorem 4.2.1 are verified. Let us also suppose that the following small gain condition holds (*Id *denotes the identity operator):*

$$\tilde{\alpha}^{-1} \circ \tilde{\sigma} \circ \left(\tilde{\alpha}^D\right)^{-1} \circ \tilde{\sigma}^D < \mathrm{Id}\ . \tag{4.12}$$

Then the team of cooperating DMs described by the interconnected dynamic equations (4.8) *and* (4.9) *is 0–GAS.* □

Proof. The proof is very simple. Owing to the Assumptions made in Theorem 4.2.1, by Lemma 4.3.1 it follows that systems (4.8) and (4.9) are provided with ISS Lyapunov functions V and V^D satisfying inequalities (4.10) and (4.11). Then, Corollary 4.2 in [30] can be directly used showing that, if the small gain condition (4.12) is verified, then the feedback system resulting from the interconnection between systems (4.8) and (4.9) is 0–GAS thus ending the proof. ■

Remark. It is worth noting that the small–gain condition (4.10) may turn out to be conservative in practice as it is typical of these kind of results. On the other hand, the generality of the problem makes it rather difficult to obtain tighter conditions without making restrictive assumptions on the structure of the agents' dynamics and on the cost function.

Chapter 5

Constrained cooperative RH control

The following chapter is dedicated to the analysis of the case where the team is composed by nonlinear agents, though we allow the presence of constraints both in the state and in the control values: this is the most important chapter of this thesis work, since it allows the general framework introduced at Chapter 1 to applicable to a vast category of problems.

The structure of this chapter is analogous to that of Chapter 4: first of all we will prove *Local ISS* properties for each DM applying the proposed cooperative distributed RH control policy, which handles delays in the communication and constraints on the state and in the controller. The shift to local analysis is also an important step towards the generalization of our framework: global properties are in fact more difficult to encounter in practical problems presenting nonlinearities.

In Section 5.2.1 we will again treat the cooperation links as interconnections subject to delays and utilize small gain theorem reasonings to show $0 - AS$ of our team. We need to thoroughly add several assumptions at Section 5.2 in order to guarantee such property, which is the price we pay to extend the generality of this approach. In the end of the chapter, an example is provided that illustrates the validity of this approach.

5.1 Problem formulation

We consider a *distributed dynamic system* made of a set of M agents denoted as $\mathcal{A} \triangleq \{\mathcal{A}^i : i = 1, ..., M\}$. Each agent $\mathcal{A}^i$ is described by the nonlinear time-invariant state equation:

$$x_{t+1}^i = f^i(x_t^i, u_t^i), \ t \geq 0, \ x_0^i = \bar{x}^i \tag{5.1}$$

where, for each $i = 1, ..., M$, $x_t^i \in \mathbb{R}^{n^i}$ denotes the local state vector and $u_t^i \in \mathbb{R}^{m^i}$ denotes the local control vector of agent $\mathcal{A}^i$ at time t, and where we assume that $f^i(0,0) = 0$, $i = 1, ..., M$. We also suppose that the dynamics of all M agents evolve on the same discrete-time space (that is, the DMs are synchronized).

The state vector x_t^i of each agent $\mathcal{A}^i : i = 1, ..., M$ is constrained to belong to a compact set X^i, that is,

$$x_t^i \in X^i \subset \mathbb{R}^{n^i} . \tag{5.2}$$

Analogously, the control vector u_t^i is constrained to take values in a compact set U^i, that is,

$$u_t^i \in U^i \subset \mathbb{R}^{m^i}. \tag{5.3}$$

As usual in our setting, each agent $\mathcal{A}^i$ exchanges an information vector w_t^i with a given set of neighboring agents $\mathcal{G}^i \triangleq \{\mathcal{A}^j : j \in G^i\}$.

For each $i = 1, \ldots, M$ and for given values of the state vector $x_t^i \in X^i$ and of the information vector $\bar{w}_t^i \in W^i$ at time–instant t, we introduced a FH cost function which is a slight modification of 1.4.

$$\begin{aligned} & J_{FH}^i(x_t^i, w_t^i, d_{t,t+N_p^i}^{h^i}, d_{t,t+N_p^i-1}^{q^i}, u_{t,t+N_c^i-1}^i, N_c^i, N_p^i) \\ & \qquad = \sum_{l=t}^{t+N_p^i-1} \left[h^i(x_l^i, u_l^i, d_l^{h^i}) + q^i(x_l^i, w_l^i, d_l^{q^i}) \right] + h_f^i(x_{t+N_p^i}^i, d_{t+N_p^i}^{h^i}) , \end{aligned}$$

where, for a generic vector r_t, we define $r_{t,\tau} \triangleq \operatorname{col}(r_t, \ldots, r_\tau)$ for both finite and infinite values of τ. Now the positive integers N_c^i and N_p^i, $i = 1, \ldots, M$ denote the lengths of the so-called control and prediction horizons, respectively, according to the framework proposed in [50]. As usual in our framework, the control variables u_l^i, $l = t, \dots, t + N_c^i - 1$ will be the argument of a suitable optimization problem, but now the control variables u_l^i, $l = t + N_c^i, \dots, t + N_p^i - 1$ will be obtained through some auxiliary control law $u_l^i = \kappa_f^i(x_l^i)$.

A new element is the introduction of a "forward-forgetting-factor" for the information vector: in fact now vector w_l^i denotes the state of the dynamic system

$$w_{l+1}^i = A_w^i w_l^i,\ l = t, \dots, t + N_p^i - 2 ; \quad w_t^i \triangleq \bar{w}_t^i \tag{5.4}$$

where $A_w^i \triangleq \alpha_w^i I_{n_w^i}$ with $\alpha_w^i < 1$ and with $I_{n_w^i}$ denoting the identity matrix of dimension n_w^i. The dynamic system (5.4) is indeed introduced in order to decrease the "importance" of the information vector in the FH cost function along the prediction horizon. It is worth noting once more that at time–instant t, vectors w_l^i can be considered as known external inputs in the cost function.

The local cost function is still composed of two terms: a partial cost term given by $\sum_{l=t}^{t+N_p^i-1} h^i(x_l^i, u_l^i, d_l^{h^i}) + h_f^i(x_{t+N_p^i}^i, d_{t+N_p^i}^{h^i})$, where h^i is a transition cost function and h_f^i is a terminal cost function, and a "cooperation" cost term given by $\sum_{l=t}^{t+N_p^i-1} q^i(x_l^i, w_l^i, d_l^{q^i})$; the quantities $d_l^{h^i}$, $d_l^{q^i}$, $i = 1, \dots, M$ denote some given vectors of appropriate dimensions. The role of vectors $d_l^{h^i}$ and $d_l^{q^i}$ has been commented at Chapter 1.2

In this chapter, for the sake of simplicity, we suppose that by a suitable change of state coordinates, it is possible to consider an equivalent formulation where the cost function (with straightforward re-definitions of the symbols) can be re-written in the simpler form

$$J_{FH}^i(x_t^i, w_t^i, u_{t,t+N_c^i-1}^i, N_c^i, N_p^i) = \sum_{l=t}^{t+N_p^i-1} \left[h^i(x_l^i, u_l^i) + q^i(x_l^i, w_l^i) \right] + h_f^i(x_{t+N_p^i}^i), \tag{5.5}$$

where $h^i(0,0) = 0$, $q^i(0,0) = 0$, and $h_f^i(0) = 0$. Moreover, the origin is an interior point of the sets X^i and U^i.

The local control law is always designed according to a RH strategy which now needs to include several constraints, and will here be stated according to

[50] (see also the well–known survey paper [56]).

