



Tolling, collusion and equilibrium problems with equilibrium constraints

Andrew Koh^{1*}, Simon Shepherd¹

¹ Institute for Transport Studies, University of Leeds, Leeds, LS2 9JT, United Kingdom

Abstract

An Equilibrium Problem with an Equilibrium Constraint (EPEC) is a mathematical construct that can be applied to private competition in highway networks. In this paper we consider the problem of finding a Nash Equilibrium in a situation of competition in toll pricing on a network utilising two alternative algorithms. In the first algorithm, we utilise a Gauss Seidel fixed point approach based on the cutting constraint algorithm for toll pricing. The second algorithm that we propose, a novel contribution of this paper, is the extension of an existing sequential linear complementarity programming approach for finding the competitive Nash equilibrium when there is a lower level equilibrium constraint. Finally we develop an intuitive approach to represent collusion between players and demonstrate that as the level of collusion goes from none to full collusion so the solution maps from the Nash to monopolistic solution. However we also show that there may be local solutions for the collusive monopoly which lie closer to the second best welfare toll solution than does the competitive Nash equilibrium.

Keywords: Sequential Linear Complementarity Programming (SLCP); EPEC; Competition; Nash Equilibrium; Collusion.

1. Introduction

The motivation of the research in this paper stems from the observation that in recent years there has been increasing amount of private sector participation within areas that are conventionally the privy of the public purse. The driving force behind this change is the assumed higher efficiency of the private sector coupled with increasing public pressures on governments for accountability and the corresponding need to derive value for money from their various budgetary commitments which are ultimately funded by the tax paying public.

In highway transportation, privately operated roads are not novel concepts (Viton, 1995). However there has been little analysis on this topic in terms of the competition between private sector providers and the equilibrium outcomes, save for theoretical

* Corresponding author: Andrew Koh (a.koh@its.leeds.ac.uk)

studies often restricted to networks with two parallel links (e.g. Verhoef *et al.*, 1996; de Palma and Lindsey, 2000). In reality, there have already been examples of private sector involvement in road construction and operation around the world (Fisher and Babbar, 1996; Roth, 1996)¹. In return for the private capitalists funding large amounts of initial capital investments for the construction of the road, they are contractually allowed to collect tolls, for some agreed duration from users when the road is finally opened (Engel *et al.*, 2002). In an era when government budgets are becoming increasingly tight and with traffic congestion becoming more of a problem in many major cities, the private sector is recognised as having an increasing role to play in the provision of traditional highway transportation investment. When the private sector is tasked with the provision of such services and in competition with others simultaneously doing the same, the concept of Nash equilibrium (Nash, 1950) can be used to model the equilibrium decision variables offered to the market.

In this paper we consider the problem of toll optimisation in modelling the situation of private sector participation in the operation of transportation services. In the case of toll only competition, we provide two heuristics for the solution of the problem. The first is simply a Gauss Seidel fixed point approach based on the cutting constraint algorithm for toll pricing which builds on our previous work (Koh *et al.*, 2009). The second algorithm that we propose, a novel contribution of this paper is an extension of the existing Sequential Linear Complementarity Programming Approach (SLCP) for finding the competitive Nash equilibrium when there is a lower level equilibrium constraint. Although SLCP has previously been formulated for a general Nash game (Kolstad and Mathiesen, 1991), we believe this to be the first application of the approach to an EPEC problem where the toll operators compete at the upper level but are bound by the lower level equilibrium of the user response in terms of demand and route choice. This application of SLCP to an EPEC should be useful in other fields such as the electricity market. We present various examples from the same network to illustrate the performance of these heuristics and compare the competitive solutions with both monopolistic and second best welfare maximising regimes. These examples include both parallel and serial links in competition. Finally, we consider how to model collusive behaviour between operators and propose an intuitive structure which allows this response to be modelled. With this natural structure we show that when moving from no collusion through partial collusion to full collusion a path is drawn between the Nash and monopoly solutions. Where an un-priced substitute link is present local optima may exist which we suggest may be more likely than a true global solution which has interesting implications for policy-makers and the behaviour of toll operators.

The structure of this paper is as follows. In the next section, we define the problem considered along with the concept of Nash Equilibrium from Nash (1950) which serves as the foundation of non-cooperative games that we discuss. Section 3 then outlines two heuristic algorithms for the problem. Section 4 utilises numerical examples to illustrate the performance of the algorithms. In Section 5 we relax the notion of non-cooperative behaviour and consider if it is possible for the players to signal, through their selection of strategic variables, to their competitor, their intention to collude such that they end up in a monopolistic equilibrium. Finally in Section 6, we summarise our results and provide directions for further research.

¹ There are in fact real world examples of such competition, in Australia, private operators compete with public operators for toll revenues on the Sydney Orbital Motorway Network. See <http://www.rta.nsw.gov.au/usingroads/motorwaysandtolling/index.html>, accessed July 2009.

2. Problem Context

Our problem is to find an optimal equilibrium toll for each private operator² who separately controls a predefined link on the traffic network under consideration. We can consider this problem to be a Cournot-Nash game between these individual operators. The equilibrium decision variables can be determined using the concept of Nash equilibrium (Nash, 1950) which we define as follows:

2.1. Nash Equilibrium

In a single shot normal form game with N players indexed by $i, j \in \{1, 2, \dots, N\}$, each player can play a strategy $u_i \in U_i$ which all players are assumed to announce simultaneously. Let $u = (u_1, u_2, \dots, u_N) \in U$ be the combined strategy space of all players in this game and let $\psi_i(u)$ be some payoff or profit function to player $i \in \{1, 2, \dots, N\}$ if the combined strategy is played. The combined strategy tuple is a Nash Equilibrium $u^* = (u_1^*, u_2^*, \dots, u_N^*) \in U$ for the game if the following holds

$$\psi_i(u_i^*, u_j^*) \geq \psi_i(u_i, u_j^*) \quad \forall u_i \in U_i, \forall i, j \in \{1, 2, \dots, N\}, i \neq j \quad (1)$$

Equation (1) states that a Nash equilibrium is attained when no player in the game has an incentive to deviate from his current strategy. She is therefore doing the best she can given what her competitors are doing (Pyndyck and Rubinfeld, 1992).

2.2. Problem Definition

We now outline the problem we wish to solve as viewed by each operator with equilibrium conditions imposed on the users' route choice.

Define:

A : the set of directed links in a traffic network,

B : the set of links which have their tolls, $B \subset A$

K : the set of origin destination (O-D) pairs in the network

\mathbf{v} : the vector of link flows $\mathbf{v} = [v_a], a \in A$

$\boldsymbol{\tau}$: the vector of link tolls $\boldsymbol{\tau} = [\tau_a], a \in B$

$\mathbf{c}(\mathbf{v})$: the vector of monotonically non decreasing travel costs as a function of link flows on that link only

$\mathbf{c} = [c_a(v_a)], a \in A$

$\boldsymbol{\mu}$: the vector of generalized travel cost for each OD pair $\boldsymbol{\mu} = [\mu_k], k \in K$

\mathbf{d} : the continuous and monotonically decreasing demand function for each O-D pair as a function of the generalized travel cost between OD pair k alone, $\mathbf{d} = [d_k], k \in K$ and

\mathbf{D}^{-1} : the inverse demand function

$\boldsymbol{\Omega}$: feasible region of flow vectors, defined by a linear equation system of flow conservation constraints.

