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Noncommutative Phenomena in Measure Theory on Operators Algebras

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SUMMARY. - *The aim of this paper is to to review the basic principles of the measure theory built on operator algebras and to summarize recent progress in this field. We concentrate on those results which demonstrate considerable difference between the classical measure theory and its operator-algebraic counterpart. In particular, we show that the functional–analytic structure of a given operator algebra A, the lattice–theoretic properties of measures on the projection lattice of A, the facial structure of the convex set of all states on A, and the continuity properties of states on A are connected in a way which has no analogy in the standard measure theory.*

1. Introduction and preliminaries

In the classical measure theory the basic object is the structure of measurable sets which forms a Boolean algebra and the structure of complex measurable functions which forms a commutative involutive function algebra when endowed with the usual multiplication and complex conjugation. In the noncommutative measure theory the algebra of measurable functions is replaced by an algebra of operators

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acting on some Hilbert space. This algebra is not commutative in general. The rejection of commutativity has led to an interesting mathematical structure with a wide range of applications to quantum physics and elsewhere.

Many principles of the standard measure theory can be extended to the noncommutative context more or less directly, even if methods of proving the generalizations have to be largely new. On the other hand, there is a great variety of results of the noncommutative measure theory which have no analogy in the classical case. We concentrate mainly on these phenomena. Especially, our goal is to show that there is a nice interplay between the following structures and properties associated with operator algebras and measures on them:

- algebraic and functional-analytic structure of the operator algebra
- lattice–theoretic properties of measures and homomorhisms on the projection lattice of the operator algebra
- facial structure of the state space (convex set of all positive normalized functionals) on the operator algebra
- continuity properties of measures and states on operator algebras.

These structures are interrelated more then in the classical measure theory and we would like to show some results demonstrating this fact.

We shall briefly recall basic results and notions of the theory of operator algebras. To a more detailed account of this theory we refer the reader to the monographs [17, 18, 34, 35, 38, 40, 44, 46]. Original results concerning noncommutative measure theory may be found in [6, 8, 7, 9, 5, 23, 24, 25, 26, 27, 29, 28].

We start our discussion by showing how the classical measure theory can be imbedded into noncommutative one naturally. Consider the following situation. Let μ be a positive continuous functional on the algebra $C_0(X)$ of all continuous functions on a locally compact Hausdorff space X vanishing at infinity and endowed with the supremum norm and usual algebraic operations. According to the well known Riesz-Markov theorem μ can be identified with a finite Radon measure, i.e. with a regular measure on the σ-algebra generated by the compact subsets of X. The algebra $C_0(X)$ can be realized naturally as an operator algebra in the following way. Take a Hilbert space $H = L^2(X, \mu)$, denote by $B(H)$ the algebra of all bounded operators acting on H, and define a representation $\pi_\mu : C_0(X) \to B(H)$ by

$$\pi_\mu(f)(g) = fg \text{ for all } f \in C_0(X), g \in H.$$

It is straightforward to show that π_f preserves the algebraic operations, sends conjugate functions to adjoint operators and decreases the norm. In that way $C_0(X)$ can be embedded into $B(H)$. It can be proved that by making direct sum of all representations π_μ, where μ runs over all probability Radon measures on X, we get the isomorphisms of the corresponding algebras

$$C_0(X) \sim \sum_\mu \oplus \pi_\mu(C_0(X)).$$

Therefore $C_0(X)$ can be viewed as a norm closed selfadjoint $*$-subalgebra of the algebra $B(H)$. (In the sequel we shall mean by a $*$-algebra of operators a subalgebra of $B(H)$ which is closed under forming adjoints of operators.) The algebra $\pi_\mu(C_0(X))$ is closed in the norm topology but it is not necessarily closed in the weak operator topology. (We say that a net (T_α) of operators on a Hilbert spaces H converges to an operator T if $(T_\alpha \xi, \nu) \to (T\xi, \nu)$ for all $\xi, \nu \in H$.) It can be derived by the standard measure theoretic arguments that the weak operator closure of the algebra $\pi_\mu(C_0(X))$ is isomorphic to the algebra $L^\infty(X, \mu)$ of all essentially bounded μ-measurable functions on X equipped with the essential supremum norm. The isomorphism is given by the representation π_μ introduced above (see e.g. [46]). So the $L^\infty(X, \mu)$-algebra can be realized as an abelian weakly operator closed operator algebra. Operator algebras generalize this situation.

DEFINITION 1.1. *Let H be a Hilbert space. A norm closed $*$-algebra of operators acting on H is called a (concrete) C^*-algebra. A weakly operator closed $*$-algebra of operators acting on H is called a von Neumann algebra.*

The passage from a C^*-algebra of operators to the von Neumann algebra it generates is analogous to the transition from continuous functions to bounded measurable functions.

The simplest example of the noncommutative C^*-algebra is the algebra of all operators acting on two–dimensional Hilbert space, which is nothing but the algebra of all 2x2 complex matrices. The following theorem says that the algebras $C_0(X)$ constitute the only examples of the abelian C^*-algebras:

THEOREM 1.2. *Any abelian C^*-algebra is isomorphic to some algebra $C_0(X)$. Any abelian von Neumann algebra is isomorphic to some algebra $L^\infty(X, \mu)$.*

The algebra $C_0(X)$ uniquely determines the underlying locally compact space X. For that reason C^*-algebras are often viewed as a generalization of the locally compact spaces and many topological constructions can be done algebraically. For example, adjoining unity corresponds to one-point (minimal) compactification of X, making multiplier algebra corresponds to the Stone-Čech (maximal) compactification (se e.g [35]), etc. This approach is also the basic strategy of the noncommutative geometry (see e.g. [16]). Similarly, the L^∞-algebras generalize to von Neumann algebras. Therefore the theory of states on von Neumann algebra is sometimes referred to as the noncommutative measure theory. We will follow mainly this line.