Problem 5.1.1 (FH Optimal Control Problem) *At every time instant* $t \geq 0$, *for every agent* $\mathcal{A}^i$, $i = 1, \ldots, M$ *described by* (5.1), *for given control and prediction horizons lengths* N_c^i *and* N_p^i, *for given auxiliary control laws* κ_f^i, *for given transition, cooperation and terminal cost functions* h^i, q^i, h_f^i, *for given terminal sets* X_f^i, *and for given values* $x_t^i \in X^i$ *and* $\bar{w}_t^i \in W^i$ *of the state and the information vectors, find the optimal FH control sequence* $\{u_t^{i^{FH^\circ}}, \ldots, u_{t+N_c^i-1}^{i^{FH^\circ}}\}$ *that minimizes cost* (5.5) *subject to:*

1. *the agent's dynamics* (5.1) *with* x_t^i *as initial state and the vectors* w_l^i, $l = t, \ldots, t + N_p^i - 1$ *given by* (5.4) *with* $\bar{w}_t^i$ *as initial condition;*
2. *the auxiliary control law* $u_l^i = \kappa_f^i(x_l^i)$, $l = t + N_c^i, \ldots, t + N_p^i - 1$;
3. *the constraints (5.2) and (5.3), that is* $x_l^i \in X^i$, $u_l^i \in U^i$, $l = t, \ldots, t + N_p^i - 1$;
4. *the terminal state constraint* $x_{t+N_p^i}^i \in X_f^i$.

□

Clearly, by definition, the optimal FH control sequence $\{u_t^{i^{FH^\circ}}, \ldots, u_{t+N_c^i-1}^{i^{FH^\circ}}\}$ solving Problem 5.1.1 is such that, when applied to (5.1), the constraints (5.2), (5.3), and the terminal constraint $x_{t+N_p^i}^i \in X_f^i$ are simultaneously satisfied. Indeed, the following definition regarding a generic control sequence $u_{t,t+N_c^i-1}^i$ will be useful in the analysis reported in Section 5.2.

Definition 5.1.1 (Admissible control sequence) *Given an initial state* x_t^i, *the sequence* $u_{t,t+N_c^i-1}^i$ *is said to be an admissible control sequence for the FH optimal control Problem 5.1.1 if its application to* (5.1) *under the action of the auxiliary control law* $u_l^i = \kappa_f^i(x_l^i)$, $l = t+N_c^i, \ldots, t+N_p^i-1$ *allows simultaneous satisfaction of* (5.2), (5.3) *and of the terminal constraint* $x_{t+N_p^i}^i \in X_f^i$. □

Now, the RH procedure can be described in the usual way as follows. When the controlled agent $\mathcal{A}^i$ is in the state x_t^i at stage t, the FH optimal control

Problem 5.1.1 is solved, thus obtaining the sequence of optimal control vectors, $\{u_t^{iFH^\circ}, \dots, u_{t+N_c^i-1}^{iFH^\circ}\}$. The first control action of this sequence becomes the control action $u_t^{iRH^\circ}$ generated by the RH local controller at time–instant t (i.e., $u_t^{iRH^\circ} \triangleq u_t^{iFH^\circ}$). This procedure is repeated stage after stage and a *feedback–feedforward* control law $\gamma_{RH^\circ}^i(x_t^i, w_t^i)$ is obtained, as the control vector $u_t^{iFH^\circ}$ depends on the local current state x_t^i and on the vector of delayed states w_t^i communicated to the agent $\mathcal{A}^i$ by the cooperating agents $\mathcal{G}^i = \{\mathcal{A}^j,\, j \in G^i\}$. The system (5.1) under the action of the *RH* optimal control law can thus be rewritten as

$$x_{t+1}^i = \tilde{f}^i(x_t^i, w_t^i) \triangleq f^i(x_t^i, \gamma_{RH^\circ}^i(x_t^i, w_t^i)),\ t \geq 0,\ x_0^i = \bar{x}^i \qquad (5.6)$$

which is of the same form of the general system (3.1) considered in Section 3.1. It is worth noting that, from well-known results on RH control (see, for instance, [56] and the references cited therein), we have $\gamma_{RH^\circ}^i(0,0) = 0$ and hence $\tilde{f}^i(0,0) = 0$, that is, the origin is an equilibrium state for agent $\mathcal{A}^i$ when $w_t^i = 0,\ t \geq 0$.

5.2 Stability of the team of cooperating agents

Let us consider a generic agent $\mathcal{A}^i$ whose dynamics is described by (5.1). By exploiting the results recalled at Section 3.2, we will now show that each agent $\mathcal{A}^i$, with $i = 1, 2, \dots, M$ is regionally ISS with respect to the inputs represented by the information vectors w_t^i received from its cooperating DMs at each time–step t. Clearly, in this context, we are considering each DM as a "separate" dynamic system in the team, in the sense that the input vectors w_t^i are "external" variables that are assumed not to depend on the behavior of the other cooperating agents (i.e., at the present stage, the coupling between the DMs is not directly taken into account). Let now introduce some further useful assumptions and definitions.

Assumption 2 *A terminal cost function h_f^i, a final constraint set X_f^i, and an auxiliary control law κ_f^i are given such that:*

1. $X_f^i \subset X^i$, X_f^i *closed*, $0 \in X_f^i$;

2. $\kappa_f^i(x^i) \in U^i$, $\left|\kappa_f^i(x^i)\right| \leq L_{\kappa_f}^i \left|x^i\right|$, $L_{\kappa_f}^i > 0$, $\forall x^i \in X_f^i$;
3. $\left|f^i(x^i, \kappa_f^i(x^i))\right| \leq L_{f_c}^i \left|x^i\right|$, $L_{f_c}^i > 0$, $\forall x^i \in X_f^i$;
4. $f^i(x^i, \kappa_f^i(x^i)) \in X_f^i$, $\forall x^i \in X_f^i$;
5. $\alpha_{h_f^i}(|x^i|) \leq h_f^i(x^i) \leq \beta_{h_f^i}(|x^i|)$, $\forall x^i \in X_f^i$, *where* $\alpha_{h_f^i}$ *and* $\beta_{h_f^i}$ *are* $\mathcal{K}_\infty$*-functions;*
6. $h_f^i(f^i(x^i, \kappa_f^i(x^i))) - h_f^i(x^i) \leq -h^i(x^i, \kappa_f^i(x^i)) - q^i(x^i, \tilde{w}^i) + \psi^i(|\tilde{w}^i|)$, $\forall x^i \in X_f^i$, $\forall w^i \in W^i$, *where* ψ^i *is a* $\mathcal{K}$*-function and* $\tilde{w}^i \triangleq \left(A_w^i\right)^{N_p^i - 1} w^i$. □

Assumption 3 *The partial cost function* h^i *is such that* $\underline{r}^i(|x^i|) \leq h^i(x^i, u^i)$, $\forall x^i \in X^i$, $\forall u^i \in U^i$ *where* $\underline{r}^i$ *is a* $\mathcal{K}_\infty$*-function. Moreover,* h^i *is Lipschitz with respect to* x^i *and* u^i *in* $X^i \times U^i$*, with Lipschitz constants denoted as* L_h^i *and* L_{hu}^i*, respectively.* □

Assumption 4 *The cooperation cost function* q^i *is such that* $0 \leq q^i(x^i, w^i)$, $\forall x^i \in X^i$, $\forall w^i \in W^i$*. Moreover* q^i *is Lipschitz with respect to* x^i *and* w^i *in* $X^i \times W^i$*, with Lipschitz constants denoted as* L_q^i *and* L_{qw}^i*, respectively.* □

Assumption 5 *Let* $X^{i\kappa_f}$ *denote the set of states* x_t^i *of the system* (5.1) *for which* $\tilde{u}_{t,t+N_c^i-1}^i \triangleq \mathrm{col}\left[\kappa_f^i(x_t^i), \kappa_f^i(x_{t+1}^i), \ldots, \kappa_f^i(x_{t+N_c^i-1}^i)\right]$ *is an admissible control sequence for the FH optimal control Problem 5.1.1 and for which Points 2 and 3 of Assumption 2 are satisfied. Moreover, suppose*[1] *that* $L_{f_c}^i \neq 1$ *and let* $V^i(x_t^i, w_t^i) \triangleq J_{FH}^i(x_t^i, w_t^i, u_{t,t+N_c^i-1}^{i^{FH^\circ}}, N_c^i, N_p^i)$, $\Omega = X^{i\kappa_f}$, $\alpha_1 = \underline{r}^i$, $\alpha_2(|x_t^i|) = (L_h^i + L_{hu}^i L_{\kappa_f}^i + L_q^i)\dfrac{(L_{f_c}^i)^{N_p^i} - 1}{L_{f_c}^i - 1}|x_t^i| + \beta_{h_f^i}((L_{f_c}^i)^{N_p^i}|x_t^i|)$, $\alpha_3 = \underline{r}^i$, $\sigma_1 = L_{qw}^i \dfrac{(\alpha_w^i)^{N_p^i} - 1}{\alpha_w^i - 1}$, $\sigma_2(|w_t^i|) = \alpha_w^i L_{qw}^i \dfrac{(\alpha_w^i)^{N_p^i} - 1}{\alpha_w^i - 1}|w_t^i| + \psi^i((\alpha_w^i)^{N_p^i - 1}|w_t^i|)$, *and* $\sigma_3 = L_{qw}^i \dfrac{(\alpha_w^i)^{N_p^i} - 1}{\alpha_w^i - 1}$*. The set* W^i *is such that (3.9) is satisfied.* □

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[1] The very special case $L_{f_c}^i = 1$ can be trivially addressed by a few suitable modifications to the proof of Theorem 5.2.1.