² As the research transcends both game theory and market structures in the context of highway transportation, we will use the terms "private operators" and "players" interchangeably throughout.

If we assume that each player is able to control³ only a single i^{th} link in the network then, following Yang *et al* (2009), the optimisation problem for each i^{th} player, which represents the maximisation of the profit for the operator,⁴ is formulated as follows:

$$\text{Max}_{\tau_i} \psi_i(\boldsymbol{\tau}) = v_i(\boldsymbol{\tau})\tau_i, \forall i \in N \quad (2)$$

Where v_i is obtained by solving the variational inequality (see Smith, 1979; Dafermos, 1980)

$$\mathbf{c}(\mathbf{v}^*, \boldsymbol{\tau})^T \cdot (\mathbf{v} - \mathbf{v}^*) - \mathbf{D}^{-1}(\mathbf{d}^*, \boldsymbol{\tau})^T \cdot (\mathbf{d} - \mathbf{d}^*) \geq 0 \text{ for } \forall (\mathbf{v}, \mathbf{d}) \in \Omega \quad (3)$$

The objective for each firm (payoff) is the toll revenue obtained by charging tolls on the link operated by the i^{th} player.

Note that the vector of link flows can only be obtained by solving the variational inequality given by (3). This variational inequality represents Wardrop's user equilibrium condition which states that no road user on the network can unilaterally benefit by changing routes at the equilibrium (Wardrop, 1952). Throughout this paper, we make the additional simplifying (yet not uncommon) assumption that the travel cost of any link in the network is dependent only on flow on the link itself so that the above variational inequality in (3) can be solved by means of a convex optimisation problem (Beckmann *et al*, 1956).

3. Two Heuristic Algorithms for EPECs

The problem we have defined in the previous section is in fact an Equilibrium Problem with Equilibrium Constraints (EPEC) (Mordukhovich, 2005, 2006; Su 2005). In essence the EPEC's are problems of finding equilibrium points of players when they are bound by constraints specifying an overall system equilibrium. The study of EPECs has only recently surfaced as an important research area within the field of mathematics but has significant practical applications e.g. in deregulated electricity markets (Ralph and Smeers, 2006).

In this paper, we propose two alternative heuristics for the resolution of the problem. The first algorithm is the diagonalisation algorithm which is a modified version of the non-linear Gauss-Seidel method (as discussed in e.g. Ortega and Rheinboldt (1970); Judd, 1998). The second algorithm is a novel heuristic derived from reformulating the standard Nash game from economics as a complementarity problem and solving it using a Sequential Linear Complementarity Programming approach (SLCP). The extension is to apply this SLCP approach within the EPEC setting as described earlier.

³ Control is used as a short hand to imply that the firm has been awarded some franchise for operating the link.

⁴ Here we have assumed no costs of toll operation for ease of presentation, however these can be introduced quite easily.

3.1. Diagonalization Algorithm (Algorithm 1)

One of the first algorithms introduced for this problem was that of decomposing the problem into a series of interrelated optimisation problems and subsequently solving each individually. This is also known as a fixed point iteration algorithm which has also been referred to as the Gauss-Jacobi algorithm. In economics, Harker (1984) popularised this algorithm for solving a Cournot-Nash game. In a similar fashion, Cardell *et al* (1997) and Hobbs *et al* (2000) have used the diagonalisation algorithm to solve EPECs arising in the deregulated electricity markets.

The algorithm is presented as follows:

Diagonalisation Algorithm

- Step 0: Set iteration counter $k=0$. Select a convergence tolerance parameter, $\varepsilon(\varepsilon>0)$. Choose a strategy for each player. Let the initial strategy set be denoted $u^k = (u_1^k, u_2^k, \dots, u_N^k)$. Set $k=k+1$ and go to Step 1.
- Step 1: For the i^{th} player $i \in \{1, 2, \dots, N\}$, solve the following optimization problem:

$$u_i^{k+1} = \max_{u_i \in U_i} \psi_i(u_i, u_j^k) \quad \forall i, j \in \{1, 2, \dots, N\}, i \neq j$$
- Step 2: If $\sum_{i=1}^N |u_i^{k+1} - u_i^k| \leq \varepsilon$ terminate, else set $k = k + 1$ and return to Step 1.
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In step 1, we utilise the Cutting Constraint Algorithm (CCA) (Lawphongpanich and Hearn, 2004) to solve the optimisation problem for each player holding the other player's strategic variables fixed. Further details regarding the CCA are provided in the appendix to this paper.

The convergence proof of the diagonalisation algorithm when applied to single level Nash equilibrium problems can be found in Pang and Chan (1982) or Dafermos (1983). However the proof depends on certain conditions that may not be satisfied in an EPEC, particularly the concavity of the payoff functions. In fact, convergence of the algorithm relies on the concept of diagonal dominance of the Jacobians of the payoff functions (Gabay and Moulin, 1980, Theorem 4.1 p. 280), which intuitively requires that a player has more control over his own payoff than do his competitor(s). Therefore we propose this algorithm to be a heuristic approach for the EPEC at hand.

3.2. Sequential Linear Complementarity Programming Algorithm (Algorithm 2)

Since the game between the operators in this paper is akin to a Nash game, the second algorithm reformulates the Nash game as a complementarity problem. Adopting this approach, Kolstad and Mathiesen (1991) developed a Sequential Linear Complementarity Programming (SLCP) approach to solve the resulting reformulation. The extension below is to re-formulate this within the EPEC framework whereby the Nash game is played out at the upper level between the toll operators while the user equilibrium conditions are respected at the lower level – hence and equilibrium problem with equilibrium constraints. At each iteration, the main problem is linearised (using a

first order Taylor expansion) at a given starting point. Then the sub problem is solved as a linear complementarity problem for which the algorithm of Lemke (1965) can be applied. As far as we are aware, this is the first application of the algorithm to the EPEC and should be useful in other fields.

