ON CRITICAL GROUPS AND THE HOMOTOPY INDEX IN MORSE THEORY ON HILBERT MANIFOLDS (*)

by KRZYSZTOF P. RYBAKOWSKI (in Freiburg) (**)

SOMMARIO. - *Sia U un aperto nello spazio di Hilbert H, $\varphi \in C^{2-}(U, \mathbf{R})$, $\xi \in U$ un punto critico isolato di φ, e π il flusso generato dalle soluzioni di $\dot{u} = -\nabla\varphi(u)$. Se ξ ha un intorno fortemente ammissibile, allora i gruppi critici di (φ, ξ) nel senso di Rothe sono isomorfi ai gruppi di omologia dell'indice di omotopia di $(\pi, \{\xi\})$ (Teorema 2). Se $\varphi \in C^2(U, \mathbf{R})$, $\varphi''(\xi)$ è un'applicazione di Fredholm, ma ξ non ha un intorno fortemente ammissibile, allora tutti i gruppi critici di (φ, ξ) sono uguali a zero (banali) (Teorema 4).*

SUMMARY. - *Let U be open in the Hilbert space H, $\varphi \in C^{2-}(U, \mathbf{R})$, $\xi \in U$ be an isolated critical point of φ and π be the flow generated by the solutions of $\dot{u} = -\nabla\varphi(u)$. If ξ has a strongly admissible neighborhood, then the critical groups of (φ, ξ) are isomorphic to the homology groups of the homotopy index of $(\pi, \{\xi\})$ (Theorem 2). If $\varphi \in C^2(U, \mathbf{R})$, $\varphi''(\xi)$ is a Fredholm operator, but ξ does not have a strongly admissible neighborhood then all critical groups of (φ, ξ) are trivial (Theorem 4).*

1. - Introduction

Let M be a Riemannian manifold of class C^2 modelled on a Hilbert space H. If $\varphi : M \to \mathbf{R}$ is a function of class C^{2-0}, i.e. if $\varphi \in C^1(M, \mathbf{R})$ and the gradient $\nabla\varphi : M \to TM$ is locally Lipschitzian, then φ induces a local flow π_φ on M generated by the solutions of the *ODE*

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(**) Indirizzo dell'Autore: Institut für Angewandte Mathematik der Albert-Ludwigs-Universität - Hermann Herder Strasse 10 - 7800 Freiburg im Breisgau - West Germany.

$$\dot{x}(t) = -\nabla\varphi(x(t)) \tag{1}$$

Critical points of φ are just the equilibria of π_φ.

If ξ is an isolated critical point of φ, then the critical groups $C_q(\varphi, \xi)$, $q \in \mathbf{Z}$ are defined as

$$C_q(\varphi, \xi) = \hat{H}_q(\varphi^c \cap B, \varphi^c \cap B \setminus \{\xi\}).$$

Here $\hat{H}_q$ is the singular homology theory with coefficients in a field F, $c = \varphi(\xi)$, $\varphi^c = \{x \in M \mid \varphi(x) \le c\}$, and B is an arbitrary closed neighborhood of ξ ([6], [20]).

The notion of critical groups extends that of the Morse index of ξ. In fact, if $\varphi''(\xi)$ exists and is nondegenerate, then the Morse index of (φ, ξ) is the dimension m of the unstable manifold of ξ with respect of (1). If m is finite, then $C_q \cong F$ for $q = m$, and $C_q = \{0\}$ otherwise.

If m is infinite, then $C_q = \{0\}$ for all q. ([6]).

Critical groups can be used to prove the generalized Morse inequalities for degenerate critical points ([6]). To do this, one must impose an asymptotic compactness assumption on φ called the Palais-Smale condition.

Every isolated critical point ξ of φ forms an isolated invariant set $K = \{\xi\}$ with respect to the flow π_φ. If K has a strongly π_φ-admissible isolating neighborhood, then the homotopy index $h(\pi, K)$ is defined ([8], [13]).

Using the homotopy index theory one can prove the generalized Morse inequalities for arbitrary Morse decomposition of an arbitrary isolated π-invariant set S. Here, π is a local semiflow on X and we only have to assume that S has a strongly π-admissible isolating neighborhood ([19]).

The strong π-admissibility is an asymptotic compactness condition related to the Palais-Smale condition ([21]).

Therefore one is naturally led to ask about the relation between critical groups and the homotopy index.

The aim of this note is to show that, under the admissibility assumption, the critical groups are nothing else but the homology groups of the homotopy index (Theorem 2).

This extends earlier well-known results ([2], [11]) to the case of degenerate critical points.

Moreover, if the admissibility assumption is not satisfied, then, under some additional assumption, all critical groups are zero, so they play no role in the Morse inequalities (Theorem 4).

These two theoretical results show that, essentially, the Morse theory (with Palais-Smale), both for nondegenerate and for degenerate critical points, can yield no better results than the extended homotopy index theory. On the other hand it is known that the homotopy index has several advantages over the Morse index or critical groups, the most important one being its homotopy invariance property. This makes the homotopy index a flexible tool for perturbation problem as opposed to the «static» nature of Morse index or critical groups.