The main result can now be stated.

Theorem 5.2.1 *Under Assumptions 1-5, the locally-controlled agent* $\mathcal{A}^i$, $i = 1, \ldots, M$, *whose closed-loop dynamics are described by* (5.6), *subject to constraints* (5.2), (5.3), *and* (1.2), *is ISS with robust output admissible set* $X^{i^{MPC}}$, *where* $X^{i^{MPC}}$ *denotes the set of states of system* (5.1) *for which a solution of the FH optimal control Problem 5.1.1 does exist.* □

Proof 1 *First, by Assumption 2, for any* $x_t^i \in X_f^i$, *the sequence*

$$\tilde{u}^i_{t,t+N_c^i-1} = \operatorname{col}\left[\kappa_f^i(x_t^i), \kappa_f^i(x_{t+1}^i), \ldots, \kappa_f^i(x_{t+N_c^i-1}^i)\right]$$

is an admissible control sequence for the FH optimal control Problem 5.1.1 (see Definition 5.1.1). Then $X^{i^{MPC}} \supseteq X^{i^{\kappa_f}} \supseteq X_f^i$. *By Theorem 3.2.1, if system admits an ISS-Lyapunov function in* $X^{i^{MPC}}$, *then it is ISS in* $X^{i^{MPC}}$.

In this respect, in the following it will be shown that

$$V^i(x_t^i, w_t^i) \triangleq J_{FH}^i(x_t^i, w_t^i, u^{i^{FH^\circ}}_{t,t+N_c^i-1}, N_c^i, N_p^i)$$

is an ISS-Lyapunov function in $X^{i^{MPC}}$. *Moreover, in view of Point 5 of Assumption 2 and Assumptions 3-4*

$$\begin{aligned}
V^i(x_t^i, w_t^i) &\leq J_{FH}^i(x_t^i, w_t^i, \tilde{u}^i_{t,t+N_c^i-1}, N_c^i, N_p^i) \\
&\leq \sum_{l=t}^{t+N_p^i-1} \left[L_h^i |x_l^i| + L_{hu}^i |\kappa_f\left(x_l^i\right)| + L_q^i |x_l^i| + L_{qw}^i |w_l^i|\right] + \\
&\quad + \beta_{h_f^i}(|x_{t+N_p^i}^i|) \\
&\leq \sum_{l=t}^{t+N_p^i-1} \left[(L_h^i + L_{hu}^i L_{\kappa_f}^i + L_q^i)|x_l^i| + L_{qw}^i |w_l^i|\right] + \\
&\quad + \beta_{h_f^i}(|x_{t+N_p^i}^i|)
\end{aligned}$$

so that in view of Point 3 of Assumption 2 and owing to (5.4), *we have*

$$\begin{aligned} V^i(x_t^i, w_t^i) \ \leq & \sum_{l=t}^{t+N_p^i-1} \left[(L_h^i + L_{hu}^i L_{\kappa_f}^i + L_q^i)(L_{f_c}^i)^{l-t} |x_t^i| + L_{qw}^i (\alpha_w^i)^{l-t} |w_t^i| \right] + \\ & + \beta_{h_f^i}((L_{f_c}^i)^{N_p^i} |x_t^i|) \\ \leq & \ (L_h^i + L_{hu}^i L_{\kappa_f}^i + L_q^i) \frac{(L_{f_c}^i)^{N_p^i} - 1}{L_{f_c}^i - 1} |x_t^i| + L_{qw}^i \frac{(\alpha_w^i)^{N_p^i} - 1}{\alpha_w^i - 1} |w_t^i| + \\ & + \beta_{h_f^i}((L_{f_c}^i)^{N_p^i} |x_t^i|) \end{aligned}$$

Hence there exist two $\mathcal{K}_\infty$*-functions* β^i *and* σ^i *such that the following upper bound is verified:*

$$V^i(x_t^i, w_t^i) \leq \beta^i(|x_t^i|) + \sigma^i(|w_t^i|), \ \forall x_t^i \in X^{i\kappa_f}, \forall w_t^i \in W^i \tag{5.7}$$

The lower bound on $V^i(x_t^i, w_t^i)$ *is easily obtained using Assumption 3:*

$$V^i(x_t^i, w_t^i) \geq \underline{r}^i(|x_t^i|), \ \forall x_t^i \in X^i, \forall w_t^i \in W^i \tag{5.8}$$

Now, in view of Assumption 2, it turns out that

$$\bar{u}_{t+1,t+N_c^i}^i \triangleq \operatorname{col}\left(u_{t+1,t+N_c^i-1}^{iFH^\circ}, \kappa_f^i(x_{t+N_c^i}^i)\right) \tag{5.9}$$

is an admissible (in general, suboptimal) control sequence for the FH optimal control Problem 5.1.1 at time $t+1$ *with cost*

$$\begin{aligned} & J_{FH}^i(x_{t+1}^i, w_{t+1}^i, \bar{u}_{t+1,t+N_c^i}, N_c^i, N_p^i) \\ & \quad = V^i(x_t^i, w_t^i) - h^i(x_t^i, u_{t,t}^{iFH^\circ}) - q^i(x_t^i, w_t^i) \\ & \quad + \sum_{l=t+1}^{t+N_p^i-1} \left[h^i(x_l^i, \bar{u}_l^i) + q^i\left(x_l^i, \left(A_w^i\right)^{l-(t+1)} w_{t+1}^i\right) - h^i(x_l^i, u_l^{iFH^\circ}) - q^i\left(x_l^i, \left(A_w^i\right)^{l-t} w_t^i\right) \right] \\ & \quad + h^i(x_{t+N_p^i}^i, \kappa_f^i(x_{t+N_p^i}^i)) + q^i\left(x_{t+N_p^i}^i, \left(A_w^i\right)^{N_p^i-1} w_{t+1}^i\right) \\ & \quad + h_f^i(f^i(x_{t+N_p^i}^i, \kappa_f^i(x_{t+N_p^i}^i))) - h_f^i(x_{t+N_p^i}^i) \end{aligned}$$

Noting that, using Assumption 4

$$\begin{aligned}
&q^i(x_l^i, (A_w^i)^{l-(t+1)} w_{t+1}^i) - q^i(x_l^i, (A_w^i)^{l-t} w_t^i) \\
&\quad \leq \left| q^i(x_l^i, (A_w^i)^{l-(t+1)} w_{t+1}^i) - q^i(x_l^i, (A_w^i)^{l-t} w_t^i) \right| \\
&\quad \leq L_{qw}^i \left| (A_w^i)^{l-(t+1)} w_{t+1}^i - (A_w^i)^{l-t} w_t^i \right| \\
&\quad = L_{qw}^i \left(\alpha_w^i\right)^{l-(t+1)} \left| w_{t+1}^i - A_w^i w_t^i \right| \\
&\quad \leq L_{qw}^i \left(\alpha_w^i\right)^{l-(t+1)} \left(\left| w_{t+1}^i \right| + \alpha_w^i \left| w_t^i \right| \right) ,
\end{aligned}$$

and by using Point 6 of Assumption 2, we obtain

$$\begin{aligned}
&J_{FH}^i(x_{t+1}^i, w_{t+1}^i, \bar{u}_{t+1,t+N_c^i}, N_c^i, N_p^i) \\
&\quad \leq V^i(x_t^i, w_t^i) - h^i(x_t^i, u_{t,t}^{iFH^\circ}) - q^i(x_t^i, w_t^i) \\
&\qquad + \sum_{l=t+1}^{t+N_p^i-1} L_{qw}^i \left(\alpha_w^i\right)^{l-(t+1)} \left(\left| w_{t+1}^i \right| + \alpha_w^i \left| w_t^i \right| \right) + \psi^i(\left| \left(A_w^i\right)^{N_p^i-1} w_t^i \right|) \\
&\quad \leq V^i(x_t^i, w_t^i) - h^i(x_t^i, u_{t,t}^{iFH^\circ}) - q^i(x_t^i, w_t^i) + \varphi_1^i(|w_t^i|) + \varphi_2^i(|w_{t+1}^i|) ,
\end{aligned}$$

where:

$$\begin{aligned}
\varphi_1^i(|w_t^i|) &\triangleq \alpha_w^i L_{qw}^i \frac{(\alpha_w^i)^{N_p^i} - 1}{\alpha_w^i - 1} |w_t^i| + \psi^i(\left(\alpha_w^i\right)^{N_p^i-1} |w_t^i|), \\
\varphi_2^i &\triangleq L_{qw}^i \frac{(\alpha_w^i)^{N_p^i} - 1}{\alpha_w^i - 1}
\end{aligned}$$

are $\mathcal{K}_\infty$-functions.