To demonstrate the approach, recall that the profit of the firm i is given by (2). The first order conditions of a profit maximum for each firm are therefore given by (4)-(6) as follows:

$$f_i = -\frac{\partial \psi_i}{\partial \tau_i} \geq 0 \tag{4}$$

$$\frac{\partial \psi_i}{\partial \tau_i} \tau_i = 0 \tag{5}$$

$$\tau_i \geq 0 \tag{6}$$

These first order conditions define a complementarity problem (CP) as characterized by the system (7) which is to:

Find $\tau \in \mathfrak{R}_+^N$ given $f : \mathfrak{R}_+^N \rightarrow \mathfrak{R}^N$ such that

$$\begin{aligned} \mathbf{f}(\tau) &\geq 0 \\ \tau^T \mathbf{f}(\tau) &= 0 \\ \tau &\geq 0 \end{aligned} \tag{7}$$

If we linearise f at τ^0 (some arbitrary starting vector of tolls) using the first order Taylor expansion, then we obtain $Lf(\tau / \tau^0) = f(\tau^0) + \nabla f(\tau^0)(\tau - \tau^0)$. Hence, following (Kolstad and Mathiesen, 1991), the resulting Linear Complementarity Program (LCP) is to find $\tau \in \mathfrak{R}_+^N$ such that

$$\begin{aligned} Lf(\tau / \tau^0) &= q + M\tau \geq 0, \\ \tau^T (q + M\tau) &= 0, \\ \tau &\geq 0 \end{aligned} \tag{8}$$

Where $q = f(\tau^0) - \nabla f(\tau^0)\tau^0$ and $M = \nabla f(\tau^0)$.

In summary the proposed algorithm is as follows:

Sequential Linear Complementarity Programming Algorithm

- Step 0: Choose some starting vector of tolls τ^0 . Select a convergence tolerance parameter, $\epsilon(\epsilon > 0)$, and set $k=0$ and go to Step 1.
 - Step 1: Solve the traffic assignment problem (3) with τ^k .
 - Step 2: Employ finite differencing to approximate $f(\tau^k)$ and $\nabla f(\tau^k)$
 - Step 3: Solve the LCP (8) to obtain τ^{k+1}
 - Step 4: Check convergence: If $\max |f(\tau^{k+1})| < \epsilon$, terminate else set $k=k+1$ and go to Step 1.
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Note that in order to solve the LCP⁵, we require both the Jacobian of the profit function $f(\tau^k)$ for each firm in the game at iteration k and the Hessian (M). To do so, we solve a traffic assignment problem at τ^k and perturb the tolls by using the method of central differences (i.e. via a combination of forward and backward differencing) to approximate the gradients. The underlying assumption here is that the derivatives exist and can be approximated in this way⁶.

As with the diagonalisation approach, the convergence proof of this algorithm relies specifically on the concavity of the payoff functions of each firm (Kolstad and Mathiesen, 1991, Theorem 1, p 741). While this assumption is usually acceptable in modelling the classical Nash game for which it was developed, it may not be satisfied in a general EPEC setting.

4. Numerical Examples

In this section, we provide examples of how the proposed heuristics are used to solve for the optimal tolls. In addition, we compare the equilibrium outputs under the scenarios of competition, monopoly and under the policy of (second best) social welfare maximisation.

In the case of monopoly, we assume there is a single private operator controlling the predefined links in the network. Hence this is a simpler problem that can be solved directly using the CCA (see the Appendix) or any derivative free direct search method (e.g. Hooke Jeeves direct search (Hooke and Jeeves, 1961) or Nelder Mead Simplex algorithm (Nelder and Mead, 1965)). For the results presented here, the CCA was utilised.

Similarly in the case of social welfare maximisation by levying toll charges only, the central planner solves the following problem.

$$\begin{aligned} \text{Max}_{\tau} \quad & \sum_{k \in K} \int_0^{d_k} d^{-1}(x) - \sum_{a \in A} c_a(v_a)v_a \\ \text{s.t.} \quad & (\mathbf{v}, \mathbf{d}) \in \Omega \\ & 0 \leq \tau \leq \bar{\tau} \end{aligned} \tag{9}$$

Where $\bar{\tau}$ is the pre-specified upper bound on tolls on tolled links, $\bar{\tau} = [\bar{\tau}_a]$, $a \in B$. The CCA algorithm can be utilised for this problem as in Koh *et al* (2009).

The example used is taken from Koh *et al* (2009). The link specific parameters and the elastic demand functions can be found therein. This network has 18 one way links with 6 origin destination pairs (1 to 5, 1 to 7, 5 to 1, 5 to 7, 7 to 1 and 7 to 5).

⁵ In terms of implementation, to solve the SLCP in Step 2, we used the PATH solver (Ferris and Munson, 2000) within MATLAB.

⁶ We have made use of finite differencing to obtain first and second order derivatives despite there being alternative methods based on sensitivity analysis (Yang and Huang, 2005) proposed for obtaining these derivatives. We apply finite differences as this is simple to apply and is often used in the estimation of gradients in much numerical work.

Two parallel link scenarios are considered in this numerical example. In Scenario 1, Links 3 and 4 in Figure 1 are the only links in this network that are subject to tolls. In Scenario 2, Links 7 and 10 are the only links subject to tolls in the network. Note that in all which follows we set the maximum allowable toll to be 5000 seconds. The bound of 5000 was chosen as this translates into a practical toll level of approximately £6 which is considered to be reasonable maximum on the basis of acceptability for a toll on one link. As we demonstrate later this upper bound will only apply to one link in the monopolistic case. We also look at serial link scenarios with Scenario 3 tolling links 3 and 7 where there is an element of route choice for the users to avoid both links and Scenario 4 tolling links 1 and 3 where there is no route choice to avoid the toll on link 1.

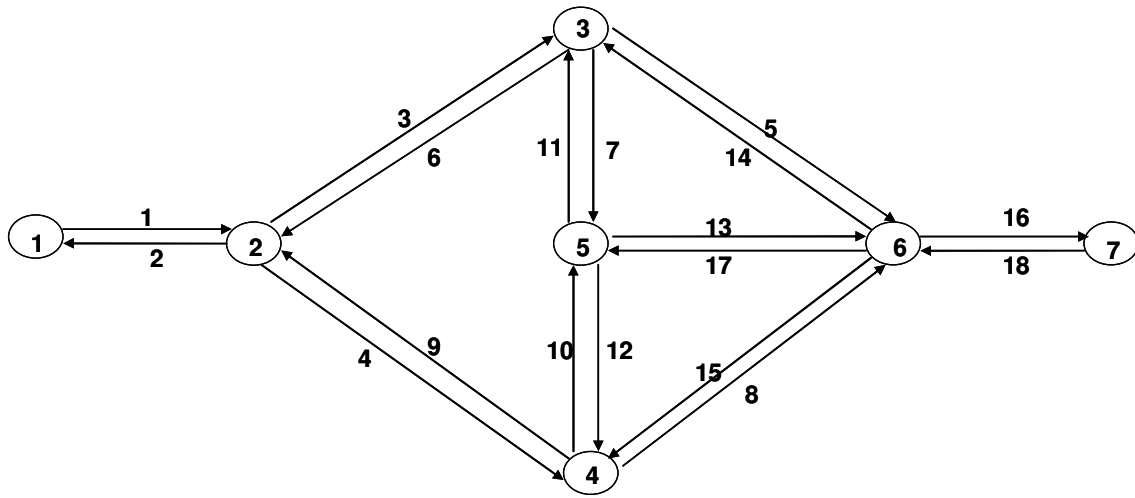


Figure 1: Network for Example 1.
Source: Koh *et al* (2009).