For the sake of completeness we mention that both C^*-algebras and von Neumann algebras can be defined as abstract $*$-Banach algebras in the following way:

DEFINITION 1.3. *A complex Banach algebra A with an involution $*$ satisfying $||x^*x||^2 = ||x||^2$ is called an (abstract) C^*-algebra. A C^*-algebra which has a predual is called a von Neumann algebra.*

As we have seen in connection with the Riesz-Markov theorem, the measure in the classical measure theory can be generalized to the positive functional on an operator algebra.

DEFINITION 1.4. *A norm one linear functional ϱ on a C^*-algebra A is called a* state *if $\varrho(a^*a) \geq 0$ for all $a \in A$.*

There is a basic example of a state on a C^*-algebra A, called vector state, given by

$$\omega_\xi(a) = (a\xi, \xi), \text{ for all } a \in A,$$

where ξ is a unit vector in the underlying Hilbert space H. It can be verified easily that given a $*$-homomorphism π of A into $B(K)$ then $\omega_\xi \circ \pi$ is again a state on A. The famous Gelfand-Naimark-Segal theorem says that any state arises in this way.

THEOREM 1.5 (GELFAND-NAIMARK-SEGAL). *For any state ϱ on a C^*-algebra A there is a unique (up to an isomorphism) triple $(H_\varrho, \pi_\varrho, \xi_\varrho)$, where H_ϱ is a Hilbert space, π_ϱ is a $*$-homomorphism of A into $B(H_\varrho)$, and $\xi_\varrho \in H_\varrho, ||\xi_\varrho|| = 1$, such that*

$$\varrho(a) = (\pi_\varrho(a)\xi_\varrho, \xi_\varrho), \; a \in A,$$

and $\overline{lin}\{\pi_\varrho(a)\xi_\varrho \mid a \in A\} = H_\varrho$.

Specifying the Gelfand-Naimark-Segal theorem to the classical measure-theoretic situation $(C_0(X), \mu)$, we see that the G.N.S. triple is $(L^2(X,\mu), \pi_\mu, \mathbf{1})$, where $\mathbf{1}$ is a constant function taking value 1 at all points of X. Indeed, $(f \cdot \mathbf{1}, \mathbf{1}) = \int_X f d\mu = \mu(f)$ for all functions $f \in C_0(X)$.

After reviewing basic definitions and results on C^*-algebras we concentrate mainly on von Neumann algebras. Throughout the paper M will always denote a von Neumann algebra. The *center* $Z(M)$ of M is the set of all elements z in M such that $zx = xz$ for all $x \in M$. The algebra M is abelian if and only if $M = Z(M)$. The basic structure for the measure theory on von Neumann algebra is the projection lattice of M. An element $e \in M$ is called a *projection* if $e = e^2 = e^*$. The set of all projections in M will be denoted by $P(M)$. If one introduces an order on $P(M)$ by $e \leq f$ if $ef = e$, then $P(M)$ becomes a complete lattice. We shall denote the infimum and the supremum of elements in $P(M)$ by $e \wedge f$ and $e \vee f$, respectively. The projection lattice is always rich, meaning that its closed linear span is the whole algebra. If $M = L^\infty(X,\mu)$ then $P(M)$ can be identified with the structure of measurable subsets of X. There is one important equivalence relation, $\sim$, on the lattice $P(M)$ given by $e \sim f$ if $e = v^*v$ and $f = vv^*$ for some $v \in M$. In that case v is a partial isometry mapping the range of e onto the range of f. For example, two projections in $B(H)$ are equivalent if and only if their ranges have the same dimension. The concept of equivalence is trivial for abelian algebras and enables to describe to what extend a given von Neumann algebra differs from abelian one.

DEFINITION 1.6. *Let e be a projection in a von Neumann algebra M. Then e is*
(1) abelian *if $eMe = \{exe \mid x \in M\}$ is an abelian von Neumann algebra (iff the segment $[0, e]$ in P(M) is a distributive lattice),*
(2) finite *if $e \sim f \leq e$ implies $e = f$ (iff the segment $[0, e]$ is a modular lattice),*
(3) infinite *if it is not finite,*
(4) properly infinite *if ze is infinite for any projection $z \in Z(M)$.*

For example, let e be a projection in $B(H)$. Then e is abelian iff $dim\ e(H) = 1$; e is finite iff $dim\ e(H) < \infty$, and e is properly infinite iff $dim\ e(H) = \infty$.