Now, from inequality

$$V^i(x_{t+1}^i, w_{t+1}^i) \leq J_{FH}^i(x_{t+1}^i, w_{t+1}^i, \bar{u}_{t+1,t+N_c^i}, N_c^i, N_p^i)$$

it follows that

$$
\begin{aligned}
&V^i(x^i_{t+1}, w^i_{t+1}) - V^i(x^i_t, w^i_t) \\
&\qquad \leq -\underline{r}^i(|x_t|) + \varphi^i_1(|w^i_t|) + \varphi^i_2(|w^i_{t+1}|),\ \forall x^i_t \in X^i,\ \forall w^i \in W^i \quad (5.10)
\end{aligned}
$$

Finally, in view of the admissible control sequence (5.9), *it follows that* $X^{i^{MPC}}$ *is a robust positively invariant set for the closed loop (5.6). Therefore, by* (5.8), (5.7), (5.10) *and Assumption 5, the optimal cost* $J^i_{FH}(x^i_t, w^i_t, u^{i^{FH^\circ}}_{t,t+N^i_c-1}, N^i_c, N^i_p)$ *is an ISS-Lyapunov function for the closed-loop system* (5.6) *in* $X^{i^{MPC}}$ *and hence, owing to Assumption 1, the closed-loop system is ISS in* $X^{i^{MPC}}$.

It is worth noting that, from the perspective of determining regionally ISS stabilizing control laws, a key aspect is the design of an auxiliary control law $\kappa^i_f(x^i)$ such that Assumption 2 holds. In this respect, under slightly more restrictive hypotheses on the agents' dynamic models and on the FH cost function, we give the following useful result (the proof is reported in the Appendix).

Lemma 5.2.1 *Assume that* $f^i \in C^2$, $h^i(x^i, u^i) = x^{i^\top} Q^i x^i + u^{i^\top} R^i u^i$ *and* $q^i(x^i, w^i) \leq x^{i^\top} \tilde{S}^i x^{i^\top} + \psi^i(|w^i|)$ *with* Q^i, R^i, *and* $\tilde{S}^i$ *being positive definite matrices and* ψ^i *being a* $\mathcal{K}$*-function.*
Furthermore, suppose that there exists a matrix K^i *such that* $A^i_{cl} = A^i + B^i K^i$ *is stable with* $A^i \triangleq \left.\dfrac{\partial f^i}{\partial x^i}\right|_{x^i=0;\, u^i=0}$, $B^i \triangleq \left.\dfrac{\partial f^i}{\partial u^i}\right|_{x^i=0;\, u^i=0}$. *Let* $\tilde{Q}^i \triangleq \beta^i(Q^i + K^{i^\top} R^i K^i + \tilde{S}_i)$ *with* $\beta^i > 1$, *and denote by* Π^i *the unique symmetric positive definite solution of the following Lyapunov equation:*

$$
A^{i^\top}_{cl} \Pi^i A^i_{cl} - \Pi^i + \tilde{Q}^i = 0. \tag{5.11}
$$

Then, there exist a constant $\Upsilon^i \in \mathbb{R}_{\geq 0}$, $\Upsilon^i > 0$ *and a finite integer* $\bar{N}_p$ *such that for all* $N_p \geq \bar{N}_p$ *the final set* $X^i_f \triangleq \left\{x^i \in \mathbb{R}^{n^i} : x^{i^\top} \Pi^i x^i \leq \Upsilon^i\right\}$ *satisfies Assumption 2 with* $\kappa^i_f(x^i) = K^i x^i$, $h^i_f = x^{i^\top} \Pi^i x^i$. □

In the next subsection, the stability analysis of the whole team of agents will be addressed.

5.2.1 Stability properties of the team of agents

In this subsection, the coupling effects due to the exchange of the delayed state information between the cooperating DMs will be taken into account in the context of the stability analysis of the whole team of agents. In this respect, let us consider the team $\mathcal{A} = \{\mathcal{A}^i,\, i = 1, ..., M\}$ where each cooperating agent $\mathcal{A}^i$ is controlled by the regionally ISS–stabilizing RH control local law solving Problem 5.1.1 for each $i = 1, \dots, M$.

We can indeed follow the same procedure as in Section 4.3, and describe the team of DMs as a suitable feedback interconnection of two systems: one accounting for the agents dynamics (now coupled through the control action) and one accounting for the delays. Hence the following state equations can be written:

$$\mathcal{X}_{t+1} = \tilde{F}(\mathcal{X}_t, \bar{\mathcal{W}}_t)\,, \tag{5.12}$$

$$\begin{cases} \mathcal{Z}_{t+1} = A\,\mathcal{Z}_t + B\,\mathcal{X}_t \\ \bar{\mathcal{W}}_t = C\,\mathcal{Z}_t, \end{cases} \tag{5.13}$$

where these equations correspond respectively to (4.8) and (4.9) in Section 4.3.

We will now show that an ISS-Lyapunov function can be defined for each of these systems, which implies that both will turn out to be regionally ISS. After this step, the stability properties of the team of DMs will be analyzed by resorting to *nonlinear small–gain theorem* arguments. First, we let $\mathcal{W} \triangleq W^1 \times \cdots \times W^M$, $\mathcal{X} \triangleq X^1 \times \cdots \times X^M$, $\mathcal{X}_f \triangleq X_f^1 \times \cdots \times X_f^M$, $\mathcal{X}^{MPC} \triangleq {X^1}^{MPC} \times \cdots \times {X^M}^{MPC}$, $D \triangleq D^1 \times \cdots \times D^M$ and $\hat{\mathcal{W}} \triangleq \max\limits_{\bar{\mathcal{W}}}\{|\bar{\mathcal{W}}| : \bar{\mathcal{W}} \in \mathcal{W}\}$. The following intermediate result can now be proved.

Lemma 5.2.2 *Under Assumptions 1-5, dynamic systems* (5.12) *and* (5.13) *are provided with suitable ISS Lyapunov functions* $V(\mathcal{X}_t, \bar{\mathcal{W}}_t)$ *in* $\mathcal{X}^{MPC}$ *and* $V^D(\mathcal{Z}_t)$ *in* $\mathbb{R}^{nz}$*, respectively.*

□

Proof 2 *This proof is analogous to that of Lemma 4.3.1. Let us consider the*

ISS Lyapunov function candidate

$$V(\mathcal{X}_t,\bar{\mathcal{W}}_t)\triangleq\sum_{i=1}^{M}V^i(x_t^i,w_t^i)$$

for system (5.12).[2] *From* (5.8) *and* (5.7), *it follows that*

$$\sum_{i=1}^{M}\underline{r}^i(|x_t^i|)\leq V(\mathcal{X}_t,\bar{\mathcal{W}}_t)\leq\sum_{i=1}^{M}\beta^i(|x_t^i|)+\sum_{i=1}^{M}\sigma^i(|w_t^i|)$$

Clearly $|x_t^i|\leq|\mathcal{X}_t|$ *and* $|w_t^i|\leq|\bar{\mathcal{W}}_t|$, $\forall i=1,\dots,M$ *and thus*

$$\begin{aligned}V(\mathcal{X}_t,\bar{\mathcal{W}}_t)\leq\sum_{i=1}^{M}\beta^i(|x_t^i|)+\sum_{i=1}^{M}\sigma^i(|w_t^i|)\leq\sum_{i=1}^{M}\beta^i(|\mathcal{X}_t|)+\sum_{i=1}^{M}\sigma^i(|\bar{\mathcal{W}}_t|)\leq\\ \leq\beta(|\mathcal{X}_t|)+\sigma(|\bar{\mathcal{W}}_t|)\,,\end{aligned}$$

where we set $\beta(|\mathcal{X}_t|)\triangleq\sum_{i=1}^{M}\beta^i(|\mathcal{X}_t|)$ *and* $\sigma(|\bar{\mathcal{W}}_t|)\triangleq\sum_{i=1}^{M}\sigma^i(|\bar{\mathcal{W}}_t|)$.