Table 1: Comparing Solution by Alternative Algorithms for competitive tolls.

	Link	<i>Diagonalisation</i> ⁷			<i>SLCP</i> ⁸		
		Toll (secs)	Iterations	CPU Time (secs)	Toll (secs)	Iterations	CPU Time (secs)
Scenario 1	3	530.63	25	1213.7	530.55	6	9.8
	4	505.65			505.62		
Scenario 2	7	141.37	25	1200.8	141.36	6	9.5
	10	138.29			138.29		
Scenario 3	3	248.62	23	1211.2	248.65	5	8.3
	7	97.84			98.52		
Scenario 4	1	5000	19	914.9	5000	5	8.8
	3	35.20			35.20		

Table 1 shows the resulting tolls, number of iterations and CPU⁹ times required for each algorithm to converge to the Nash solution. As shown the resulting tolls are almost identical and any differences are due to the convergence criteria used. The proposed SLCP algorithm uses fewer iterations and is much faster than the diagonalisation based approach – requiring less than 1% of the CPU time.

⁷ Using the diagonalisation algorithm with CCA (Algorithm 1) and a termination tolerance of $\epsilon = 1e-06$

⁸ Using the SLCP algorithm (Algorithm 2) with a termination tolerance of $\epsilon = 1e-06$.

⁹ CPU time refers to the time taken by the central processing unit of the computer to perform the evaluation using of the respective algorithms to the precision tolerances specified.

Table 2 shows the tolls, the revenues collected and the change in social welfare for each toll pair under (a) the competitive case, (b) the monopoly case and (c) the second-best welfare case where operators are assumed to co-operate to maximise social welfare. Firstly, the table shows that when there are no alternative routes available (as in the case of Scenario 1 where Links 3 and 4 are tolled), the monopolist can charge the maximum toll allowable for link 3. In fact the upper bound of the toll here is a binding constraint on the toll in the monopoly case. The toll on link 4 is lower due to the slightly longer free-flow travel time. A check with both tolls set at 5000 seconds showed that the total revenue was indeed lower than shown in table 2 (being only 3,555,289 seconds). As may be expected with the monopolistic case the impact on welfare is negative. However in the case of two competing operators, each player has no alternative but to succumb to the strategy charged by the other and hence ultimately both are only able to charge a much lower toll (around 10% of the monopolist's toll). The overall welfare change for Scenario 1 under competition is reasonably close to that of second best social welfare maximisation, but is as expected lower.

Table 2: Tolls, Revenues and Social Welfare under Alternative Market Structure Assumptions (Tolls in seconds, Revenue and Welfare Change in seconds per hour).

	Link	Competition: Revenue Maximisation			Monopoly: Revenue Maximisation			Second Best Welfare Maximisation		
		Toll	Revenue	Welfare Change	Toll	Revenue	Welfare Change	Toll	Revenue	Welfare Change
Scenario 1	3	530.63	461,882	87,633	5000 ¹⁰	2,543,530	-1,581,256	510.93	449,583	87,818
Parallel	4	505.65	420,293		4986.73	1,013,577		488.13	407,301	
Total Revenue			882,175			3,557,108			856,883	
Scenario 2	7	141.37	105,295	187,422	713.19	280,255	150,587	181.83	116,209	202,311
Parallel	10	138.29	100,848		709.53	266,465		179.30	110,580	
Total Revenue			206,143			546,720			226,783	
Scenario 3	3	248.65	146,756	-88,020	242.01	147,209	-76,956	0	0	95,795
Serial	7	98.52	54,309		92.54	54,275		141.89	74,027	
Total Revenue			201,065			201,482			74,027	
Scenario 4	1	5000	3,552,057	-1,590,050	5000	3,553,670	-1,585,945	488.21	836,935	87,818
Serial	3	35.20	11,122		26.73	10,513		22.74	20,008	
Total Revenue			3,563,179			3,564,184			856,943	

The more interesting case emerges in Scenario 2 when there is an un-tolled alternative (Link 17 in Figure 1) available for travel into destination Zone 5. In this situation, even a monopolist controlling both Links 7 and 10 together cannot charge the maximum allowed toll of 5000 seconds on each link to maximise his revenue. Here the tolls are

¹⁰ As the original network parameters included an elasticity of -0.58 for this case (with constant elasticity demand function) a true unbounded toll would tend to infinity. To create a reasonable unbounded solution with an interior solution we would have to increase the elasticity of demand. A test with elasticity of -2.0 was conducted and this gave an interior solution with tolls of 751 and 728 seconds which are well within the bound of 5000 seconds.

limited to around 700 seconds (though the impact on welfare is positive). In the case of competition, Table 2 shows that the tolls charged and the total revenue earned are even lower than under that of a central planner attempting to maximise social welfare. So we may conclude that where there is an un-tolled alternative, competition has the effect of driving tolls down below the socially optimal toll level which was not the case for scenario 1 where there was no un-tolled alternative. The change in social welfare is still as expected lower under competition.

Furthermore, the tolls are lower under competition than under monopoly. Since parallel links are the equivalent of substitutes in the route choice, this supports the general observation by Economides and Salop (1992) that competition between substitute products would lead to lower prices (*vis-à-vis* monopoly).

The first two scenarios have focused on parallel competing links. In the case of serial links, economic theory suggests that the tolls would be higher under competition than under a single monopoly (Economides and Salop, 1992; Small and Verhoef, 2007). Under scenarios 3 and 4 where we have serial links in competition we verify this result that the tolls are indeed higher under the competitive solution than under the single monopoly and hence result in a greater loss in welfare. It is also worth noting that where there are alternative free routes as in scenario 3 then the tolls are relatively low even under monopoly, whereas under scenario 4 where the toll on link 1 has no free substitute route, then the upper bound constrains the solution and the operator of link 1 exerts power over the operator of link 3. In these serial cases, both the competitive and monopolistic solution results in a negative welfare change compared to the second best welfare maximising tolls implying that with competition, society is worse off which has crucial implications for policy makers. They should consider carefully whether to allow direct competition in the serial link case.

5. Possibilities for Collusion between Operators

This section of the paper investigates collusion and considers whether it is possible for operators to receive signals from a competitor to achieve the revenues associated with monopoly control over their networks. In this section of the paper, our examples are restricted to games with two players. To this effect, we introduce a scalar, α ($0 \leq \alpha \leq 1$) which represents the degree of co-operation between the players when they optimise their toll revenues for links under their control.