The von Neumann algebra M is called *finite, infinite, and properly infinite*, if its largest projection, **1**, which is a unite for M, is finite, infinite, and properly infinite, correspondingly. The von Neumann algebras are classified into types as the following table shows:

So we have a scale of types. Abelian algebras, i.e. Type I_1 algebras, are located on one side of this scheme, while Type III algebras are on the opposite side. Type I_n algebras are exactly matrix algebras $M_n(A)$ of all $n \times n$ matrices over some abelian algebra A. Type II_1 algebras can be viewed as 'continuous' matrix algebras and arise often in connection with unitary representations of some groups and by making inductive limits of finite-dimensional algebras. Type III algebras can be obtained, for example, by a special construction called cross product. Basic result of the structure theory of von Neumann algebras says that any von Neumann algebra can be obtained as a direct sum of some types stated in the table. This simplifies if M is a *factor*, i.e. the algebra with one-dimensional center. In that case M has one of the types defined in the table. Von Neumann

Type	Definition	Examples
Type I	every nonzero central projection majorizes a nonzero abelian projection	matrix algebras $B(H)$
Type I_n	$\mathbf{1}$ is a sum of n equivalent abelian projections	$M_n(A)$ A abelian
Type II	(1) M contains no nonzero abelian projection (2) any nonzero central projection majorizes a nonzero finite projection	
Type II_1	M is of Type II and finite	group algebras inductive limits
Type II_∞	M is of Type II and properly infinite	tensor products constructions
Type III	M does not contain any nonzero finite projection	cross products

algebras without Type III direct summand are called *semifinite.*

After reviewing basic properties of von Neumann algebras we return to the measure theory built on them. Given a state ϱ on a von Neumann algebra M we get by restricting ϱ to the projection lattice $P(M)$ a function which is finitely additive with respect to orthogonal projections, i.e. $\varrho(e+f)=\varrho(e)+\varrho(f)$ for $e,f\in P(M)$, with $ef=0$. Such a positive function on $P(M)$ will be called a *measure.* We say that a measure μ on $P(M)$ is *completely additive* if $\mu(\vee_{\alpha\in I}e_\alpha)=\sum_{\alpha\in I}\mu(e_\alpha)$, where (e_α) is a system of mutually orthogonal projections. A measure μ is called σ-additive (or countably additive) if $\mu(\vee e_n)=\sum_n\mu(e_n)$, whenever (e_n) is a countable system of mutually orthogonal projections. The additivity of a state ϱ on $P(M)$ embodies continuity property of ϱ. For example, ϱ is continuous with respect to the weak*-topology on M if and only if $\varrho|P(M)$ is completely additive. So, measures given by restrictions of states are not always σ-additive or completely additive. For that reason finitely additive measures are more important and natural on operator algebras then in the standard measure theory. Let us also remark that if M is abelian, i.e. isomorphic to some $L^\infty(X,\mu)$, then measure on $P(M)$ gives a finitely additive measure on the system of measurable subsets of X. In that sense the noncommutative measure theory is an extension of the classical measure theory.

Any measure μ on the σ-algebra $P(M)$, where $M = L^\infty(X,\mu)$ can be extended to a positive functional on M by taking the abstract Lebesgue integral with respect to μ. A natural question arises whether also any probability measure on the projection lattice of a general von Neumann algebra is the restriction of some state on M. This problem can be viewed as a problem of constructing a noncommutative integral. The basic and very deep theorem of the noncommutative measure theory says that measures do extend to states, except type I_2 algebras [1, 21, 15, 37, 39, 47].

THEOREM 1.7 (GENERALIZED GLEASON THEOREM). *Let M be a von Neumann algebra which does not contain any Type I_2 direct summand. Then any finitely additive probability measure on the projection lattice $P(M)$ extends (uniquely) to a state on M.*

It took more then thirty years to prove the Gleason theorem in the form stated above. Its final version for even Banach-space-valued measures has been proved in [10, 11] and it is highly nontrivial as well. To illustrate why it is relatively easy to conjecture the Gleason theorem and so difficult to prove it, let us realize that there is only one candidate for a linear extension ϱ of a measure μ on $P(M)$. Indeed, any self-adjoint element $a \in M$ can be approximated in norm by a finite linear combination of mutually commuting projections. Therefore, if ϱ does exists, then the value $\varrho(a)$ has to be the limit of $\varrho(s_n)$, where s_n are self-adjoint elements with finite spectrum (analog of the step functions). The value $\varrho(s_n)$ can be defined straightforwardly as

$$\varrho(s_n) = \sum_{i=1}^{m_n} \lambda_{i,n}\mu(e_{i,n}),$$

for $s_n = \sum_{i=1}^{m_n} \lambda_{i,n} e_{i,n}$, where $e_{i,n}$ are orthogonal spectral projections of s_n. In this way we get a function on M that is certainly additive on commuting operators. Nevertheless, it has proven to be extremely difficult to show that ϱ will become additive with respect to mutually noncommuting elements. For verifying that nontrivial functional-analytic methods have to be employed. The problem of linearity of a partially additive function is still open for the C^*-algebras (see [36, 14, 13] for an important progress).

PROBLEM 1.8 (MACKEY–GLEASON): Let A be a C^*-algebra having no representation on a two–dimensional Hilbert space. Let f be a real function on the self-adjoint part of A, bounded on the unit ball of A, and fulfilling the following two conditions for all commuting elements $a, b \in A$, and $\lambda \in R$:

$$f(\lambda a) = \lambda f(a)$$

$$f(a+b) = f(a) + f(b).$$

Is then f linear?

2. Subadditive measures

In this section we shall deal with subadditive measures on projections. We say that a measure μ on the projection lattice $P(M)$ is *subadditive* if

$$\mu(e \vee f) \leq \mu(e) \vee \mu(f) \text{ for all } e, f \in P(M).$$

The concept of subadditivity is the first example of notion which says nothing in the classical measure theory but which is very strong in the noncommutative measure theory.