Unless otherwise specified, let π be a local semiflow on a metric space X. Recall that a closed set $N \subset X$ is called *π-admissible,* if given any two sequences $x_n \in X$, $t_n \geq 0$, $n \in \mathbf{N}$, such that $t_n \to \infty$ as $n \to \infty$ and $x_n \pi [0, t_n] \subset N$ for all $n \in \mathbf{N}$, then the sequence $\{x_n \pi t_n \mid n \in \mathbf{N}\}$ has a convergent subsequence.

N is called *strongly π-admissible* if N is π-admissible and π *does not explode in N,* i.e. whenever $x \in N$ and $x\pi [0, \omega_x) \subset N$, then $\omega_x = \infty$. Here, ω_x is the supremum of all $t \geq 0$ for which $x\pi [0, t]$ is defined.

We will now state without proof a sufficient condition for strong π-admissibility, which is a special case of results proved in [11], [14].

Proposition 1 - *Let X be a Banach space and U be open in X. Let $A: X \to X$ be a bounded linear operator and $f: U \to X$ be locally Lipschitzian. Let π be the local flow on U generated by the ODE*

$$\dot{x} = Ax + f(x). \tag{2}$$

Suppose that there is a direct sum decomposition $X = X_1 \oplus X_2$ with $A(X_i) \subset X_i$, $A_i := A \mid X_i$, $i = 1, 2$, and such that $\dim X_2 < \infty$ and $\operatorname{re} \sigma(A_1) < -\delta$ for some $\delta > 0$.

Then the following properties are satisfied:

1. *If $f(0) = 0$, $f'(0)$ exists and $f'(0) = 0$, then $K = \{0\}$ has a strongly π-admissible neighborhood.*
2. *If f is compact, i.e. takes bounded sets into relatively compact sets, then every bounded set $N \subset U$ which is closed in X is strongly π-admissible.*

Lastly we will recall the definition of an isolating block.

Let B be closed in the metric space X, and $x \in \partial B$.

x is called an *exit* (resp. *entrance,* resp. *bounce-off*) point of B if for every solution $\sigma: [-\delta_1, \delta_2] \to X$ through $x = \sigma(0)$ with $\delta_1 \geq 0$ and $\delta_2 > 0$ the following properties hold:

(i) there is an ε_2, $0 < \varepsilon_2 \leq \delta_2$ such that $\sigma(t) \notin B$ (resp. $\sigma(t) \in \operatorname{Int} B$, resp. $\sigma(t) \notin B$) for $0 < t \leq \varepsilon_2$.

(ii) if $\delta_1 > 0$, then there is an ε_1, $0 < \varepsilon_1 \leq \delta_1$ such that $\sigma(t) \in \operatorname{Int} B$ (resp. $\sigma(t) \notin B$, resp. $\sigma(t) \notin B$) *for* $-\varepsilon_1 \leq t < 0$.

B is called an *isolating block* if:

(i) each boundary point x of B is an entrance, an exit or a bounce-off point of B.

(ii) the set B^- consisting of all exit and all bounce-off points of B is closed.

REMARKS - The homotopy index theory for two-sided flows on compact spaces is due to Conley [2].

This theory was extended by the present author to «admissible» semiflows on noncompact spaces. For more information on this extended theory see [2] and [8] - [19].

2. - The main results

Unless otherwise specified, let π be a local semiflow on a metric space X.

For the rest of this paper, let H_q, $q \in \mathbf{Z}$, be an arbitrary homology or cohomology theory with values in an arbitrary R-module G. We assume that $H_q = \{0\}$ for $q < 0$, and that H_q is defined for all pairs of topological spaces.

By E we denote the set of all equilibria of π. If $\varphi : X \rightarrow \mathbf{R}$ is any function and $c \in \mathbf{R}$, we write

$$\varphi^c = \{x \in X \mid \varphi(x) \leq c\}$$

$$E_c = \{x \in E \mid \varphi(x) = c\}.$$

If $\varphi : X \rightarrow \mathbf{R}$ is continuous, then φ is called a *quasi-potential of* π if for every $x \in X \setminus E$ there is an $\varepsilon > 0$ such that the function $t \rightarrow \varphi(x\pi t)$, $t \in [0, \varepsilon)$, is strictly decreasing.

Note that if φ is continuous and $\limsup\limits_{t \to 0^+} \frac{1}{t}(\varphi(x\pi t) - \varphi(x)) < 0$ for all $x \in X \setminus E$ then φ is a quasi-potential of π_φ.

In particular, if $\varphi \in C^{2-0}(M, \mathbf{R})$, then φ is a quasi-potential of π_φ.

For the rest of this section, let φ be a quasi-potential of π and ξ be an isolated critical point of φ with $\xi \in E_c$.

The *critical groups* $C_q(\varphi, \xi)$ are defined as

$$C_q(\varphi, \xi) = H_q(\varphi^c \cap B, \varphi^c \cap B \setminus \{\xi\}), \; q \in \mathbf{Z} \tag{3}$$

where B is any closed neighborhood of ξ.

