GROUP ACTIONS SATISFYING THE DCP: UNIQUENESS(*)

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SOMMARIO.- *L'autore dimostra che se le azioni fra un gruppo finito e il suo gruppo di automorfismi sono tali da soddisfare la proprietà del doppio centralizzante, allora esse sono univocamente determinate (a meno di isomorfismi).*

SUMMARY.- *The autor show that if a finite group is acted upon by a group of its automorphisms in such a way that the action satisfies the double centralizer property, then the action is unique (up to isomorphism).*

We wish to show that actions satisfying the double centralizer property [1] are unique up to isomorphism. This will be accomplished by showing that we may choose the generators of A and G so that the action (A,G) can be completely described in a very precise way. Throughout this discussion, B_n will denote the group of $n \times n$ matrices over Z_p of the form

$$\begin{pmatrix} I & * \\ 0 & t \end{pmatrix}.$$

Here, I is the $(n-1) \times (n-1)$ identity matrix, t is an arbitrary element of Z_p satisfying $t^q = 1$, and both p and q are prime.

THEOREM 1. *Suppose that* $|A| = p^{n-1}q, p > q$, $|G| = p^n, n \geq 1$, *and that the action* (A,G) *satisfies the DCP. Then the action* (A,G) *is isomorphic to the action of* B_n *on the row vector space* $(Z_p)^n$ *by right multiplication.*

Proof. Think of G as an n-dimensional vector space over Z_p. Let P be the Sylow p-subgroup of A and let Q be a Sylow q-subgroup of A.

Claim: There exists a basis for G, $v_0, \ldots, v_{n-1}$, and a basis for P, $x_1, \ldots, x_{n-1}$, such that

(a) $v_i^{x_i} = v_i + v_0$, $1 \leq i \leq n-1$, and

(b) $v_i^{x_j} = v_i$, $i \neq j$, $0 \leq i \leq n-1$, $1 \leq j \leq n-1$.

(*) Pervenuto in Redazione il 25 maggio 1989.

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The claim will be proven by induction on n. The claim is vacuously true when $n = 1$.

Now suppose $n > 1$. Let X be a subgroup of P of index p. Let y_{n-1} be an element of P which is not in X. The induced action $(A/ <y_{n-1}>, C_G(y_{n-1}))$ satisfies the DCP and thus so does $(XQ, C_G(y_{n-1}))$. Therefore, by the induction hypothesis, there exist $v_0, v_1, ..., v_{n-2}$, a basis for $C_G(y_{n-1})$, and $x_1, ..., x_{n-2}$, a basis for X, such that

$$v_i^{x_i} = v_i + v_0 \quad \text{if } 1 \leq i \leq n-2, \text{ and}$$

$$v_i^{x_j} = v_i \quad \text{if } i \neq j, 0 \leq i \leq n-2, 1 \leq j \leq n-2.$$

Since $y_{n-1} \notin X$, $C_G(X) \nsubseteq C_G(y_{n-1})$. Let v_{n-1} be an element of $C_G(X)$ that does not belong to $C_G(y_{n-1})$. Observe that $C_G(X) \cap C_G(y_{n-1}) = C_G(<X,y_{n-1}>) = C_G(P)$. However, C_G is a subgroup lattice antiisomorphism [1] and so $C_G(y_{n-1})$ is a maximal subgroup of G and $C_G(P)$ is a minimal subgroup of G. Thus, $C_G(X)$ is a 2-dimensional subspace of G and must be generated by v_0 and v_{n-1}.

Since y_{n-1} normalizes $C_G(X)$ and centralizes v_0, it acts on $C_G(X)/<v_0>$. But $<y_{n-1}>$ and $C_G(X)/<v_0>$ are both of order p. Therefore, y_{n-1} must centralize $C_G(X)/<v_0>$.

Hence, there exists $a \in Z_p$ such that $v_{n-1}^{y_{n-1}} = v_{n-1} + av_0$. Note that $a \neq 0$, since y_{n-1} does not centralize v_{n-1}. Let $b = a^{-1} \pmod{p}$ and let $x_{n-1} = y_{n-1}^b$. Then $v_{n-1}^{x_{n-1}} = v_{n-1} + abv_0 = v_{n-1} + v_0$ and $C_G(x_{n-1}) = C_G(y_{n-1})$. This completes the proof of the claim.

We will now consider the $n = 1$ and $n \geq 2$ cases separately.

Case 1. $n \geq 2$.