Moreover $\sum_{i=1}^{M}|x_t^i|\leq\sum_{i=1}^{M}|\mathcal{X}_t|=M|\mathcal{X}_t|$.

Then $|\mathcal{X}_t|\geq\frac{1}{M}\sum_{i=1}^{M}|x_t^i|$ *and* $|\mathcal{X}_t|\leq\sum_{i=1}^{M}|x_t^i|$. *Now, recall that, for any* $\mathcal{K}$ *function* γ, *we have* $\gamma\left(\sum_{i=1}^{M}a_i\right)\leq\sum_{i=1}^{M}\gamma(Ma_i)$ *where* $a_i>0, i=1,\dots,M$ *are arbitrarily chosen positive scalars). Therefore, considering the* $\mathcal{K}$ *function* $\underline{r}^i$, *for a generic* $i\in\{1,\dots,M\}$, *we have*

$$\underline{r}^i(|\mathcal{X}_t|)\leq\underline{r}^i\left(\sum_{i=1}^{M}|x_t^i|\right)\leq\sum_{i=1}^{M}\underline{r}^i(M|x_t^i|)\leq\sum_{i=1}^{M}\underline{r}^i(M|\mathcal{X}_t|)$$

---

[2] It is worth noting that, instead of the above definition of V, a weighted sum of Lyapunov functions could be used along the reasoning provided in [34] in the framework of composite systems.

and hence

$$\underline{r}^i\left(|\mathcal{X}_t|/M\right) \leq \underline{r}^i\left(\frac{1}{M}\sum_{i=1}^{M}|x_t^i|\right) \leq \sum_{i=1}^{M}\underline{r}^i(|x_t^i|).$$

Therefore, letting $\underline{r}\left(|\mathcal{X}_t|\right) \triangleq \underline{r}^i\left(|\mathcal{X}_t|/M\right)$ *for an arbitrarily chosen index* i, *we showed that*

$$\underline{r}(|\mathcal{X}_t|) \leq V(\mathcal{X}_t, \bar{\mathcal{W}}_t),\ \forall \mathcal{X}_t \in \mathcal{X},\ \forall \bar{\mathcal{W}}_t \in \mathcal{W} \tag{5.14}$$

$$V(\mathcal{X}_t, \bar{\mathcal{W}}_t) \leq \beta(|\mathcal{X}_t|) + \sigma(|\bar{\mathcal{W}}_t|),\ \forall \mathcal{X}_t \in \mathcal{X}_f,\ \forall \bar{\mathcal{W}}_t \in \mathcal{W} \tag{5.15}$$

From (5.10) *it follows that*

$$\Delta V \triangleq \sum_{i=1}^{M} V^i(x_{t+1}^i, w_{t+1}^i) - \sum_{i=1}^{M} V^i(x_t^i, w_t^i) \leq$$
$$\leq -\sum_{i=1}^{M}\underline{r}^i(|x_t|) + \sum_{i=1}^{M}\varphi_1^i(|w_t^i|) + \sum_{i=1}^{M}\varphi_2^i(|w_{t+1}^i|)$$

Moreover, $-\sum_{i=1}^{M}\underline{r}^i(|x_t|) \leq -\underline{r}^i\left(|\mathcal{X}_t|/M\right)$ *and* $\sum_{i=1}^{M}\varphi_1^i(|w_t^i|) \leq \sum_{i=1}^{M}\varphi_1^i(|\bar{\mathcal{W}}_t|)$ *and* $\sum_{i=1}^{M}\varphi_2^i(|w_{t+1}^i|) \leq \sum_{i=1}^{M}\varphi_2^i(|\bar{\mathcal{W}}_{t+1}|)$.

Then, letting $\varphi_1(|\bar{\mathcal{W}}_t|) \triangleq \sum_{i=1}^{M}\varphi_1^i(|W_t^i|)$ *and* $\varphi_2(|\bar{\mathcal{W}}_{t+1}|) \triangleq \sum_{i=1}^{M}\varphi_2^i(|W_{t+1}^i|)$, *it follows that*

$$\Delta V \leq -\underline{r}\left(|\mathcal{X}_t|\right) + \varphi_1(|\bar{\mathcal{W}}_t|) + \varphi_2(|\bar{\mathcal{W}}_{t+1}|) \ \leq\ -\underline{r}\left(|\mathcal{X}_t|\right) + \varphi_{tot}(\|\bar{\mathcal{W}}\|_{t+1}), \tag{5.16}$$

$\forall\, \mathcal{X}_t \in \mathcal{X}, \forall\, \bar{\mathcal{W}} \in \mathcal{M}_{\mathcal{W}}$, *where* $\varphi_{tot}(s) \triangleq \varphi_1(s) + \varphi_2(s)$. *Therefore, by (5.14), (5.15) and (5.16),* $V(\mathcal{X}_t, \bar{\mathcal{W}}_t)$ *is an ISS-Lyapunov function in* $\mathcal{X}^{MPC}$ *for system* (5.12) *and hence this system is ISS in* $\mathcal{X}^{MPC}$.

As far as system (5.13) *is concerned (we recall that this system describes the effects of the time–delays in the information exchange variables), the proof that it is ISS is obviously trivial since* (5.13) *is an asymptotically stable discrete-*

time linear system. We only very briefly sketch some parts of the proof just for the purpose of introducing a few quantities that will be used subsequently. A candidate ISS Lyapunov function for system (5.13) *is* $V^D(\mathcal{Z}_t) \triangleq |\mathcal{Z}_t|$. *It is immediate to determine two* $\mathcal{K}$*−functions* $\underline{r}^D(|\mathcal{Z}_t|)$ *and* $\beta^D(|\mathcal{Z}_t|)$ *such that*

$$\underline{r}^D(|\mathcal{Z}_t|) \leq V^D(\mathcal{Z}_t) \leq \beta^D(|\mathcal{Z}_t|)$$

Moreover, it is straightforward to obtain

$$\Delta V^D \triangleq V^D(\mathcal{Z}_{t+1}) - V^D(\mathcal{Z}_t) = |A\mathcal{Z}_t + B\mathcal{X}_t| - |\mathcal{Z}_t| \leq -\underline{r}^D(|\mathcal{Z}_t|) + \varphi_1^D(|\mathcal{X}_t|) \tag{5.17}$$

with a suitable definition of the $\mathcal{K}$*−function* φ_1^D.