EXAMPLES 2.1. *(1) If M is abelian then any measure on $P(M)$ is subadditive. Indeed, if projections e and f commute then $e \wedge f = ef$ and $e \vee f = e + f - ef$. Whence*

$$\begin{aligned} \mu(e \vee f) &= \mu(e + f - ef) = \mu(e) + \mu(f) - \mu(e \wedge f) \\ &\leq \mu(e) + \mu(f). \end{aligned}$$

(2) Any tracial state ϱ on M restricts to a subadditive measure on $P(M)$. A state ϱ on M is called tracial *if one of the following equivalent conditions is satisfied:*
(i) $\varrho(ab) = \varrho(ba)$ for all $a, b \in M$,
(ii) $\varrho(u^* a u) = \varrho(a)$ for all $a \in M$ and all unitary elements $u \in M$.
Any tracial state has to be constant on equivalent projection and so,

when restricted to the projection lattice, it can be regarded as a generalized dimension. For that reason tracial states play an important role in the structure theory of operator algebras. The subadditivity of such states is a consequence of the Kaplansky formula

$$e \vee f - f \sim e - e \wedge f$$

which implies, for a tracial state ϱ, that

$$\varrho(e \vee f) = \varrho(e) + \varrho(f) - \varrho(e \wedge f) \leq \varrho(e) + \varrho(f).$$

The following elementary observation characterizes subadditive measures on the simplest noncommutative algebra. Even if the proof is elementary, it might give some insight.

OBSERVATION 2.2. *Any subadditive state ϱ on the algebra M_n of all $n \times n$ matrices is the normalized trace.*

Proof. Take $e, f \in P(M_n)$, with $dim\ e = dim\ f = 1, ef \neq 0$. Then $e \vee (1 - f) = 1$. By the subadditivity

$$1 \leq \varrho(e) + \varrho(1 - f) = \varrho(e) + 1 - \varrho(f).$$

In other words, $\varrho(f) \leq \varrho(e)$ and, by symmetry, $\varrho(e) = \varrho(f)$. It follows $\varrho(e) = \varrho(f)$ for all one-dimensional projections e, f. Using standard linear algebra arguments it follows that ϱ is a trace. □

It turns out that the the foregoing example can be generalized considerably to all von Neumann algebras [8, 42, 41]. In particular, we succeeded in showing that for subadditive measures the generalized Gleason theorem holds without any restriction upon the type I_2 direct summand [8].

THEOREM 2.3. *Let μ be a subadditive (finitely additive) probability measure on the projection lattice $P(M)$ of a von Neumann algebra M. Then μ extends uniquely to a tracial state of M.*

3. Countably additive $*$-homomorphisms

The aim of this part is to describe an interplay between the σ-additivity and the lattice properties of $*$-homomorphisms between von Neumann algebras. A linear map $\pi : M \to N$ between von Neumann algebras M and N is called a $*$-*homomorphism* if $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a^*) = \pi(a)^*$ for all $a, b \in M$. It implies that π maps a projection lattice $P(M)$ into a projection lattice $P(N)$ and it is finitely additive on orthogonal projections. A homomorphism π is called σ-additive if

$$\pi(\sum e_n) = \sum \pi(e_n)$$

for any orthogonal sequence of projections (e_n) in M. (The sums are considered in the weak-operator topology.) The σ-additivity represents a higher degree of continuity. This can be clearly demonstrated for σ-finite algebras. A von Neumann algebra is called *σ-finite* if any orthogonal system of nonzero projections in M is at most countable. If M is σ-finite then $\pi : M \to N$ is σ-additive if and only if it is weak*-continuous. In order to relate additivity of homomorphisms to additivity of states let us remark that a state ϱ is σ-additive if and only if the $*$-homomorphisms π_ϱ appearing in the Gelfand-Naimark-Segal Theorem (Theorem 1.5) is σ-additive. One of the charms of the theory of operator algebras are theorems on automatic continuity. It is interesting to see how the algebraic structure itself forces functional-analytic continuity properties. In the following theorem we have summarized results of this type due to Feldmann and Fell, and Takesaki [20, 45].

THEOREM 3.1. *Let M and N be σ-finite von Neumann algebras. A $*$-homomorphism $\pi : M \to N$ is σ-additive if one of the following conditions is satisfied:*

(i) *M is of Type II_1 and N acts on a separable Hilbert space*

(ii) *M is properly infinite.*

It should be noted that in the results stated above it is not enough to assume only that the domain algebra is σ-finite. For example, the quotient map $\pi : B(H) \to B(H)/K$ of the algebra $B(H)$ into its quotient by the ideal K of all compact operators on H is never σ-additive even if H is separable. So the restriction upon the size of the von Neumann algebras in question as well as restrictions upon their type are indispensable for σ-additivity of homomorphisms. On the other hand, we show that simple algebraic condition borrowed from the lattice theory characterizes σ-additivity without posing any further restrictions. We shall approach to this problem from the other side. Every $*$-homomorphism π preserves projections and so also their order. A natural question arises when π preserves also the suprema of projections. If it does preserve suprema then it preserves infima by the linearity and so it has to be a lattice homomorphism. Let us look at two important classes of lattice homomorphisms:

EXAMPLES 3.2. *(1) If M is abelian then any $*$ homomorphism $\pi : M \to N$ is a lattice homomorphism. Indeed,*

$$\begin{aligned} \pi(e \vee f) &= \pi(e) + \pi(f) - \pi(ef) = \\ &= \pi(e) + \pi(f) - \pi(e)\pi(f) = \pi(e) \vee \pi(f). \end{aligned}$$

(2) It turns out that the σ-additivity always implies that π is a lattice homomorphism. For this let us take $e, f \in P(M)$ and assume that $\pi : M \to N$ is σ-additive. Pick a vector ξ in the underlying Hilbert space H such that $[\pi(e) \vee \pi(f)]\xi = 0$. By the spectral theorem, applied to the self-adjoint operator $e+f$, there are projections (p_n) such that $p_n \nearrow e \vee f = range(e+f)$ and $e+f \geq 1/n\, p_n$. From the σ-additivity of π we infer $\pi(p_n) \nearrow \pi(e \vee f)$ and so $\pi(p_n)\xi \to \pi(e \vee f)\xi$. But

$$0 = ((\pi(e) + \pi(f))\xi, \xi) \geq 1/n(\pi(p_n)\xi, \xi),$$

which implies $\pi(p_n)\xi = 0$ and so $\pi(e \vee f)\xi = 0$. We have proved that $\pi(e \vee f) \leq \pi(e) \vee \pi(f)$. Since the reverse inequality holds for any $*$-homomorphism the proof is completed.

Therefore, σ-additivity implies that given $*$-homomorphism is a lattice homomorphism. Perhaps surprisingly it turns out that also the converse holds (see [7]).

THEOREM 3.3. *Let $\pi : M \to N$ be a $*$-homomorphism between von Neumann algebras, where M has no nonzero abelian direct summand. Then the following statements are equivalent:*

(i) $\pi(e \vee f) = \pi(e) \vee \pi(f)$ *for all projections e and f.*

(ii) π *is countably additive on $P(M)$.*

Theorem 3.3 indicates that there is an interesting interplay between the ordered structure of projections and the weak* topology on M. This interplay is a purely noncommutative phenomenon, since if M is abelian then any $*$-homomorphism on M is manifestly a lattice homomorphism.

Also, Theorem 3.3 might have the following consequences for the axiomatic foundations of quantum physics. Any projection operator has a two point spectrum and therefore it corresponds to an observable, or experiment, with two possible outcomes. For that reason the projection is regarded as a basic proposition about the quantum system which is true (if we measure value 1) or false (if we measure value 0). The largest projection $\mathbf{1}$ is always true. The supremum $e \vee f$ of the projections e and f is interpreted as the proposition 'e or f.' Now a simple algebraic computation implies that condition (ii) in Theorem 3.3 is in fact equivalent to

$$e \vee f = 1 \text{ implies } \pi(e) \vee \pi(f) = 1.$$

In physical interpretation it means that

if 'e or f' is tautologically true, then '$\pi(e)$ or $\pi(f)$' is also tautologically true.

This natural conditions implies that π has to be countably additive, thereby justifying the countable additivity on a simple physical ground.

Theorem 3.3 has further the following ramifications [7]. We say that a homomorphism $\pi : M \to N$ is called a *$*$-Jordan homomorphism* if π is $*$-linear and $\pi(x^2) = \pi(x)^2$ for all $x \in M$.

COROLLARY 3.4. *If $\pi : M \to N$ is a $*$-Jordan homomorphism between von Neumann algebras M and N where M has no nonzero abelian direct summand, then the following statements are equivalent:*

(i) $\pi(e \vee f) = \pi(e) \vee \pi(f)$ *for all projections e and f*

(ii) π *is countably additive on $P(M)$.*

Another stronger version of Theorem 3.3 can be obtained for the lattice morphisms themselves. A map $\varphi : P(M) \to P(N)$ between the projection lattices of von Neumann algebras is said to be *lattice morphism* if the following two conditions are satisfied for all $e, f \in P(M)$:
(1) $\varphi(e), \varphi(f)$ are orthogonal whenever e and f are orthogonal,
(2) $\varphi(e \vee f) = \varphi(e) \vee \varphi(f)$ for all e, f.

A lattice morphism is a σ-*morphism* if

$$\varphi(\vee_{n=1}^{\infty} e_n) = \vee_{n=1}^{\infty} \varphi(e_n)$$

for *every* sequence $(e_n) \subset P(M)$.

COROLLARY 3.5. *Let M and N be von Neumann algebras where M has neither nonzero abelian nor Type I_2 direct summand. Then every lattice morphism $\varphi : P(M) \to P(N)$ is a σ-morphism.*

The proof is based on the fact that by the Generalized Gleason Theorem any lattice morphism extends to a Jordan homomorphism on M, which explains the presence of the assumption on type I_2 direct summand. Again, in purely noncommutative case we get σ-additivity for free.

4. Jauch-Piron states

In this part we shall analyze another important property of states, called the Jauch-Piron property, which is automatically satisfied in the standard measure theory but has unexpected consequences in the noncommutative framework.

DEFINITION 4.1. *A state ϱ on a von Neumann algebra M is called* Jauch-Piron *if $\varrho(e \vee f) = 0$ whenever e, f are projections in M with $\varrho(e) = \varrho(f) = 0$.*

Expressing this definition in the lattice theoretic language we can say that a state ϱ is Jauch-Piron if and only if $Ker\,\varrho \cap P(M)$ is a sublattice of $P(M)$. Therefore the Jauch-Piron property has a natural mathematical structure. Nevertheless, this concept has its origin in the mathematical foundations of quantum mechanics. It was introduced and studied by J.M.Jauch and C.Piron [30, 31, 32, 43] in the context of the propositional calculus of quantum theory. The value $\varrho(e)$ of a state ϱ at a projection e represents probability of the event given by e if the ambient physical system is prepared in the state ϱ. The Jauch-Piron property then translates as follows: If events e and f have both probability zero, then the probability of occurring event e or event f is also zero. States without this property are too pathological to find applications to quantum mechanics.