Let $v_n = v_0$. Since A acts faithfully on G, there is a natural embedding of A into $GL(n,p)$ with respect to the ordered basis $v_1, ..., v_n$. Denote this embedding with the symbol $\overline{}$. Therefore, $\bar{x}_i = \begin{pmatrix} I & d_i \\ 0 & 1 \end{pmatrix}$, $1 \leq i \leq n - 1$, where I is the $(n-1) \times (n-1)$ identity matrix and d_i is the $(n-1)$-dimensional column vector with a one in position i and zeroes elsewhere. Therefore, $\bar{P}$ is the group of all $n \times n$ matrices of the form $\begin{pmatrix} I & D \\ 0 & 1 \end{pmatrix}$ where D is an arbitrary $(n-1)$-dimensional column vector over Z_p.

Let y be an element of order q in A. Since A is a P-group, there exists $k \in Z_p$, $k^q = 1$, $k \neq 1$, such that $x^y = x^k$ for all $x \in P$. Write $\bar{y} = \begin{pmatrix} M & C \\ R & t \end{pmatrix}$.

Then

$$\begin{pmatrix} M+DR & C+tD \\ R & t \end{pmatrix} = \begin{pmatrix} I & D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M & C \\ R & t \end{pmatrix} = \begin{pmatrix} M & C \\ R & t \end{pmatrix} \begin{pmatrix} I & kD \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} M & kMD+C \\ R & kRD+t \end{pmatrix}$$

for arbitrary D. Hence, $R = 0$ and $M = k^{-1}tI$.

Since A does not centralize $v_n, P = C_A(v_n)$. Thus, y does not centralize v_n and $t \neq 1$. Since y does not generate A, $C_G(y) \neq C_G(A) = 1$. Therefore, $\bar{y}$ must have 1 as an eigenvalue and so $t = k$.

Hence, $\bar{A} = \langle\bar{P}, \bar{y}\rangle = B_n$. Map G into $(\mathbf{Z}_p)^n$ by sending $\sum_{i=1}^{n} c_i v_i$ into $(c_1, ..., c_n)$. This map and the map $\overline{}$ define an isomorphism from the action (A,G) into the action $(B_n, (\mathbf{Z}_p)^n)$ by right multiplication.

Case 2. $n = 1$.

Let $a \in A$. Then there exists a unique $t_a \in \mathbf{Z}_p$, $t_a \neq 0$, such that $v_1^a = t_a v_1$. Now map A into B_1 by sending a into (t_a) and G into $(\mathbf{Z}_p)^1$ by sending cv_1 into (c). These maps define an isomorphism from the action (A,G) into $(B_1, (\mathbf{Z}_p)^1)$.

Hence, the action (A,G) is isomorphic to the action $(B_n, (\mathbf{Z}_p)^n)$ by right multiplication for all $n \geq 1$. ◇

THEOREM 2. *Let p, q, and r be prime numbers. Suppose A and G are both P-groups of orders $p^n q$ and $p^n r$, respectively, $p \neq q$, $p \neq r$, and $n \geq 1$. Let the action (A,G) satisfy the DCP. Then the action (A,G) is unique up to isomorphism.*

Proof. Fix $s \in \mathbf{Z}_p$, $o(s) = r(\text{mod } p)$. Let $G = PR$, $|P| = p^n$ and $|R| = r$. Let S be the Sylow p-subgroup of A. Since P is characteristic in G, $C_A(P) \triangleleft A$.

Therefore, $C_A(P) \subseteq S$. (A proper subgroup of a P-group is normal if and only if it is a p-group.) Observe that $C_A(P)C_A(R) = C_A(P \cap R) = C_A(1) = A$ and that $C_A(P) \cap C_A(R) = C_A(PR) = C_A(G) = 1$. Hence, A is the semidirect product of $C_A(R)$ and $C_A(P)$.

Observe that $(A/C_A(P), P)$ and $(C_A(R), P)$ are isomorphic actions. Thus, by Lemma 1 [1], $(C_A(R), P)$ satisfies the DCP. Therefore, by Theorem 1, $(C_A(R), P)$ is isomorphic to $(B_n, (\mathbf{Z}_p)^n)$. Clearly, the actions $(C_A(P),$

P) and $(C_A(R), R)$ are trivial. Hence, we only need to determine what effect the elements of $C_A(P)$ have on the elements of R.

Let e be the element of P that corresponds to $(0, \ldots, 0,1)$ in $(Z_p)^n$. Since $o(s) = r(\text{mod } p)$, there exists a nontrivial $z \in R$ such that $v^z = v^s$ for all $v \in P$.