With α , we can consider a more general form of the expression for the payoff function (2) as given in (10)

$$\psi_i(\boldsymbol{\tau}, \boldsymbol{\beta}) = v_i(\boldsymbol{\tau}, \boldsymbol{\beta})\tau_i + \alpha(v_j(\boldsymbol{\tau}, \boldsymbol{\beta})\tau_j), \forall i, j \in N, i \neq j \quad (10)$$

Equation (10) reduces to the familiar form of (2) when $\alpha = 0$; and when $\alpha = 1$ the objective of each player is to maximise the total toll revenue of both players. Note that she can only however change tolls on links under her control and continues to take the other player's toll as exogenous. Thus whilst the i th player is in the process of optimising her revenue, she is taking into account a proportion represented by α of the j th player's toll revenue. In doing so via the diagonalisation algorithm, she is

effectively “signalling” to her competitor that she wishes to “collude” to maximise total revenue. It is implicitly assumed that players reciprocate the actions of their competitors and would do likewise. Thus the α term represents some intuitive level of collusion between players, ranging from no collusion, through partial collusion to full collusion.

Consider the network shown in Figure 1 and recall the two separate scenarios developed therein with Scenario 1 being toll revenue competition on links 3 and links 4 while Scenario 2 represented toll revenue competition on links 7 and links 10.

5.1. Collusion in Scenario 1

Figure 2 shows, for the case depicted in Scenario 1, how the toll solution moves from the non-cooperative Nash Solution when ($\alpha = 0$) towards the monopoly solution ($\alpha = 1$) as the level of collusion is increased. In particular, when $\alpha = 1$, we obtain exactly the same solution as the monopoly operator’s tolls as shown in Table 2.

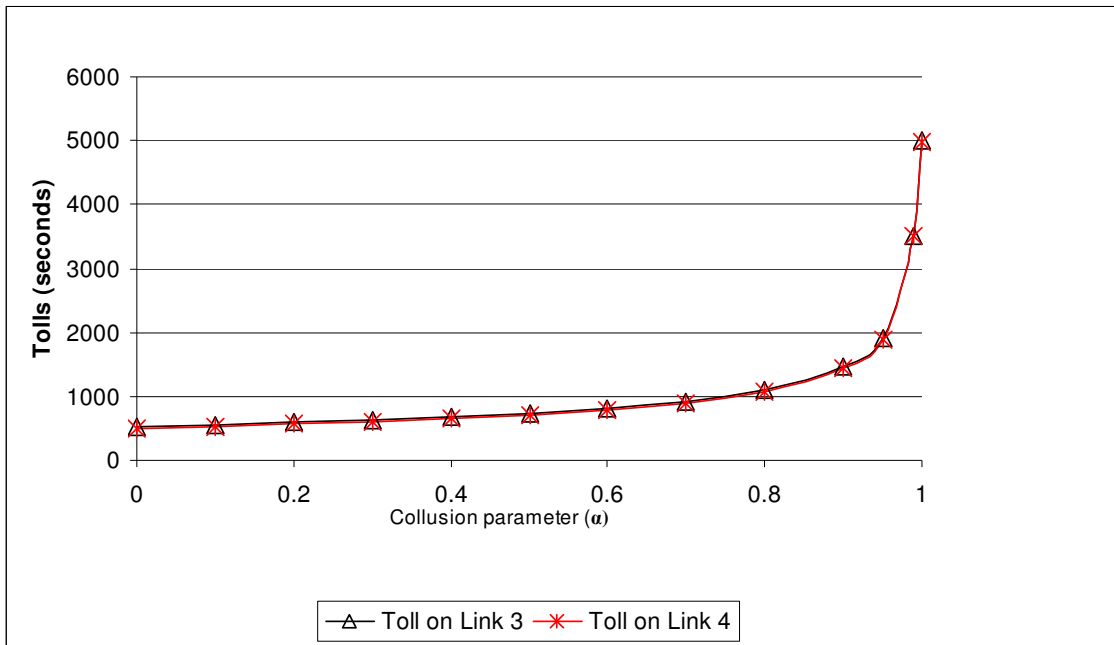


Figure 2: Scenario 1: Tolls set by each operator as collusion parameter (α) varies.

It can also be observed that with slightly “less than full collusion” ($\alpha= 0.95$ or 0.99) the toll levels are also much lower thereby suggesting that less than full collusion can lead to substantial losses (in revenues) for both players. Figure 3 shows the revenues for operators of links 3 and 4 and the total revenue as the collusion parameter is varied. Notice that when going from a collusion level of 0.99 to 1.0 the upper bound on the toll becomes active which means that to generate more total revenue the operator of link 4 must accept a reduction in revenue. This is due to the higher free flow travel time on link 4 and the interaction with the upper bound on tolls on link 3. Obviously this upper bound is theoretic here but in practical applications, there could be some upper bound set as part of some franchise agreement with a regulator. If this were the case then it would not be in the interests of the second operator to collude unless there was some contract to share out the resulting revenues. We therefore suggest that under such cases

there may be a limited collusion aspect which would bring down the tolls compared to the true monopolistic solution.

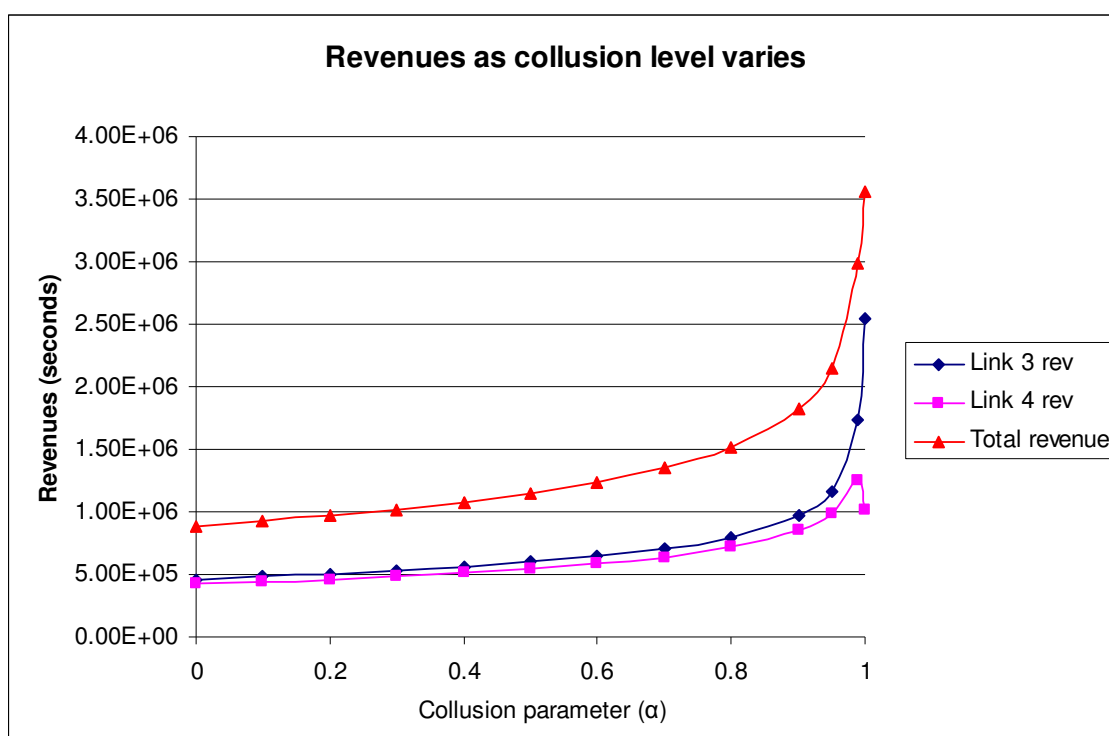


Figure 3: Scenario 1: Revenues as collusion parameter (α) varies.