EXAMPLES 4.2. *(1) Any subadditive state is Jauch Piron. Especially, the traces and the states on abelian algebras are Jauch-Piron.*
(2) Any σ-additive state is Jauch-Piron. (This can be obtained by using the same ideas as in the proof of Example 3.2 (2).)
(3) It is not so straightforward to give an example of a state which is not Jauch-Piron. We shall outline this construction briefly (see e.g. [4]). Denoting by M_2 the algebra of all 2x2 complex matrices let us form an l^∞-direct sum of these algebras $M = M_2 \oplus M_2 \oplus \cdots$. Fix a state φ_1 on the algebra l^∞ with $\varphi_1 | c_0 = 0$ and a vector state φ_2 on M_2 given by a unit vector ξ in a two-dimensional Hilbert space underlying the von Neumann algebra M_2. The basic theory of tensor products of von Neumann algebras guarantees that there is a state φ on M such that

$$\varphi(\sum_{n=1}^{\infty} \oplus \, a_n x) = \varphi_1((a_n))\varphi_2(x)$$

for all $(a_n) \in l^\infty$ and $x \in M_2$. For a given unit vector $\nu \in H_2$ we shall denote by p_ν the projection onto the one-dimensional subspace generated by ν. Define projections $e = \sum_{n=1}^{\infty} \oplus \, p_\xi$ and $f = \sum \oplus_{n=1}^{\infty} p_{\xi_n}$, where $\xi_n \to \xi, \xi_n \neq \xi$. Then $\varphi(e) = \varphi_1((1))\varphi_2(p_\xi) = 1$ and $\varphi(f) = \varphi(efe) = \varphi(\sum_{n=1}^{\infty} \oplus \lambda_n p_\xi)$, where $\lambda_n \to 1$. So $(1-\lambda_n)_n \in c_0$ and $\varphi_1((\lambda_n)) = 1$. It follows

$$\varphi(f) = \varphi_1((\lambda_n))\varphi_2(p_\xi) = 1.$$

Hence, $e \wedge f = \sum \oplus (p_\xi \wedge p_{\xi_n}) = 0$, or equivalently, $(1-e) \vee (1-f) = 1$, while $\varphi(1-e) = \varphi(1-f) = 0$. Thus the Jauch-Piron property of ϱ is violated for projections $1-e, 1-f$. This construction can be modified *for general essentially noncommutative algebra. In particular, it can be proved [6, 22] that direct sums of abelian and finite-dimensional algebras constitutes all examples of von Neumann algebra for which each state is Jauch-Piron (see also Theorem 5.2).*

For further analysis of the Jauch-Piron states we shall need the following notions. A state ϱ on M is called *singular* if for any projection $p \in M$ such that $\varrho(p) > 0$ there exists a nonzero projection $q \in M$, $q \leq p$, with $\varrho(q) = 0$. This is equivalent to saying that the identity can be decomposed into the sum $\mathbf{1} = \sum e_\alpha$, where (e_α) is an orthogonal system of projections with $\varrho(e_\alpha) = 0$ for all α. Similarly to the classical measure theory any state on a von Neumann algebra can be decomposed into completely additive and singular part. We say that a state ϱ on M is *regular* if $\varrho(\sum e_n) = 0$ for any orthogonal sequence of projections (e_n) with $\varrho(e_n) = 0$ for all n. Regularity is a weaker form of the $\sigma-$additivity. On the other hand, it can be proved that ϱ is regular if and only if the kernel of ϱ is closed with respect to suprema of countably many projections [5]. Therefore regularity is a strong σ-additive version of the Jauch-Piron property. Hence, the following implications hold

$$\varrho \text{ is } \sigma\text{-additive} \implies \varrho \text{ is regular} \implies \varrho \text{ is Jauch-Piron.} \tag{1}$$

None of these implications can be reversed.

COUNTEREXAMPLE 4.3. *If M is a finite von Neumann algebra with an infinite-dimensional center, then there is a tracial (and therefore Jauch-Piron) state ϱ on M which is singular and nonregular on the center.*

However, it turns out that in many cases the Jauch-Piron condition is strong enough to imply regularity, non-singularity or even σ-additivity. At first we summarize results concerning the σ-finite algebras [6].

THEOREM 4.4. *Let M be a σ-finite von Neumann algebra. Then*

(i) *if M is properly infinite, then all Jauch-Piron states on M are non-singular,*

(ii) *if M is of Type III, then all Jauch-Piron states on M are regular,*

(iii) *if M is semifinite and ϱ is a Jauch-Piron state on M, then ϱ is regular if and only if ϱ is nonsingular on $Z(eMe)$ for all finite projections e of M.*

The assumptions of Theorem 4.4 are not superfluous. It is therefore interesting that the continuity properties of the Jauch-Piron states depend on the type of the von Neumann algebra and the size of its center.