We will now show that $C_G(S) = <e> = [C_A(P), z]$. This will enable us to find a y in $C_A(P)$ such that $[y,z] = e$. Finally, we will use this to compute the action of an arbitrary element of A on an arbitrary element of G.

Let $S_0 = S \cap C_A(R)$, the Sylow p-subgroup of $C_A(R)$. Therefore, $C_p(S_0) = C_G(S_0) \cap P = C_G(S_0) \cap C_G C_A(P) = C_G(S_0 C_A(P)) = C_G(S)$. However, by Theorem 1, $C_p(S_0) = <e>$. Hence, $C_G(S) = <e>$.

Observe that $|S : C_S(G/P)|$ must divide $|\text{Aut}(G/P)| = r - 1$. But $r < p$ and so $|S : C_S(G/P)| = 1$. Thus, $[G/P,S] = 1$ and $[G,S] \subseteq P$. But $P = C_G C_A(P)$ and so $[G,S,C_A(P)] = 1$. Since $R \subseteq G$, it follows that $[R,S,C_A(P)] = 1$.

Since $C_A(P) \subseteq S$ and S is abelian, $[S,C_A(P), R] = 1$. Therefore, by the Three Subgroups Lemma, $[C_A(P),R,S] = 1$. Thus, $[C_A(P), z] \subseteq C_G(S) = <e>$. But $[C_A(P), z] \neq 1$, and so $[C_A(P), z] = <e>$.

Suppose $y_1, y_2 \in C_A(P)$ such that $[y_1, z] = [y_2, z]$. Then $(z^{-1})^{y_1} z = (z^{-1})^{y_2} z$ and $y_1 y_2^{-1} \in C_A(z^{-1}) \cap C_A(P) = C_A(R) \cap C_A(P) = 1$. Therefore, $y_1 = y_2$. But $o(e) = p = |C_A(P)|$. Hence, the map from $C_A(P)$ into $<e>$ which sends y into $[y,z]$ is a bijection. Thus, there exists $y \in C_A(P)$ such that $[y,z] = e$. Hence, $z^y = ze^{-1}$ and $(z^k)^y = e^{f(k)} z^k$ where $f(k) = (s^{-k} - 1)/(s - 1)$ for all $k \in Z_r$.

Let $x \in Z_p$. Then since $[y,e] = 1$, $(z^k)^{y^x} = e^{xf(k)} z^k$. Let $a \in C_A(R)$ and $v \in P$. Then $(vz^k)^{ay^x} = (v^a)^{y^x} (z^k)^{ay^x} = (v^a)(z^k)^{y^k} = v^a e^{xf(k)} z^k$. Thus, the action (A, G) must be unique up to isomorphism. ◇

We will now show that a group may be written uniquely as a coprime direct product of coprime indecomposable groups. This will give us enough information to prove that actions satisfying the *DCP* are essentially unique.

THEOREM 3. *Let* $G = G_1 \times \ldots \times G_m = H_1 \times \ldots \times H_n$ *be two coprime direct products of nontrivial coprime indecomposable groups. Then* $m = n$ *and* $G_i = H_i$, $1 \leq i \leq n$, *after a suitable rearrangement of the subscripts.*

Proof. Let s be an integer such that $1 \leq s \leq m$. Then $G_s = (G_s \cap H_1) \times \ldots \times (G_s \cap H_n)$. However, G_s is coprime indecomposable. Therefore,

$G_s = G_s \cap H_t$ for some t, $1 \le t \le n$, and $G_s \subseteq H_t$. By the symmetry of the argument $H_t \subseteq G_k$ for some k, $1 \le k \le m$. Hence, $G_s \subseteq G_k$. But $|G_i|$, $1 \le i \le m$, are pairwise relatively prime and $G_s \ne 1$. Thus, $s = k$ and $G_s = H_t$.

This implies that each G_i is one of the H_j's. So, $m \le n$. But the symmetry of the argument implies $n \le m$. Hence, $m = n$. ◇

THEOREM 4. *Let the actions* (A_1,G_1) *and* (A_2,G_2) *satisfy the DCP. Suppose that* $A_1 \cong A_2$ *and* $G_1 \cong G_2$. *Then the actions* (A_1,G_1) *and* (A_2,G_2) *are isomorphic.*

Proof. This theorem follows directly from Theorems 1, 2, and 3, and Theorems 2 and 5 [1]. ◇

REFERENCES

[1] R. CALCATERRA, *Group actions with inverting centralizers*. Arch. Math., *49*, (1987), 465-469.