5.2. Collusion in Scenario 2

In the case of Scenario 2, where there is an additional route (Link 17) that is not subject to tolls, this form of implicit collusion however does not obtain the solution under monopoly. In particular, consider the situation when $\alpha=1$, then employing the diagonalisation algorithm, the equilibrium tolls obtained are as shown in Table 3.

Table 3: Toll and Revenues for Scenario 2 considering collusion with $\alpha=1$.

Link	Toll (seconds)	Revenues (seconds)
Link 7	189.76	116,816
Link 10	186.58	111,216
Total Revenues		227,402

Figure 4 illustrates however that the above solution is in fact a local optimum of the total revenue function. The results reported in Table 3 are plotted together in Figure 3 where it is compared against the global optimum which is in fact the solution obtained under monopoly (see Table 2).

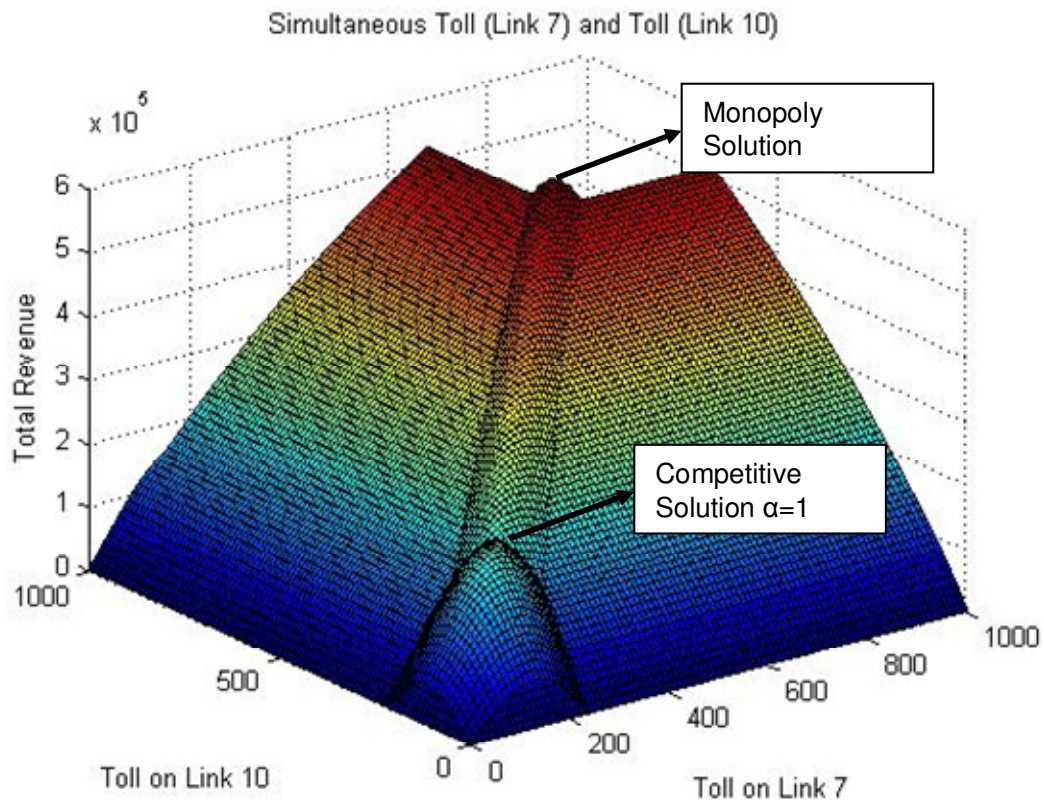


Figure 4: Total Revenue Surface as Tolls on Link 7 and 10 vary.

This illustrates the general difficulty with optimisation algorithms and the potential for a local equilibrium to be located. There is also a possibility that in Scenario 2, there continues to be a link (number 17 in Figure 1) available that is in competition with the tolled links and hence even under collusion, there could exist an incentive to capture that untolled traffic by reducing the toll charge and breaking the collusion which may result in an alternative local solution rather than a global one. In Koh (2008), we presented a global optimisation algorithm to locate this global optimum based on particle swarm optimisation (PSO). However it is not clear whether the behaviour of the operators would allow them to find the global optimum in this case. PSO is based on some learning scheme – but to learn how such a surface operates requires an element of trust in your competitor. It is our view that the diagonalisation approach better reproduces realistic behaviour of operators and tests with other initial toll levels or starting points all result in the local rather than global solution. This suggests that the local monopoly solution may be the more likely of the two outcomes which has significant implications for both policy-makers and operators as here the tolls are lower and closer to second best welfare maximisation.

To reinforce this claim we show in Figure 5 that as the collusion parameter is increased then the tolls on links 7 and 10 do in fact move from the Nash tolls to this local monopolistic toll set. This time, as the tolls remain within bounds, the move towards the monopoly solution is smooth and the revenues increase for both operators as α is increased.

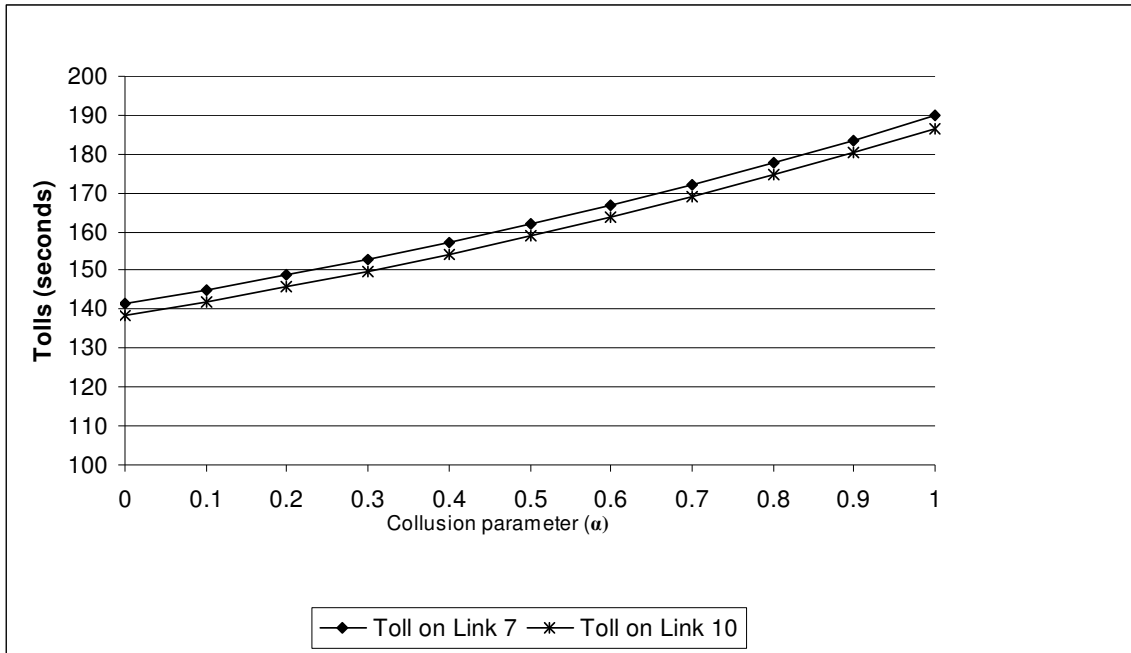


Figure 5: Scenario 2: Tolls as collision parameter (α) varies.