The excision axiom of (co)homology easily implies that $C_q(\varphi, \xi)$ is independent of the choice of B.

We can now state

THEOREM 2 - *Let $\xi \in E_c$ be an isolated equilibrium of π admitting a strongly π-admissible closed neighborhood N.*

Then the homotopy index $h(\pi, K)$, where $K = \{\xi\}$, is defined and

$$H_q(h(\pi, K)) \cong C_q(\varphi, \xi), \quad q \in \mathbf{Z}. \tag{4}$$

In other words, the critical groups of (φ, ξ) are just the (co)homology groups of the homotopy index of $(\pi, \{\xi\})$.

Proof. Using the definition of a quasi-potential it is easily proved that N is an isolating neighborhood of the isolated invariant set $K = \{\xi\}$. By results in [8], the homotopy index $h(\pi, K)$ is defined. Moreover, given any isolating block $B \subset N$, $\xi \in B$, it follows from results in [8], [19] that

$$H_q(h(\pi, K)) \cong H_q(B, B^-), \quad q \in \mathbf{Z}. \tag{5}$$

Here, B^- is the set of all exit and bounce-off points of B.

We will show that there exists an isolating block $B \subset N$ with

$$H_q(B, B^-) \cong H_q(\varphi^c \cap B, \varphi^c \cap B \backslash \{\xi\}), \quad q \in \mathbf{Z} \tag{6}$$

thereby completing the proof:

We need the following

LEMMA 3 - *There exists an $\alpha < c$ and an isolating block $B \subset N$, $\xi \in B$, such that $B^- \subset \varphi^\alpha$.*

Lemma 3 is proved in the Appendix.

Choose α and B as in the lemma.

Recall that $A^+(B)$ denotes the set of all $x \in X$ such that $x\pi t \in B$ for all $t \geq 0$ for which $x\pi t$ is defined. Since π does not explode in N (by the strong admissibility assumption), it follows that for $x \in A^+(B)$, $x\pi t$ is defined for all $t \geq 0$.

Similarly $A^-(B)$ is the set of all $x \in X$ for which there is a solution $\sigma : \mathbf{R}^- \to B$ with $\sigma(0) = x$.

For $x \in B \backslash A^+(B)$ define $s_B^+(x) = \sup\{t \geq 0 \mid x\pi[0, t]$ is defined and $x\pi[0, t] \subset B\}$.

It is easily seen that s_B^+ is continuous (see Lemma 13 in [19]).

Let $C = \varphi^c \cap B \backslash \{\xi\}$. Then $C \cap A^+(B) = \phi$. In fact suppose

$x \in C \cap A^+(B)$. Then $x\pi t \to \xi$ as $t \to \infty$. Therefore $x \neq \xi$ implies $\varphi(x) > \varphi(\xi) = c$, a contradiction.

Define $D : [0,1] \times C \to C$ by

$$D(\tau, x) = x\pi(\tau \cdot s_B^+(x)). \tag{7}$$

D is well-defined and continuous. Moreover, our choice of B implies that $B^- \subset C$ and, clearly, $s_B^+(x) = 0$ for $x \in B^-$, so $D(\tau, x) \equiv x$ for $x \in B^-$. Moreover, $D(0, x) \equiv x$ and $D(1, x) \in B^-$, for $x \in C$.

This proves that D is a strong deformation retraction of C onto B^-.

It follows that

$$H_q(B, B^-) \cong H_q(B, \varphi^c \cap B \backslash \{\xi\}), \quad q \in \mathbf{Z}. \tag{8}$$

Now notice that if $x \in B \backslash A^+(B)$ and $\varphi(x) > c$, then

$$\varphi(x\pi s_B^+(x)) \leq \alpha < c$$

so that there is a unique $t(x) < s_B^+(x)$ such that $\varphi(x\pi t(x)) = c$.

Define for $(\tau, x) \in [0,1] \times B$

$$\rho(\tau, x) = \begin{cases} x & \text{if } \varphi(x) \leq c, \\ x\pi(\tau/(1-\tau)) & \text{if } \varphi(x) > c, x \in A^+(B),\ 0 \leq \tau < 1, \\ \xi & \text{if } \varphi(x) > c, x \in A^+(B),\ \tau = 1, \\ x\pi(\tau/(1-\tau)) & \text{if } \varphi(x) > c, x \notin A^+(B),\ 0 \leq \tau < 1, \\ & \quad \tau/(1-\tau) \leq t(x) \\ x\pi t(x) & \text{if } \varphi(x) > c, x \notin A^+(B),\ 0 \leq \tau < 1 \\ & \quad \text{and } \tau/(1-\tau) > t(x) \\ x\pi t(x) & \text{if } \varphi(x) > c, x \notin A^+(B),\ \tau = 1 \end{cases} \tag{9}$$

ρ is a well-defined map from $[0,1] \times B \to B$. In the Appendix we prove that ρ is continuous.