Now we turn to "large" von Neumann algebras. M is called *locally σ-finite* if M is a direct sum of σ-finite algebras. For example, any finite algebra is locally σ-finite. By the structure theory of von Neumann algebras [44] any von Neumann algebra M can be written as $M = M_0 \oplus M_1$, where M_0 is the largest locally σ-finite direct summand of M and M_1 is a part containing no σ-finite direct summand. The part M_1 can be considered as a "big portion" of M (it has to live on some large Hilbert space). A projection e in M is called *locally σ-finite* if the hereditary subalgebra eMe is locally σ-finite. As we have seen in Theorem 4.4 the position of regularity and the Jauch-Piron property is complicated for σ-finite and thereby also for locally σ-finite algebras. Nevertheless, for large algebras the situation simplifies surprisingly (see [6]).

THEOREM 4.5. *Let M be a von Neumann algebra with no nonzero σ-finite direct summand. Then all Jauch-Piron states on M are regular.*

This theorem is counterintuitive on first reading because for algebras with large systems of orthogonal projections one might expect the situation to be much worse then in the case of σ-finite algebras. Combining both σ-finite and non-σ-finite case we get the following theorem [6]:

THEOREM 4.6. *Let ϱ be a Jauch-Piron state on a von Neumann algebra M. Then ϱ is regular on M if ϱ is nonsingular on the centers of hereditary subalgebras $Z(eMe)$ for all projections e in M for which the central cover $c(e)$ is locally σ-finite. Especially, every Jauch-Piron state on factor is regular.*

In summary, the more noncommutative structure we have the more distinct is the relationship between algebraic and continuity properties of states.

5. Jauch-Piron property and σ-additivity.

In the concluding part of this note we shall study σ-additivity of Jauch-Piron states. The σ-additivity is a convenient mathematical idealization of the Jauch-Piron property which lacks a priori physical justification. However, it turns out that the σ-additivity coincide with the Jauch-Piron property for states having a nice geometrical structure.

We recall a few notions and fix the notation. For a von Neumann algebra M the symbol $S(M)$ will stand for the convex set of all states on M. Endowed with the weak*-topology, $S(M)$ becomes a compact topological space. A state ϱ is called *pure* if it is an extreme point of the state space $S(M)$. For example, any vector state on $B(H)$ is pure. The following theorem was proved for algebras with separable preduals in [4] and for the general case in [6].

THEOREM 5.1. *Let M be a von Neumann algebra containing no non-zero abelian part. A pure state ϱ on M is Jauch-Piron if and only if it is σ-additive.*

For a global explanation of this interplay between the Jauch-Piron condition and the convex geometry we shall need some further concepts. A convex subset F of $S(M)$ is called a *face* if $\varrho_1, \varrho_2 \in F$ whenever $\frac{1}{2}(\varrho_1 + \varrho_2) \in F$. Any singleton subset of the state space is a face exactly when it consists of a pure state. A face F of $S(M)$ is called a *split face* if there is a face $F^\#$ (called complementary face of F) such that $S(M)$ is the direct convex sum of F and $F^\#$, i.e. any state ϱ can be written uniquely as a convex combination of states $\varrho_1 \in F$ and $\varrho_2 \in F^\#$. There are deep theorems in the theory of von Neumann algebras describing the facial structure of the state space in terms of projections in the double dual (see e.g. [3, 19]). It is known that the double dual M^{**} of a von Neumann algebra M is again a von Neumann algebra whose algebraic operations extend operations in M after embedding M into M^{**}. Then F is a norm closed face of $S(M)$ if and only if $F = \{\varrho \in M^* \mid \varrho(p) = 1\}$, where p is a uniquely determined projection in M^{**}. Therefore the structure of norm closed faces is isomorphic to the projection lattice $P(M^{**})$. Moreover, a set F is a split face of $S(M)$ if and only if $F = \{\varrho \in M^* \mid \varrho(z) = 1\}$, where z is a uniquely determined central projection in M^{**}. So split faces correspond to projections in the center of the double dual. It should be noted that the center of the double dual is always large, so the structure of split faces is rich. Given a state ϱ on M and an element $a \in M$ with $\varrho(a^*a) \neq 0$ we can define a transformed state

$$\varrho_a(x) = \frac{\varrho(a^*xa)}{\varrho(a^*a)}.$$

It can be proved that a norm closed face is a split face if $\varrho_a \in F$ whenever $\varrho \in F$ (see [5]). According to this one can verify easily that the following sets are split faces: the set of all σ-additive states, the set of all completely additive states, the set of all singular states.

Let us now investigate the convex properties of the Jauch-Piron state space, i.e. of the set $S_J(M)$ of all Jauch-Piron states on a von Neumann algebra M. The $S_J(M)$ is always a convex subset of the state space. Nevertheless, it turns out that $S_J(M)$ is almost never a face and that it is almost never norm closed, expect the case of all states being Jauch-Piron [5].

THEOREM 5.2. *Let M be a von Neumann algebra. The following statements are equivalent:*

(i) *$S_J(M)$ is a face*

(ii) *$S_J(M)$ is norm closed*

(iii) *$S_J(M) = S(M)$*

(iv) *$M = M_1 \oplus M_2$, where M_1 is abelian and M_2 is finite-dimensional.*

Despite this, if we make a norm closure of the set $S_J(M)$, we get even a split face.