5.3. Collusion in Scenario 3

For the serial links 3 and 7 the benefits of collusion are small as the users have the opportunity to avoid the second toll and so there is little common traffic between the links. The tolls move smoothly between the competitive solution and the monopoly solution reported in table 2.

In passing it is worth mentioning that serial links 7 and 12 or 10 and 11 result in the same optimal tolls for the competitive and monopolistic cases. This is because the links have no common traffic and can be considered as independent so there is no possibility of monopoly here. This shows that at the extreme the serial link result is that the competitive tolls are greater than *or equal* to the monopoly tolls.

5.4. Collusion in Scenario 4

In this case, as the collusion parameter (α) varies, the toll on Link 1 remains constant at 5000 seconds (the upper bound). However, the toll for Link 3 decreases smoothly towards the monopolistic toll as shown in Figure 6. Figure 6 also shows the revenue for link 3 which decreases as the level of collusion increases. This is the same effect as was seen for scenario 1 as the stronger operator is bound by the upper limit on the toll. In order to generate more revenue in total the weaker operator must accept a lower revenue, so there is no incentive to collude here without a revenue sharing agreement.

Thus we can say for both the serial and parallel case where one operator becomes limited by some upper bound on the toll level then there will be no incentive to collude for the second operator.

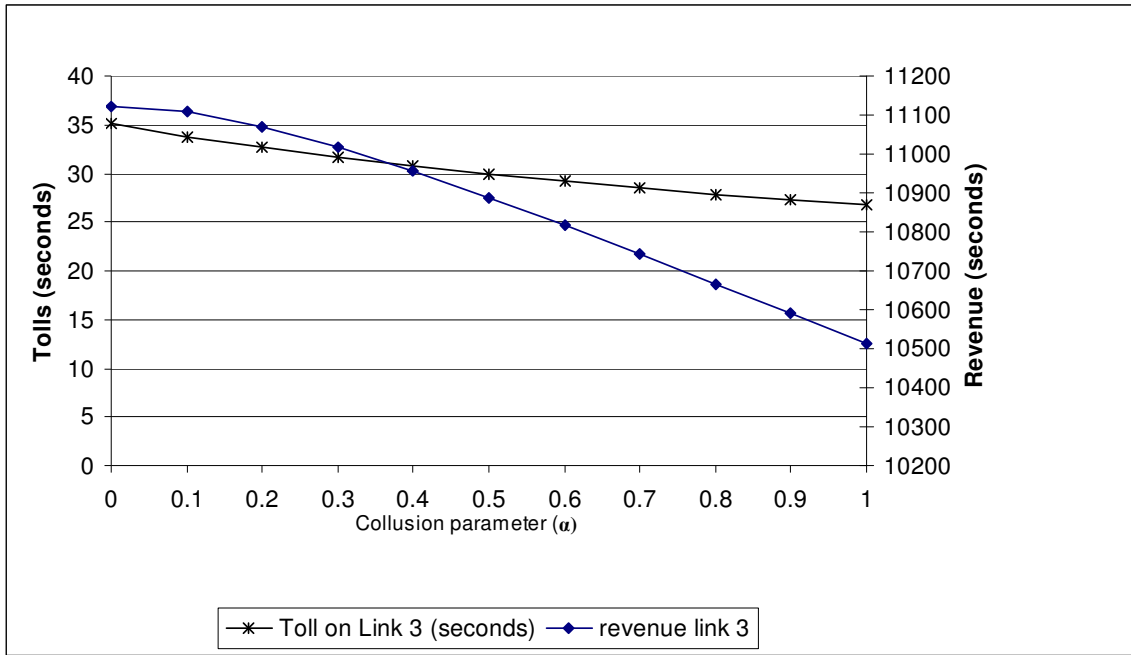


Figure 6: Scenario 4: Toll and revenue on link 3 as collision parameter (α) varies.

6. Summary and Conclusions

The first element of this paper demonstrated the use of two heuristics to solve the toll competition problem which is basically an Equilibrium Problem with Equilibrium Constraints or EPEC. The first was simply a Gauss Seidel fixed point approach based on the cutting constraint algorithm for toll pricing which builds on our previous work (Koh *et al.*, 2009). The second algorithm which is a novel contribution of this paper is the extension of an existing Sequential Linear Complementarity Programming approach for finding the competitive Nash equilibrium when there is a lower level equilibrium constraint. Although SLCP has previously been formulated for a general Nash game (Kolstad and Mathiesen, 1991), we believe this to be the first application to an EPEC problem where the toll operators compete at the upper level but are bound by the lower level equilibrium of the user response in terms of demand and route choice. This application of SLCP to an EPEC was shown to give the same solution as the diagonalisation approach but with significantly lower computation time. (We believe this approach could prove useful also in other fields such as analysing deregulated electricity markets.)

The second element of the paper compared the competitive, monopolistic and second best welfare maximising solutions for both parallel and serial link toll operation. These tests confirmed that in the parallel link case competitive tolls are lower than monopoly tolls and are reasonably close to second best welfare tolls. In the serial link case we also confirmed that the competitive tolls were greater than (or equal to) the monopoly tolls and that the welfare level is lower under competition than under monopoly. This would suggest that regulators should not allow direct competition in the serial link case. However for both the serial and parallel cases it was also shown that the presence of un-

toll alternatives reduces the toll levels and so reduces opportunities for monopolistic behaviour.

The third element of the paper introduced an intuitive formulation of collusive behaviour. To this effect, we introduced a collusion parameter to reflect the degree of cooperation between operators. Implicit in the assumption was that operators would be willing to reciprocate the action of the other and we have ignored the associated issues of stability of coalitions formed. Nevertheless, even for the simple examples presented in this paper, we have found the potential for multiple equilibria to be obtained. Furthermore we demonstrated that in all cases as collusion increases from none through to full collusion then the tolls map from the Nash solution to the monopoly solution in a smooth manner. Where a local monopoly solution exists then the collusive behaviour can also map toward this local monopoly rather than the global one which is by definition more acceptable to the public in terms of welfare change.