Clearly $\rho(\tau, x) \equiv x$ for $x \in \varphi^c \cap B$, and $\rho(0, x) \equiv x$ and $\rho(1, x) \in \varphi^c \cap B$ for all $x \in B$.

Consequently ρ is a strong deformation retraction of B onto $\varphi^c \cap B$.

Thus we obtain that

$$H_q(B, \varphi^c \cap B \backslash \{\xi\}) \cong H_q(\varphi^c \cap B,\ \varphi^c \cap B \backslash \{\xi\}), \quad q \in \mathbf{Z} \tag{10}$$

(8) and (10) imply (6) and the theorem is proved.

We shall now specialize to the local flow π_φ of section 1. If ξ is an isolated critical point of (i.e. an isolated equilibrium of) π_φ then the critical groups of (φ, ξ) are defined even if there is no strongly π_φ-admissible neighborhood of ξ.

However, using a result of Mawhin and Willem, we will prove that in such a case, under some additional assumption, all critical groups are zero, so they play no role in the Morse inequalities. Since all considerations are of local character, we may assume that M is an open set in a Hilbert space H, $\xi = 0$ and $\varphi(\xi) = 0$.

Consider the following assumption (S):

(S) U is an open neighborhood of 0 in a Hilbert space H, $\varphi : U \to \mathbf{R}$ is a C^2-function with $\nabla\varphi(0) = 0$, $\nabla\varphi(u) \neq 0$ for $u \neq 0$, and $L = \varphi''(0) : H \to H$ is a Fredholm operator.

We use some arguments from [7]. Let $R(L)$ (resp. $N(L)$) be the range (resp. the kernel) of L. Let $Q : H \to H$ be the orthogonal projector of H onto $R(L)$. Write $u = v + w$ where $v = Qu$ and $w = (I - Q)u$. Then $v \in R(L)$ and $w \in N(L)$.

Now the implicit function theorem implies that there is a ball $\Omega \subset H$ centered at zero and a unique C^1-map $g : \Omega \cap N(L) \to R(L)$ such that $g(0) = 0$ and

$$Q\nabla\varphi(w + g(w)) = 0 \text{ for } w \in \Omega \cap N(L). \tag{11}$$

Let

$$\hat{\varphi}(w) = \varphi(w + g(w)). \tag{12}$$

Then $\hat{\varphi}$ is a C^2-function (!) and

$$\nabla\hat{\varphi}(w) = (I - Q)\nabla\varphi(w + g(w)) \tag{13}$$

$$\hat{\varphi}''(w) = (I - Q)\varphi''(w + g(w))(Id + g'(w)) \tag{14}$$

for $w \in \Omega \cap N(L)$.

By the generalized Morse Lemma due to Mawhin and Willem [5], there is an open neighborhood $\tilde{U} \subset U$ of 0 in H, an open neighborhood $W \subset \Omega \cap N(L)$ of 0 in $N(L)$, and a homeomorphism h from U into $\tilde{U}$ such that $h(0) = 0$ and

$$\varphi(h(u)) = 1/2(Lv, v) + \hat{\varphi}(w) \tag{15}$$

for $u = v + w \in \tilde{U}$.

Let $R(L) = H^+ \oplus H^-$ be the direct sum decomposition of $R(L)$ into the subspaces H^+ and H^- on which L is positive— (resp. negative—) definite, and let $v = v^+ + v^-$ be the corresponding decomposition of $v \in R(L)$.

We can now state

THEOREM 4 - *Let (S) be satisfied and* $\hat{\varphi}$ *be as in* (12). *Let* $m = \dim H^-$. *Then the following properties hold:*

1. *If $m = \infty$, then all critical groups of $(\varphi, 0)$ are trivial.*

2. *If $m < \infty$, then $K = \{0\}$ has a strongly π-admissible isolating neighborhood, $h(\pi_\varphi, K)$ is defined and for all $q \in \mathbf{Z}$*

$$C_q(\varphi, 0) \cong H_q(h(\pi_\varphi, \{0\}) \cong H_{q-m}(h(\pi_{\hat{\phi}}, \{0\})) \cong C_q(\hat{\varphi}, 0) \qquad (16)$$

Here, $\pi_{\hat{\phi}}$ is the local semiflow on $\Omega \cap N(L)$ generated by $\dot{w} = -\nabla\hat{\varphi}(w)$.

Proof. Let B be any closed bounded neighborhood of 0 in H, $B \subset \tilde{U}$. Then

$$C_q(\varphi, 0) \cong H_q(\varphi^0 \cap h(B), \varphi^0 \cap h(B) \backslash \{0\}) \cong H_q(\psi^0 \cap B, \psi^0 \cap B \backslash \{0\} \cong = C_q(\psi, 0) \qquad (17)$$

where

$$\psi(u) = 1/2(Lv, v) + \hat{\varphi}(w). \qquad (18)$$

Consider the *ODE*

$$\dot{u} = -\nabla\psi(u) \qquad (19)$$

(19) is uncoupled and has the form

$$\left.\begin{aligned} \dot{v}^+ &= -Lv^+ \\ \dot{v}^- &= -Lv^- \\ \dot{w} &= -\nabla\hat{\varphi}(w) \end{aligned}\right\} \qquad (20)$$