THEOREM 5.3. *Let M be a von Neumann algebra. Then $\overline{S_J(M)}$ is a split face and*

$$\overline{S_J(M)}^{\#} \subset \{\varrho \in S(M) \mid \varrho(e) = 0 \text{ for all } \sigma\text{-finite projections } e\}.$$

Theorem 5.3 says that the complementary face of the norm closure of the Jauch-Piron state space consists of highly nonsingular states. They have to vanish at any σ-finite projection. In particular, if M is σ-finite then $S_J(M)$ is norm dense in the state space.

No we state a criterion of the σ-additivity in terms of the Jauch-Piron property. The proof is based on the theory of states on von Neumann algebras [33] and our results on the σ-additivity of $*$-homomorphisms between von Neumann algebras (see [7] and section 3).

THEOREM 5.4. *Let M be a von Neumann algebra with no nonzero abelian direct summand. A state ϱ is σ-additive if and only if the split face F_ϱ generated by ϱ consists of Jauch-Piron states.*

An individual Jauch-Piron state ϱ need not to be σ-additive. Nevertheless, if any transformed state of ϱ is Jauch-Piron then ϱ has to be already σ-additive. This also explains the fact that every pure Jauch-Piron state ϱ is σ-additive. Indeed, if ϱ is a pure state, then a smallest split face F_ϱ containing ϱ is equal to $\{\varrho(u^* \cdot u) \mid u \in M \text{ is unitary}\}$. Since all unitary transformations of ϱ are Jauch-Piron we see by Theorem 5.4 that ϱ is σ-additive. Besides, Theorem 5.4 has the following interesting consequence:

THEOREM 5.5. *If M is a von Neumann algebra with no nonzero abelian part then $S_\sigma(M)$, the split face of all σ-additive states on M, is the largest split face contained in $S_J(M)$.*

It means that, for essentially noncommutative algebras, the Jauch-Piron structure in the state space determines uniquely the σ-additivity. Another result of this type can be obtained for homomorphisms:

THEOREM 5.6. *Let $\pi : M \to N$ be a $*$-homomorphism between von Neumann algebras M and N where M has no nonzero abelian direct summand. Then π is σ-additive if and only if $\varrho \circ \pi$ is a Jauch-Piron state on M for each Jauch-Piron state ϱ of N.*

In summary, using the concept of Jauch-Piron state one can justify the use of the σ-additive states in foundations of quantum physics on elementary physical ground. The results on σ-additivity of pure Jauch-Piron state can also be extended in another direction. Namely, it turns out that conditions much weaker then pureness imply, for Jauch-Piron states, the σ-additivity. For formulating these results we shall need the following notions. A state $\varrho \in S(M)$ is called *countably singular* if there is a sequence of orthogonal projections (e_n) in M such that $\sum e_n = \mathbf{1}$ and $\varrho(e_n) = 0$ for all n. By the symbol $S_{s\sigma}(M)$ we shall denote the set of all countably singular states on M. It is clear that any countably singular state lies in the complementary split face of the split face of all σ-additive states because it can not majorize any positive multiple of a σ-additive state. Moreover, countably additive states generate this split face.

THEOREM 5.7. *The complementary split face $S_\sigma(M)^\#$ of the split face $S_\sigma(M)$ of all σ-additive states on a von Neumann algebra M is the smallest split face containing all countably singular states of M.*

With the aid of the continuum hypothesis the foregoing theorem simplifies considerably:

THEOREM 5.8. *Let M be a von Neumann algebra. If M is σ-finite or if the continuum hypothesis is true then*

$$S_\sigma(M)^\# = S_{s\sigma}(M).$$

In the proof of Theorem 5.8 the continuum hypotheses is used to show that countably singular states form a norm closed face. This theorem can be viewed as a noncommutative result on the decomposition of the state space into σ-additive and countably singular part. Provided that the following question has a positive answer the Theorem 5.8 holds without any set theoretic assumption.

PROBLEM 5.9: Let μ be a finitely additive probability measure on the power set $Exp(R)$ of the set of all real numbers R vanishing on all singletons. Is there a sequence (A_n) of disjoint subsets of R with $R = \cup A_n$ and $\mu(A_n) = 0$ for all n?

The decomposition theorem plays an important role in proving that much general condition then pureness imply, together with the

Jauch-Piron property, the σ-additivity. A state ϱ of a von Neumann algebra M is called a *factor state* if the weak operator closure of $\pi_\varrho(M)$ in the algebra of all operators on the Hilbert space $B(H_\varrho)$, where H_ϱ is a Hilbert space appearing in the Gelfand-Naimark-Segal Theorem (Theorem 1.5), is a factor. Factor states are important for the structure theory of von Neumann algebras and arise often in physical applications. Important examples of factor states are pure states and their special convex combinations. It is known that ϱ is a factor state if and only if the canonical extension of ϱ to the second dual is concentrated on some minimal central projection. Since the central projections in the double dual correspond to split faces in the state space a factor state ϱ is either in F or in $F^\#$ for any split face F. This property of factor states together with the results above implies the following criterion of the σ-additivity [5]:

THEOREM 5.10. *Let M be a properly infinite von Neumann algebra. Suppose that M is σ-finite or the continuum hypothesis is true. Then all Jauch-Piron factor states on M are σ-additive.*

This result does not hold for finite algebras. For example, it can be shown that any Type I_n algebra with infinite-dimensional center admits a Jauch-Piron factor state which is singular on the center. However, the position of the Jauch-Piron property and $\sigma-$additivity can be fully described for finite algebras without the aid of the continuum hypothesis (for more details see [5]).

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