However for both the serial and parallel cases where the first operator is limited by an upper bound on the toll level then to increase total revenue the second operator must accept a reduction in revenue. Thus there is no incentive to collude in this case unless there is some form of agreement to share revenues set up in advance.

There is much scope to develop this work further considering the case of asymmetric collusion where one operator colludes more than the other which takes us into the area of leader-follower games such as Stackelberg. In terms of algorithms there is the obvious extension to the case of more than two operators which raises the question of whether the heuristics will converge with more complex tolling systems. In addition, the analysis presented in this paper can be employed to study competition between cities intending to introduce road pricing and/or other demand management or capacity enhancement measures. These serve as topics for further research which could build on the findings presented here.

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Appendix

The Cutting Constraint Algorithm (CCA)

Mathematical Program with Equilibrium Constraints

In the case of a single operator (operator is used here generically) who sets tolls and/or capacities to optimise some objective function which could be to maximise social welfare in the case of a local authority or to maximise profit in the case of a private firm. This optimisation problem is effectively a Mathematical Program with Equilibrium Constraints (MPEC). The economic paradigm for a generic MPEC is based on the setting of a Stackleberg game where the leader sets his strategic decision variables and the road users on the network follow. In optimising his objective the decision maker has to take into account the responses of the road users whose route choice is given by Wardrop's Equilibrium Condition. A large amount of development has occurred in this branch of mathematical optimisation (Luo *et al* 1996) which has applications in e.g. mechanics, robotics and transportation analysis. The primary difficulty with the MPEC is that they fail to satisfy certain technical conditions (known as constraint qualifications) at any feasible point (Chen and Florian, 1995; Scheel and Scholtes, 1995). In recent research, Koh *et al* (2009) investigated the use of the cutting constraint algorithm (CCA) (Lawphongpanich and Hearn, 2004) to solve an MPEC in the context of second best congestion pricing and capacity optimisation.

Reinterpretation of Variational Inequality Condition

Let us define the 2 additional variables

$\bar{\beta}_a$: a pre-specified upper bound on capacities, $\bar{\beta} = [\bar{\beta}_a]$, $a \in B$

$\bar{\tau}$: a pre-specified upper bound on tolls, $\bar{\tau} = [\bar{\tau}_a]$, $a \in B$

As we have defined in the main paper, the feasible region of flow vectors, Ω , is a linear equation system of flow conservation constraints.

From convex set theory, e.g. (Bazaraa *et al* 2008, Theorem 2.1.6 p.43), $(\mathbf{v}, \mathbf{d}) \in \Omega$ can be defined as a convex combination of a set of extreme points. Hence we can rewrite the equilibrium condition (3) using the following:

$$\mathbf{c}(\mathbf{v}^*, \tau, \beta)^T \cdot (\mathbf{u}^e - \mathbf{v}^*) - \mathbf{D}^{-1}(\mathbf{d}^*, \tau, \beta)^T \cdot (\mathbf{q}^e - \mathbf{d}^*) \geq 0 \text{ for } \forall e \in E$$

Where $(\mathbf{u}^e, \mathbf{q}^e)$ is the vector of extreme link flow and demand flow indexed by the superscript e , and E is the set of all extreme points of Ω .

A Cutting Constraint Algorithm for the MPEC

The Cutting Constraint Algorithm redefines the variational inequality using the extreme points. Together with the initial extreme point, generated by an initial shortest path problem, and the constraints defining feasible flows, the master problem is solved to find the optimal tolls and capacities at each iteration. Subsequently new extreme points (“cuts”) are found by solving a sub problem using the results for the current iteration.

The CCA Algorithm is as follows:

<i>CCA Algorithm</i>	
Step 0:	Initialise the problem by finding the shortest paths for each O-D pair; set l (iteration counter) = 0; define the aggregated link flow and demand flow $(\mathbf{u}^l, \mathbf{q}^l)$; and include $(\mathbf{u}^l, \mathbf{q}^l)$ into E .
Step 1:	Set $l=l+1$. Solve the Master Problem with all extreme points in E and obtain the solution vector $(\mathbf{v}, \mathbf{d}, \tau, \beta)$; then set $(\mathbf{v}^l, \mathbf{d}^l, \tau^l, \beta^l)$.
Step 2:	Solve the Sub Problem with $(\mathbf{v}^l, \mathbf{d}^l, \tau^l, \beta^l)$ and obtain the new extreme point $(\mathbf{u}^l, \mathbf{q}^l)$.
Step 3:	Convergence Check: If $\mathbf{c}(\mathbf{v}^l, \tau^l, \beta^l)^T \cdot (\mathbf{u}^l - \mathbf{v}^l) - (\mathbf{D}^{-1}(\mathbf{d}^l, \tau^l, \beta^l))^T \cdot (\mathbf{q}^l - \mathbf{d}^l) \geq 0$, terminate and $(\mathbf{v}^l, \mathbf{d}^l, \tau^l, \beta^l)$ is the solution, otherwise include $(\mathbf{u}^l, \mathbf{q}^l)$ into E and return to Step 1.

The Master Problem in Step 1 is defined as follows:

$$\begin{aligned}
 & \min_{(\tau, \beta, \mathbf{v}, \mathbf{d})} \psi_1(\tau, \beta, \mathbf{v}, \mathbf{d}) \\
 & \text{s.t.} \\
 & 0 \leq \tau_a \leq \bar{\tau}_a \quad \text{for given } \varepsilon \text{ and } \forall a \in B \\
 & 0 \leq \beta_a \leq \bar{\beta}_a \quad \text{for given } \gamma \text{ and } \forall a \in B \\
 & (\mathbf{v}, \mathbf{d}) \in \Omega \\
 & \mathbf{c}(\mathbf{v}^*, \tau, \beta)^T \cdot (\mathbf{u}^e - \mathbf{v}^*) - \mathbf{D}^{-1}(\mathbf{d}^*)^T \cdot (\mathbf{q}^e - \mathbf{d}^*) \geq 0 \text{ for } \forall e \in E
 \end{aligned}$$

The sub problem of Step 2 is a shortest path problem which is formulated as follows:

$$\begin{aligned}
 & \min_{(\mathbf{u}, \mathbf{q})} \mathbf{c}(\mathbf{v}, \tau, \beta)^T \cdot \mathbf{u} - (\mathbf{D}^{-1}(\mathbf{d}, \tau, \beta))^T \cdot \mathbf{q} \\
 & \text{s.t.} \\
 & (\mathbf{u}, \mathbf{q}) \in \Omega
 \end{aligned}$$

Further details of our implementation of the algorithm can be found in Koh *et al* (2009). Our numerical experiments indicate that for a small network tested in that paper, CCA obtained the global solution in a large number of test instances (as verified against a multi-start derivative free Hooke Jeeves (Hooke and Jeeves, 1961) method. Instances where it failed could be resolved by modifying the variable bounds which is recognised as a common obstacle in applying gradient based non linear programming methods to solve MPECs.