Let us prove that 0 is the only critical point of ψ in $\tilde{U}$. In fact, $\nabla\psi(0) = 0$ of course. If $\nabla\psi(u) = Lv + \nabla\hat{\varphi}(w) = 0$, then $v = 0$ and $\nabla\hat{\varphi}(w) = 0$. Now (11) and (13) imply that $\nabla\varphi(w + g(w)) = 0$, so $w + g(w) = 0$, i.e. $w = 0$. The claim follows. If $m < \infty$, then by Proposition 1, $\{0\}$ has a strongly π_φ-admissible isolating neighborhood, so $h(\pi_\varphi, \{0\})$ is defined. By the same token, $h(\pi_\psi, \{0\})$ is defined.

Here we have used the fact that $\dim N(L) < \infty$. Since π_ψ is a product of local flows, we obtain

$$h(\pi_\psi, \{0\}) = \Sigma^m \wedge h(\pi_{\hat{\phi}}, \{0\}). \qquad (21)$$

Here, Σ^m is the homotopy type of the pointed m-dimensional unit sphere. (cf. [14]).

Hence, using simple arguments from algebraic topology (cf. the proof of Proposition 3.1 in [15]), we get

$$H_q(h(\pi_\psi, \{0\})) = H_{q-m}(h(\pi_{\hat{\phi}}, \{0\})) \qquad (22)$$

(17), (21) (22) and Theorem 1 imply formula (16).

Now assume that $m = \infty$. By our assumption $B \cap N(L)$ is an isolating neighborhood of $\{0\}$ with respect to π_ϕ. Moreover, since $\dim N(L) < \infty$, $B \cap N(L)$ is trivially strongly π_ϕ-admissible.

Let $\varepsilon > 0$ be arbitrary and set

$$B_1 = \{v^+ \in H^+ \mid (Lv^+, v^+) \leq \varepsilon\} \tag{23}$$

$$B_2 = \{v^- \in H^- \mid -(Lv^-, v^-) \leq \varepsilon\}. \tag{24}$$

Since $\hat{\varphi}(0) = 0$, there is a ball W in $N(L)$ at 0 with radius $< \varepsilon$ such that

$$|\hat{\varphi}(w)| < \varepsilon / 2 \text{ for } w \in W \tag{25}$$

Choose $\varepsilon > 0$ so small that $B_1 \oplus B_2 \oplus W \subset \tilde{U}$.

By Lemma 3, there is an isolating block $B_3 \subset W$, $0 \in B_3$, with respect to π_ϕ such that $B_3^- \subset \hat{\varphi}^{-\delta}$ for some $\delta > 0$. Choose such a block B_3 and let $B = B_1 \oplus B_2 \oplus B_3$.

Define for $\tau \in [0,1]$, $u = v^+ + v^- + w \in B$

$$D_1(\tau, u) = (1 - \tau) v^+ + v^- + w \in B.$$

D_1 is a continuous map from $[0,1] \times B$ to B. Moreover $D_1([0,1] \times (\psi^0 \cap B)) \subset \psi^0 \cap B$. Let $\tau \in [0,1]$ and $u \in \psi^0 \cap B \backslash \{0\}$. Then $D_1(\tau, u) \neq 0$ if $\tau < 1$, of course. Suppose $D_1(1, u) = 0$. Then $v^- + w = 0$ so $u = v^+ \neq 0$. But then $\psi^0(u) > 0$, a contradiction.

Thus using D_1, we see that $\psi^0 \cap (B_2 \oplus B_3)$ is a strong deformation retract of $\psi^0 \cap B$ and $\psi^0 \cap (B_2 \oplus B_3) \backslash \{0\}$ is a strong deformation retract of $\psi^0 \cap B \backslash \{0\}$.

Thus

$$C_q(\psi, 0) \cong H_q(\psi^0 \cap (B_2 \oplus B_3), \psi^0 \cap (B_2 \oplus B_3) \backslash \{0\}). \tag{26}$$

As in the proof of Theorem 2, define for $u = v^- + w \in B_2 \oplus B_3$

$$s^+(u) = \sup\{t \mid u\pi_\psi[0, t] \text{ is defined and } u\pi_\psi[0, t] \subset B_2 \oplus B_3\}.$$

If $u \notin A^+(B_2 \oplus B_3)$, then $s^+(u) < \infty$. Moreover, as before, s^+ is easily seen to be continuous on $B_2 \oplus B_3 \backslash A^+(B_2 \oplus B_3)$.

Let us show that $\psi^0 \cap (B_2 \oplus B_3) \backslash \{0\} \cap A^+(B_2 \oplus B_3) = \phi$. In fact let $u = v^- + w \in \psi^0 \cap (B_2 \oplus B_3) \backslash \{0\}$.