Now, recalling from (5.13) that $\bar{\mathcal{W}}_t = C\,\mathcal{Z}_t$, from the proof of Lemma 5.2.2, from (5.16) it follows immediately that the ISS Lyapunov function $V(\mathcal{X}_t, \bar{\mathcal{W}}_t)$ satisfies

$$\begin{aligned} V(\mathcal{X}_{t+1}, \bar{\mathcal{W}}_{t+1}) &- V(\mathcal{X}_t, \bar{\mathcal{W}}_t) \leq -\alpha_4(V(\mathcal{X}_t, \bar{\mathcal{W}}_t)) + \varphi_1(|\bar{\mathcal{W}}_t|) + \varphi_2(|\bar{\mathcal{W}}_{t+1}|) \\ &\leq -\alpha_4(V(\mathcal{X}_t, \bar{\mathcal{W}}_t)) + \varphi_w(V^D(\mathcal{Z}_t)) + \varphi_{w+1}(V^D(\mathcal{Z}_{t+1})) \\ &\leq -\alpha_4(V(\mathcal{X}_t, \bar{\mathcal{W}}_t)) + \varphi_{w_{tot}}(\|V^D(\mathcal{Z})\|_{t+1}), \\ &\qquad \forall \mathcal{X}_t \in \mathcal{X}_f,\ \forall \mathcal{Z} \in \mathcal{M}_{\mathbb{R}^{n_\mathcal{Z}}} \end{aligned} \tag{5.18}$$

where α_4 is defined as in the proof of Theorem 5.2.1, whereas $\varphi_{11} \triangleq \varphi_1 + \varepsilon$, $\varphi_w \triangleq \varphi_1 \circ (\underline{r}^D)^{-1}$, $\varphi_{w+1} \triangleq \varphi_2 \circ (\underline{r}^D)^{-1}$ and $\varphi_{w_{tot}}(s) \triangleq \varphi_w(s) + \varphi_{w+1}(s)$. Moreover, as far as the ISS Lyapunov function $V^D(\mathcal{Z}_t)$ is concerned, from (5.17) it follows that

$$V^D(\mathcal{Z}_{t+1}) - V^D(\mathcal{Z}_t) \leq -\alpha_4^D(V^D(\mathcal{Z}_t)) + \varphi_w^D(V(\mathcal{X}_t, \bar{\mathcal{W}}_t)) \tag{5.19}$$

where, again, α_4^D is defined analogously to the above definition of α_4, whereas $\varphi_w^D \triangleq \varphi_1^D \circ (\underline{r}^D)^{-1}$.

Analogously to the proof of Theorem 3.2.1, given $e \in \mathbb{R}_{\geq 0}$, let $R(e) \triangleq \{\mathcal{X} : V(\mathcal{X}, \bar{\mathcal{W}}) \leq e, \forall \bar{\mathcal{W}} \in \mathcal{W}\}$. Let $\Theta \triangleq \{\mathcal{X} : V(\mathcal{X}, \bar{\mathcal{W}}) \leq \bar{e} = \max\limits_{R(e) \subseteq \mathcal{X}_f} e, \forall \bar{\mathcal{W}} \in \mathcal{W}\}$. Note that $\bar{e} > b(\hat{\mathcal{W}})$ and $D \subset \Theta$. Since the region D is reached asymptotically, the state will arrive in Θ in a finite time, that is, there exists T_θ such that $V(\mathcal{X}_k, \bar{\mathcal{W}}_k) \leq \bar{e}, \forall k \geq T_\theta$. Hence, the region Θ is a robust positively invariant set for the system (5.1). Thanks to Remark 3.7 in [30], from (5.18) and (5.19) it follows that there exist some $\mathcal{KL}$-functions $\hat{\beta}$ and $\hat{\beta}^D$ such that

$$V(\mathcal{X}_k, \bar{\mathcal{W}}_k) \leq \max\{\hat{\beta}(V(\mathcal{X}_t, \bar{\mathcal{W}}_t), k), \gamma_1(\|V^D(\mathcal{Z})\|_k)\}, \ \forall \mathcal{X}_t \in \Theta, \ \forall k \in \mathbb{Z}_{\geq 0}, \ k \geq t \tag{5.20}$$

$$V^D(\mathcal{Z}_k) \leq \max\{\hat{\beta}^D(V^D(\mathcal{Z}_t), k), \gamma_2(\|V(\mathcal{X}, \bar{\mathcal{W}})\|_k)\}, \ \forall \mathcal{X} \in \mathcal{X}, \ \forall k \in \mathbb{Z}_{\geq 0}, \ k \geq t \tag{5.21}$$

where we define

$$\gamma_1 \triangleq \alpha_4^{-1} \circ \rho^{-1} \circ \varphi_{w_{tot}} \tag{5.22}$$

and

$$\gamma_2 \triangleq (\alpha_4^D)^{-1} \circ \rho^{-1} \circ \varphi_w^D \tag{5.23}$$

with ρ any $\mathcal{K}_\infty$-function such that $(id - \rho) \in \mathcal{K}_\infty$.

Now, the following result about the stability properties of the team of cooperating agents can be proved.

Theorem 5.2.2 *Suppose that Assumptions 1-5 are verified. Moreover, assume that the following small gain condition holds:*

$$\gamma_1 \circ \gamma_2(s) < s. \tag{5.24}$$

with γ_1 *and* γ_2 *given by* (5.22) *and* (5.23) *and argument* s *takes its values from a suitable subset of* $\mathbb{R}_{\geq 0}$ *according to inequalities* (5.18)–(5.21)*. Then the team of cooperating agents described by the interconnected dynamic equations* (5.12) *and* (5.13) *is 0-AS in* $\mathcal{X}^{MPC} \times \mathbb{R}^{n_\mathcal{Z}}$. □

Proof 3 *If* $\gamma_1 \circ \gamma_2(s) < s$, *from* (5.20) *and* (5.21) *it follows that*

$$V(\mathcal{X}_k, \bar{\mathcal{W}}_k) \leq \max\{\hat{\beta}(V(\mathcal{X}_t, \bar{\mathcal{W}}_t), t), \gamma_1(\hat{\beta}^D(V^D(\mathcal{Z}_t), t))\},$$
$$\forall \mathcal{X}_t \in \Theta, \ \forall k \in \mathbb{Z}_{\geq 0}, \ k \geq t$$

$$V^D(\mathcal{Z}_k) \leq \max\{\hat{\beta}^D(V^D(\mathcal{Z}_t), t), \gamma_2(\hat{\beta}(V(\mathcal{X}_t, \bar{\mathcal{W}}_t), t))\},$$
$$\forall \mathcal{X}_t \in \Theta, \ \forall k \in \mathbb{Z}_{\geq 0}, \ k \geq t$$

and hence $V(\mathcal{X}_k, \bar{\mathcal{W}}_k)$, $V^D(\mathcal{Z}_k)$ *are bounded by initial condition. By Lemma 3.13 in [30], an asymptotic gain from* $V^D(\mathcal{Z}_k)$ *to* $V(\mathcal{X}_k, \bar{\mathcal{W}}_k)$ *is given by* γ_1

whereas an asymptotic gain from $V(\mathcal{X}_k, \bar{\mathcal{W}}_k)$ *to* $V^D(\mathcal{Z}_k)$ *is given by* γ_2. *Hence:*

$$\begin{align} \overline{\lim_{k\to\infty}} V(\mathcal{X}_k, \bar{\mathcal{W}}_k) &\leq \overline{\lim_{k\to\infty}} [\alpha_4^{-1} \circ \rho^{-1} \circ \varphi_w(V^D(\mathcal{Z}_{k-1})) + \nonumber \\ &\quad +\alpha_4^{-1} \circ \rho^{-1} \circ \varphi_{w+1}(V^D(\mathcal{Z}_k))] \tag{5.25} \\ &\leq \alpha_4^{-1} \circ \rho^{-1} \circ \varphi_w(\overline{\lim_{k\to\infty}} V^D(\mathcal{Z}_{k-1})) + \tag{5.26} \\ &\quad +\alpha_4^{-1} \circ \rho^{-1} \circ \varphi_{w+1}(\overline{\lim_{k\to\infty}} V^D(\mathcal{Z}_k)) \nonumber \end{align}$$

But

$$\overline{\lim_{k\to\infty}} V^D(\mathcal{Z}_{k-1}) = \overline{\lim_{k\to\infty}} V^D(\mathcal{Z}_k)$$

Hence

$$\begin{aligned} \overline{\lim_{k\to\infty}} V(\mathcal{X}_k, \bar{\mathcal{W}}_k) &\leq \alpha_4^{-1} \circ \rho^{-1} \circ \varphi_{w_{tot}}(\overline{\lim_{k\to\infty}} V^D(\mathcal{Z}_k)) \\ &= \gamma_1(\overline{\lim_{k\to\infty}} V^D(\mathcal{Z}_k)) \\ &\leq \gamma_1 \circ \gamma_2(\overline{\lim_{k\to\infty}} V(\mathcal{X}_k, \bar{\mathcal{W}}_k)) \end{aligned}$$

Again, the assumption that $\gamma_1 \circ \gamma_2(s) < s$ *implies that*

$$\overline{\lim_{k\to\infty}} V^D(\mathcal{Z}_k) = \overline{\lim_{k\to\infty}} V(\mathcal{X}_k, \bar{\mathcal{W}}_k) = 0$$

Thus, the system is 0-AS in $\mathcal{X}^{MPC} \times \mathbb{R}^{nz}$.