If $v^- \neq 0$, then from (20), $u \notin A^+(B_2 \oplus B_3)$. Suppose $v^- = 0$. Then $w \neq 0$ and $\hat{\varphi}(w) \leq 0$. Since $\hat{\varphi}$ is a potential of π_ϕ, it follows that the solution of π_ϕ through $x = w$ must leave B_3. The claim is proved. Define for $\tau \in [0,1]$, $u \in \psi^0 \cap (B_2 \oplus B_3 \backslash \{0\}$,

$$D_2(\tau, u) = u\pi_\psi(\tau \cdot s^+(u)) \subset B_2 \oplus B_3.$$

Since ψ decreases along solutions of π_ψ, $D_2(\tau, u) \in \psi^0 \backslash \{0\}$.

D_2 defines a strong deformation retraction onto
$\psi^0 \cap (\partial B_2 \oplus B_3 \cup B_2 \oplus B_3^-) = \partial B_2 \oplus B_3 \cup B_2 \oplus B_3^-$.
The last equality is a consequence of our choice of ε, δ and B_3.

Now (26) implies that

$$C_q(\psi, 0) \cong H_q(\psi^0 \cap (B_2 \oplus B_3), \ \partial B_2 \oplus B_3 \cup B_2 \oplus B_3^-). \qquad (27)$$

Let $\rho : [0 \ \ 1] \times B_3 \to B_3$ be defined as in (9) with φ, ξ, π, B being replaced by $\hat{\varphi}, 0, \pi_{\hat{\varphi}}, B_3$, respectively.

For $\tau \in [0, 1]$, $u = v^- + w \in B_2 + B_3$ define

$$D_3(\tau, u) = v^- + \rho(\tau, w).$$

From (9) we see that $\hat{\varphi}(\rho(\tau, w)) \leq \hat{\varphi}(w)$. Hence, if $u \in \psi^0$, then

$$\psi(D_3(\tau, u)) = \tfrac{1}{2}(Lv^-, v^-) + \hat{\varphi}(\rho(\tau, w)) \leq \psi(u) \leq 0.$$

Therefore$D_3 : [0, 1] \times (\psi^0 \cap (B_2 \oplus B_3)) \to \psi^0 \cap (B_2 \oplus B_3)$ is well-defined and continuous. D_3 is a strong deformation retraction onto

$\psi^0 \cap (B_2 \oplus (\hat{\varphi}^0 \cap B_3)) = B_2 \oplus (\hat{\varphi}^0 \cap B_3)$.

Moreover the restriction of D_3 to $[0, 1] \times (\partial B_2 \oplus B_3 \cup B_2 \oplus B_3^-)$ is easily seen to be a strong deformation retraction onto

$\partial B_2 \oplus (\hat{\varphi}^0 \cap B_3) \cup B_2 \oplus B_3^-$.

Thus from (27),

$$C_q(\psi, 0) \cong H_q(B_2 \oplus (\hat{\varphi}^0 \cap B_3), \ \partial B_2 \oplus (\hat{\varphi}^0 \cap B_3) \cup B_2 \oplus B_3). \qquad (28)$$

On H^- define the norm $|v^-| = -\varepsilon^{-1}. \ (Lv^- \ \ v^-)$.
$|v^-|$ is equivalent to the scalar product norm on H^-.

Since H^- is infinite-dimensional, a well-known result of Dugundji implies that there is a strong deformation retraction D_4 of the closed unit ball $\tilde{\Omega}$ in $(H^-, | \ \ |)$ onto $\partial\tilde{\Omega}$ (see [4], page 66 for an easy proof that there is a retraction $r : \tilde{\Omega} \to \partial\tilde{\Omega}$ and define the strong deformation retraction D_4 by $D_4(\tau, x) = (1 - \tau)\, x + \tau r(x))$.

However, $\tilde{\Omega} = B_2$, so there is a strong deformation retraction D_4 of B_2 onto ∂B_2. Define $D_5(\tau, v^- + w) = D_4(\tau, v^-) + w$.

Using D_5 and arguments as above, we get from (28)

$$C_q(\psi, 0) \cong H_q(\partial B_2 \oplus (\hat{\varphi}^0 \cap B_3), \ \partial B_2 \oplus (\hat{\varphi}^0 \cap B_3)) = \{0\}. \qquad (29)$$

Now (17) and (29) complete the proof that $C_q(\varphi, 0) = \{0\}$ for all $q \in \mathbf{Z}$.

REMARKS - Since $\hat{\varphi}$ defines a local center manifold for the uncoupled system (20), formula (16) is a variant of the index product formula. For a more general version of the index product formula see [14]. Related finite-dimensional results also appear in [1] and [3].

3. - Appendix

Proof of Lemma 2: We use arguments from the proof of Theorem 2.1 in [8].

There is an open set $\tilde{U}$, $\xi \in \tilde{U}$ with $\tilde{N} := Cl\tilde{U} \subset \operatorname{Int} N$, and functions $\tilde{g}^+ = g^+_{\tilde{U}}$, $\tilde{g}^- = g^-_{\tilde{N}}$ defined on page 360 in [8] such that for some $\delta > 0$ and all δ_1, δ_2, $0 < \delta_1, \delta_2 < \delta$ the set

$$B = B_{\delta_1,\delta_2} = Cl\{x \in \tilde{U} \mid \tilde{g}^+(x) < \delta_1, \tilde{g}^-(x) < \delta_2\} \tag{30}$$

is an isolating block with

$$B^- = \{x \in \partial B \mid \tilde{g}^+(x) = \delta_1\}. \tag{31}$$

(Note that in [8], 'B^+' is written to denote what we mean by B^-).

Fix $\delta_1 < \delta$. We will show that there is an $\alpha < c$ and a $\delta_2 < \delta$ such that $B^- \subset \varphi^\alpha$. In fact, otherwise, there is a sequence $x_n \in \tilde{U}$ with $\tilde{g}^+(x_n) = \delta_1$, $\varphi(x_n) \geq c - n^{-1}$ for $n \in \mathbf{N}$, and $\tilde{g}^-(x_n) \to 0$ for $n \to \infty$. As in the proof of Theorem 2.1 in [8], this implies that for a subsequence of $\{x_n\}$, denoted again by $\{x_n\}$, we have

$$x_n \to x \in A^-(N) \text{ as } n \to \infty.$$

There is a solution $c: \mathbf{R}^- \to N$ of π with $\sigma(0) = z$. By admissibility $\sigma(t) \to \xi$ as $t \to -\infty$. Since $\tilde{g}^+$ and $\tilde{g}^-$ are continuous on B, it follows that $\tilde{g}^+(z) = \delta_1 > 0$ so $z \neq \xi$. Therefore $\varphi(z) < \varphi(\xi) = c$. However, by our assumption $\varphi(z) \geq c$, a contradiction which completes the proof.

Now we prove that ρ defined by (9) is continuous. In fact, otherwise there are $\beta > 0$, and a sequence $(\tau_n, x_n) \in [0,1] \times B$ converging to $(\tau, x) \in [0,1] \times B$ and such

$$d(\rho(\tau_n, x_n), \rho(\tau, x)) \geq \beta \text{ for all } n \tag{32}$$

where d denotes the metric on X.

We will consider several cases, each time arriving a contradiction.

1. $\varphi(x) \leq c$: Then $\rho(\tau, x) = x$

By (32), we may assume that $\varphi(x_n) > c$ for all n. Thus $\varphi(x) = c$.

1.1. $x \in A^+(B)$:

Then, clearly, $x = \xi$ and so $x_n \to \xi$ as $n \to \infty$. Moreover, by

(9), for every n, $\rho(\tau_n, x_n) = x_n \pi t_n$ for some $t_n \geq 0$ and $\varphi(x_n \pi t_n) \geq c$, or else $\rho(\tau_n, x_n) = \xi$.

By (32) we may assume the first case for all n.

We claim that $x_n \pi t_n \to \xi$ for $n \to \infty$. In fact, this is so if $\{t_n\}$ is bounded, since $\xi \pi t \equiv \xi$ for all $t \geq 0$.

If $\{t_n\}$ is unbounded, then, w.l.o.g. $t_n \to \infty$. Then by admissibility $\{x_n \pi t_n\}$ has limit points. Let z be any such limit point, then $z \in A^-(B)$ (cf. [8]), and by our assumption, $\varphi(z) \geq c$. As in the proof of Lemma 2 we obtain $z = \xi$. The claim is proved.

It follows that $\rho(\tau_n, x_n) \to \xi = \rho(\tau, x)$, a contradiction to (32).

1.2. $x \notin A^+(B)$:

We may assume that $x_n \notin A^+(B)$ for all n since $A^+(B)$ is closed. We claim that $t(x_n) \to 0$ as $n \to \infty$. In fact let $c > 0$ be any small number. Since $x \neq \xi$, $\varphi(x) = c$. We obtain $\varphi(x \pi \varepsilon) < \varphi(x) = c$. Thus $\varphi(x_n \pi \varepsilon) < c$ for all n large enough. Hence $t(x_n) < \varepsilon$ for all such n, and the claim follows. By (9) $\rho(\tau_n, x_n) = x_n \pi s_n$ with $0 \leq s_n \leq t(x_n)$. Now the claim implies $\rho(\tau_n, x_n) \to x \pi 0 = x = \rho(\tau, x)$ a contradiction to (32).

2. $\varphi(x) > c$:

Then we may assume that $\varphi(x_n) > c$ for all n.

2.1. $x \in A^+(B)$:

Then by (32) we may assume that $x_n \notin A^+(B)$ for all n.

2.1.1. $\{t(x_n)\}$ is unbounded:

Then w.l.o.g. $t(x_n) \to \infty$: If $\tau < 1$, then

$$\tau_n(1 - \tau_n)^{-1} \to \tau(1 - \tau)^{-1}$$

as $n \to \infty$, hence, in particular, $\tau_n(1 - \tau_n)^{-1} < t(x_n)$ for n large enough. Then (9) implies $\rho(\tau_n, x_n) \to \rho(\tau, x)$, a contradiction. If $\tau = 1$, then $\rho(\tau, x) = \xi$, and $\rho(\tau_n, x_n) = x_n \pi t_n$ with $t_n \to \infty$, and $\varphi(\rho(\tau_n, x_n)) \geq c$. As in the proof of Lemma 2, this implies that $\rho(\tau_n . x_n) \to \xi$, a contradiction.