Remark 5.2.1 *It is worth noting that the small-gain condition* (5.24) *may turn out to be conservative in practice as it is typical of these kind of results. On the other hand, the generality of the problem makes it rather difficult to obtain tighter conditions without introducing more restrictive assumptions on the structure of the agents' dynamics and on the cost function. Indeed, for special classes of cooperative control problems, different conditions for the stability of the team of DMs can be obtained. For instance, we recall that in [15] stability has been shown for formation control of UAV's under different hypotheses as the knowledge of the neighbors dynamics, suitably fast information exchange and bounded error between the predicted and actuated state trajectories of each member of the team. As another example, stability of a set of decoupled systems is ensured in [32], by assuming the knowledge of feasibility regions and a* *specific hierarchical design of the decentralized RH control problem: the computations are shared by nodes with different priorities, which can impose their control decisions on the subordinate neighbors.*

Remark 5.2.2 *As expected, in the special case where the state equation* (5.1) *takes on a linear structure, the FH cost function* (5.5) *is quadratic, and no state and control constraints are present, more specialized and tight results can be found. In particular, the control law takes on an* explicit feedback–feedforward structure *and some interesting properties hold. The reader is referred to [20] for more details.*

5.3 Illustrative example

In this section we will show some simulation results concerning a team of UAVs moving in $\mathbb{R}^2$ with nonlinear dynamics; this is the same example already introduced at Chapter 1, with equation (1.3). Such a problem has reasonable simplicity but allows to ascertain the basic features and properties of the proposed cooperative control law. A team of $M = 3$ vehicles will be considered, whose continuous-time models and data are taken according to [31]:

$$\begin{aligned} m\ddot{x}^i &= -\mu_1 \dot{x}^i + (u_R^i + u_L^i)\cos(\theta^i), \\ m\ddot{y}^i &= -\mu_1 \dot{y}^i + (u_R^i + u_L^i)\sin(\theta^i), \\ J\ddot{\theta}^i &= -\mu_2 \dot{\theta}^i + (u_R^i - u_L^i) r_v. \end{aligned} \tag{5.27}$$

where $i = 1, 2, 3$. For simplicity, we assume that all the members of the team have the same physical parameters: the mass is $m = 0.75\, Kg$, the inertia is $J = 0.00316\, Kgm^2$, the linear friction coefficient is $\mu_1 = 0.15\, Kg/s$ and the rotational friction coefficient is $\mu_2 = 0.005\, Kgm^2/s$ and finally the radius of the vehicle is $r_v = 8.9cm$. The state vector of each DM will be from now on denoted as z^i, and is defined by considering the position and velocity in each direction of the plane, plus the orientation angle and rotational velocity $z^i \triangleq \mathrm{col}\,(\theta^i, \dot{\theta}^i, x^i, \dot{x}^i, y^i, \dot{y}^i)$, whereas the control vector is given by $u^i \triangleq \mathrm{col}\,(u_L^i, u_R^i)$. The continuous-time models (5.27) are discretized with a sampling time $T = 0.1s$, thus obtaining suitable discrete-time models, where the state vectors are denoted by z_t^i and the control vectors are denoted by u_t^i.

Remark 5.3.1 *In the following, the simulation trials will refer to the above approximated discrete-time model for mere illustration purposes and to show the effectiveness of the proposed cooperative control scheme. However, as shown in [59], in some cases the control law that stabilizes the approximated discrete-time model may perform quite poorly when applied to the exact model. This is clearly an important issue and we refer the reader to the above reference for more details and to the works [62, 63] for the general case of control of non-linear sampled-data systems. For a MPC algorithm where the continuous time evolution of the system is explicitly taken into account, while the optimization is performed with respect to a piece-wise constant control signal, see [52].* □

The objective of the distributed cooperative controller is to reach a certain formation following a predefined desired trajectory for each UAV. The desired trajectories have been chosen with constant velocities and null rotational velocity. At every time instant t, each agent solves Problem 5.1.1 with FH cost function

$$\begin{aligned} J^i_{FH} = & \sum_{l=t}^{t+N^i_p-1} \left(|z^i_l - \bar{z}^1_l + d^{i1}|^2_{Q^i} + |u^i_l - \bar{u}^i|^2_{R^i} \right) + |z^i_{t+N^i_p} - \bar{z}^1_{t+N^i_p} + d^{i1}|^2_{P^i} \\ & + \sum_{l=t}^{t+N^i_p-1} \sum_{j \in G^i} |z^i_l - \tilde{z}^j_l + d^{ij}|^2_{S^{ij}} \end{aligned} \tag{5.28}$$

where $\bar{z}^1_l$ represents the desired trajectory of the leader while d^{ij} are the desired distance between agent i and agent j ($d^{ii} = 0,\ \forall i = 1, \dots, M$). Hence the term $\bar{z}^1_l - d^{i1}$ represents the desired trajectory of the $i - th$ UAV. The values of d^{ij} are such that the three UAVs assume a triangle formation. The term $\bar{u}^i$ is the control vector necessary in order to maintain each UAV on the desired trajectory. For the information vector to take on a constant value within the prediction horizon, we let

$$\tilde{z}^j_{t+k} = (\bar{z}^1_{t+k} - d^{j1}) + (z^j_{t-\Delta^{ij}} - \bar{z}^1_{t-\Delta^{ij}} + d^{j1}) .$$

The delays have all been set to $\Delta^{ij} = \Delta = 5T$ and the communication topology is assumed to be stationary. Specifically, we suppose that the leader does not receive any information from the other DMs (hence $S^{1j} = 0,\ \forall j \in G^1$). Moreover agent 2 gets information from the leader and from agent 3 and, analogously, agent 3 gets information from the leader and from agent 2.

The values of the parameters used for the leader are $N_c^1 = N_p^1 = 5$, $Q^1 = 0.1 \cdot \mathrm{diag}\,(1, 50, 1, 1, 1, 1)$, $R^1 = 0.01 \cdot \mathrm{diag}\,(1, 1)$, and $S^{1j} = 0, \forall j \in G^1$. The lengths of horizons N_c^1, N_p^1, though quite small, are indeed sufficient for the leader to show a reasonably good tracking performance as it starts quite close to the desired trajectory. For the other agents, we consider the same values of the parameters, that is, we have $N_c^i = 10$, $N_p^i = 250$, $Q^i = 0.1 \cdot \mathrm{diag}\,(1, 50, 1, 1, 1, 1)$, $R^i = 0.01 \cdot \mathrm{diag}\,(1, 1)$, $S^{ij} = \mathrm{diag}\,(0.1, 0.1, 1, 0.1, 1, 0.1)$, $\alpha_w = 0.96$, $i = 2, 3$. The matrices P^i are obtained, from the choice of Q^i, R^i and S^{ij}, by the auxiliary control law designed according to Lemma 5.2.1 using $\beta^i = 3$ and $\tilde{S}^{ij} = 2S^{ij}$, $i = 1, 2, 3$. The FH Optimal Control Problem 5.1.1 is characterized by the constraints $u^i_{Lmin} \le u_t^{1i} \le u^i_{Rmax}$; $u^i_{Lmin} \le u_t^{2i} \le u^i_{Rmax}$, with $u^i_{Lmin} = 0$, $u^i_{Lmax} = 6$, $u^i_{Rmin} = 0$, and $u^i_{Rmax} = 6$, $i = 1, 2, 3$, where u_t^{1i} (u_t^{2i}) denotes the first (second) component of vector u_t^i. Moreover, the terminal constraints $|z^i_{t+N_p^i} - \bar{z}^1_{t+N_p^i} + d^{i1}|^2_{P^i} \le \Upsilon^i$, $i = 1, 2, 3$, have been obtained numerically according to Lemma 5.2.1. The values of Υ^i are constant along the trajectories and are respectively $\Upsilon^1 = 0.3$ and $\Upsilon^i = 1.2$, $i = 2, 3$. These values are not comparable since the matrices P^i are different. The control necessary in order to maintain each UAV on the desired trajectory is $\bar{u}^{1i} = 1$, $\bar{u}^{2i} = 1$. The values of the desired distances between the agents are the following:

$$d^{12} = 16col(0, 0, -\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) - 0.5\cos(\frac{\pi}{4}), 0, -\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) + 0.5\cos(\frac{\pi}{4}), 0)\,,$$

$$d^{13} = 16col(0, 0, -\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) + 0.5\cos(\frac{\pi}{4}), 0, -\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) - 0.5\cos(\frac{\pi}{4}), 0)\,,$$

$$d^{21} = 16col(0, 0, +\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) + 0.5\cos(\frac{\pi}{4}), 0, +\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) - 0.5\cos(\frac{\pi}{4}), 0)\,,$$

$$d^{23} = 16col(0, 0, \cos(\frac{\pi}{4}), 0, -\cos(\frac{\pi}{4}), 0)$$

$$d^{31} = 16col(0, 0, \sin(\frac{\pi}{3}\cos(\frac{\pi}{4}) - 0.5\cos(\frac{\pi}{4}), 0, +\sin(\frac{\pi}{3})\cos(\frac{\pi}{4}) + 0.5\cos(\frac{\pi}{4}), 0)$$

and

$$d^{32} = 16col(0, 0, -\cos(\frac{\pi}{4}), 0, \cos(\frac{\pi}{4}), 0)\,.$$

Moreover, the initial condition of the desired trajectory of the leader is:

$$\bar{z}_0^1 = \mathrm{col}\,(\frac{\pi}{4}, 0, 0, \frac{1}{m}(\bar{u}^{1i} + \bar{u}^{2i})\cos(\frac{\pi}{4}), 0, \frac{1}{m}(\bar{u}^{1i} + \bar{u}^{2i})\sin(\frac{\pi}{4}))\,.$$

The entire desired leader's trajectory is obtained, starting from the initial conditions, holding constant the velocities. Finally, the initial conditions of the UAVs are $z_0^1 = \bar{z}_0^1$, $z_0^2 = \bar{z}_0^2$, $z_0^3 = \bar{z}_0^2 + 3.8d^{23}$.

In Fig. 5.1, the team trajectories are reported in the two-dimensional space: the objective is to attain a triangle formation along a line of 45° as followers of the leader. The dotted lines depict the actual behavior of the agents. It is worth noting the cooperative behavior of the DMs when the two followers get closer to each other.

In Fig. 5.2, the behaviors of the control variables of Agents 2 and 3 are shown. In particular, in Figs. 5.2(a) and 5.2(c) the behaviors of the first component of the control variables are plotted, whereas in Figs. 5.2(b) and 5.2(d), the difference between the first and the second components of the control variables are shown. This has been done to better appreciate the differences between the first and the second components of the control variables; actually, these differences are rather small due to the small magnitude of the variations of the orientation of the two agents. In Fig. 5.2, the dashed lines depict the constraints imposed on the control variables.

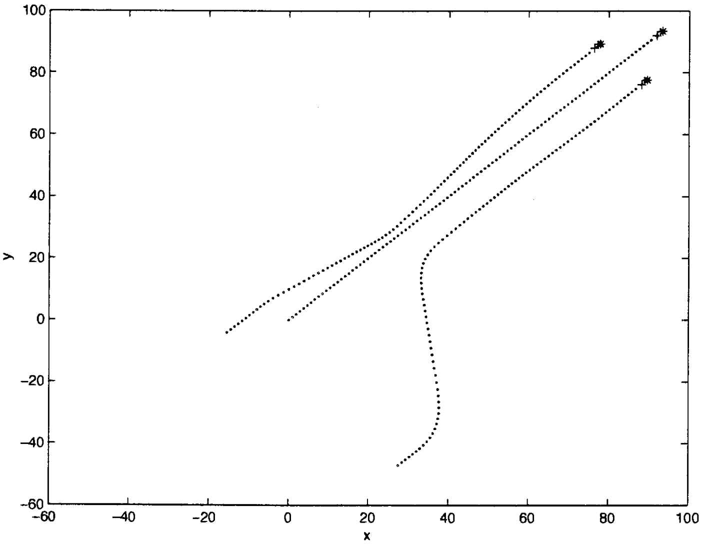

Figure 5.1: Team trajectories (dotted lines). The front of the vehicle is represented by the symbol '*' whereas the back of the vehicle is represented by the symbol '+'.

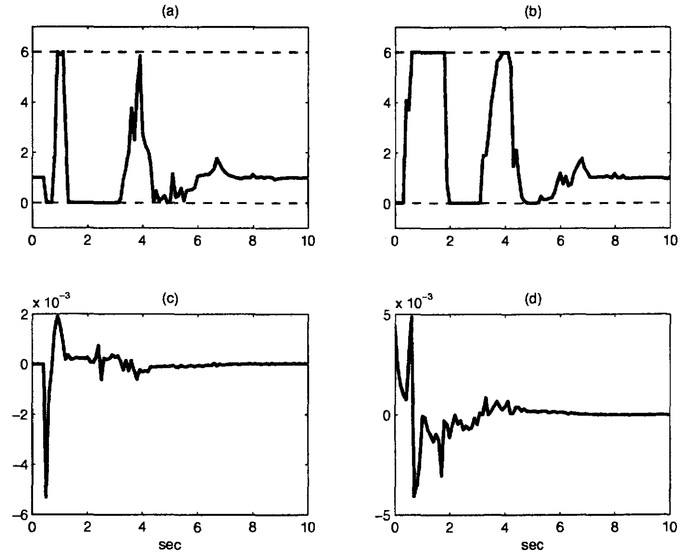

Figure 5.2: Behaviors of the control variables of Agents 2 and 3. (a) and (c) behaviors of the first component of the control variables. (b) and (d) difference between the first and the second components of the control variables. Dashed lines: control constraints.

Conclusive remarks

This thesis work presents a general scheme for Cooperative Distributed Model Predictive Control: cooperation is realized in a set of dynamically uncoupled systems through the control action. The latter is the result of a cost minimization process at the level of each Decision Maker which is part of the overall system and is in this sense distributed: each agent's cost function is composed by a stabilizing term and a cooperative term which depends on the states of the neighboring agents, which are communicated with a certain delay. The minimization and implementation of the obtained controller are performed in a Receding Horizon fashion.

The scheme is suited to the cases where the DMs are modeled through linear or nonlinear discrete time systems, and the existence of constraints on the state or the control action are considered: the latter fact brings in many issues and requires several assumptions, though greatly extends the practical applicability of the control algorithm.

The main contribution of this work is that of providing a RH based framework that is entirely distributed and allows each DM to locally find a suitable controller, allowing for both stabilization of the entire system and cooperation within the agents. This is possible without requiring each agent to know the dynamics of its neighbors. Also, no restriction in terms of the predicted and actuated trajectories is necessary in order to guarantee stability, nor is required a hierarchy among the agents.

It must be noted here that the information exchange strategy may be improved: in particular, one could argue that exchanging the predicted trajectories instead of the current state of each DM would be beneficial in terms of converging to an overall centrally computed solution. This at the expense of assuming that every agent is optimizing also for its neighbors knowing their model and that there is a hierarchy in the local optimization processes [13, 32]. In certain cases though, it is not possible to know the models of all the neighbors, nor determine a priority that can be uniquely assigned to every DM. This is true for applications where the model of a DM would be particularly complex, such as a real aircraft or autonomous ground vehicle, or not available at all. Those are the cases where this framework can be successfully applied.

The presence of delays in the communication exchange is taken into account, though issues are still arising in the fact that they are deterministic. Another fundamental and restrictive assumption is that of synchronization of all the DMs. It is still a drawback of this scheme, that of not considering stochastically varying delays in the communication, and not addressing the problem of independent computational clocks at each DM.

As far as the future work and improvements, the main issues to be solved are indeed the stochastic communication delays and the need of dropping the assumption of synchronization among the agents.

Research on how to introduce process disturbances and possible additive faulty behavior of the DMs is ongoing and aiming at formulating a fault tolerant cooperative architecture. Another interesting improvement is that of considering partial state measurement or noisy state measurements at the level of each agent, both for feedback and cooperation purposes. This case will require the introduction of state estimation techniques, which may greatly affect the performance of the RH controller, in particular in the presence of nonlinear DM models.

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