2.1.2. $\{t(x_n)\}$ is bounded:

Then, w.l.o.g. $t(x_n) \to t_0 < \infty$ as $n \to \infty$. It follows that $c = \varphi(x_n \pi t(x_n)) \to \varphi(x \pi t_0)$, so $\varphi(x \pi t_0) = c$. But $x \pi t_0 \in A^+(B)$, hence $x \pi t_0 = \xi$. (Note that this cannot happen if π is a local (two-sided) flow). Thus $x \pi t \equiv \xi$ for all $t \geq t_0$. This and (9) clearly imply that $\rho(\tau_n, x_n) \to \rho(\tau, x)$ as $n \to \infty$, a contradiction.

2.2. $x \notin A^+(B)$:

Then we may assume that $x_n \notin A^+(B)$ for all n.

We claim that $t(x_n) \to t(x)$. In fact, if $M > 0$, $M < t(x)$ is given, then $\varphi(x\pi M) > c$ for n large so $t(x_n) > M$ for n large. Similarly, $t(x) < M$ implies $t(x_n) < M$ for n large and the claim follows. Our claim and (9) now clearly imply that $\rho(\tau_n, x_n) \to \rho(\tau, x)$, again a contradiction to (32).

The proof is complete.

REFERENCES

[1] AMANN, H. and E. ZEHNDER, *Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations,* Ann. Scuola Norm. Sup. Pisa IV, Vol. VII, 539-603 (1980).

[2] CONLEY, C., *Isolated invariant sets and the Morse index,* CBMS 38, Providence, 1978.

[3] DANCER, E. N., *Degenerate critical points, homotopy indices and Morse inequalities,* Journal für Reine und Angewandte Mathematik 350, 1980, 1-22.

[4] DEIMLING, K., *Nonlinear Functional Analysis,* Springer Verlag 1985.

[5] MAWHIN, J. L. and M. WILLEM, *On the generalized Morse Lemma,* Bull. Soc. Math. Belg. série b vol. 37 fasc. II (1985), 23-29.

[6] MAWHIN, J. L. and M. WILLEM, *Lectures on Variational Methods,* preprint.

[7] MAWHIN, J. L. and M. WILLEM, *On the generalized Morse Lemma,* preprint.

[8] RYBAKOWSKI, K. P., *On the homotopy index for infinite dimensional semiflows,* Trans, Amer. Math. Soc. 269 (1982), 351-382.

[9] RYBAKOWSKI, K. P., *On the Morse index for infinite dimensional semiflows,* in «Dynamical Systems II», Academic Press, New York 1982, 633-637.

[10] RYBAKOWSKI, K. P., *The Morse index, repeller attractor pairs and the connection index for semiflows on noncompact spaces,* J. Diff. Equations (47) (1983), 66-98.

[11] RYBAKOWSKI, K. P., *Trajectories joining critical points of nonlinear parabolic and hyperbolic partial differential equations,* J. Diff. Equations (51) (1984), 182-212.

[12] RYBAKOWSKI, K. P., *Irreducible invariant sets and asymptotically linear functional differential equations,* Boll. U.M.I. (6) 3-B (1984), 245-271.

[13] RYBAKOWSKI, K. P., *An introduction to the homotopy index theory in noncompact spaces,* Appendix to: J. Hale et al. «Introduction to infinite-dimensional semiflows-geometric theory», Springer Verlag 1983, 147-191.

[14] RYBAKOWSKI, K. P., *An index product formula for the study of elliptic resonance problems,* J. Diff. Equations (56) (1985), 408-425.

[15] RYBAKOWSKI, K. P., *Nontrivial solutions of elliptic boundary value problems with resonance at zero,* Ann. Mat. Pura Appl., (IV), Vol. CXXXIX, 237-278.

[16] RYBAKOWSKI, K. P., *A homotopy index continuation method and periodic solutions of second order gradient systems,* J. Diff. Equations, (65) (1986), 203-218.

[17] RYBAKOWSKI, K. P., *On a relation between the Brouwer degree and the Conley index for gradient flows,* Bull. Soc. Math. Belgique, série b, vol. 37, fasc. II (1985), 87-96.

[18] RYBAKOWSKI, K. P., *The homotopy index and partial differential equations,* Springer - Verlag 1987.

[19] RYBAKOWSKI, K. P. and E. ZEHNDER, *A Morse equation in Conley's index theory for semiflows on metric spaces,* Erg. Theory and Dyn. Systems 5 (1985), 123-143.

[20] ROTHE, E., *Critical point theory in Hilbert spaces under regular boundary conditions,* J. Math. Anal. Appl. 36 (1971), 377-431.

[21] RYBAKOWSKI, K. P., *Some recent results in the homotopy index theory in infinite dimensions,* Rend. Ist. Mat. Univ. Trieste, vol. XVIII (1986) 83-92.