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On some Semilinear Periodic Parabolic Problems

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Summary. - *Let* $\Omega \subset \mathbb{R}^N$ *be a smooth bounded domain. We study existence and nonexistence of positive solutions for some semilinear Dirichlet periodic parabolic problems of the form* $Lu = h(x,t,u)$ *in* $\Omega \times \mathbb{R}$ *for a class of Caratheodory functions* $h : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ *such that* $h(.,0) = 0$ *and* $\lim_{\xi \to 0^+} \xi^{-1} h(.,\xi) = 0$ *or* $\pm\infty$. *All results remain true for the corresponding elliptic problems.*

1. Introduction

Let Ω be a $C^{2+\theta}$ bounded domain in $\mathbb{R}^N$, $\theta \in (0,1)$, $N \geq 2$. For $T > 0$ and $1 \leq p \leq \infty$, let L^p_T be the Banach space of T-periodic functions f on $\Omega \times \mathbb{R}$ (i.e. satisfying $f(x,t) = f(x,t+T)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$) such that $f_{|\Omega\times(0,T)} \in L^p(\Omega \times (0,T))$, equipped with the norm $\|f\|_{L^p_T} := \left\|f_{|\Omega\times(0,T)}\right\|_{L^p(\Omega\times(0,T))}$. Let C_T be the space of continuous and T-periodic functions on $\overline{\Omega} \times \mathbb{R}$ provided with the L^∞ norm, and let $C_T^{1+\theta,(1+\theta)/2}$ be the space of T-periodic functions belonging to $C^{1+\theta,(1+\theta)/2}\left(\overline{\Omega} \times \mathbb{R}\right)$.

Let $\{a_{ij}\}$, $\{b_j\}$, $1 \leq i,j \leq N$, be two families of T-periodic functions satisfying $a_{ij} \in C^{0,1}\left(\overline{\Omega} \times \mathbb{R}\right)$, $a_{ij} = a_{ji}$ and $b_j \in L^\infty_T$, and

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assume that

$$\sum a_{ij}(x,t)\,\xi_i\xi_j \geq \alpha_0 |\xi|^2$$

for some $\alpha_0 > 0$ and all $(x,t) \in \Omega \times \mathbb{R}$, $\xi \in \mathbb{R}^N$. Let A be the $N \times N$ matrix whose i,j entry is a_{ij}, let $\overline{b} = (b_1, ..., b_N)$, let $0 \leq c_0 \in L_T^\infty$ and let L be the parabolic operator given by

$$Lu = u_t - div\,(A\nabla u) + \langle \overline{b}, \nabla u \rangle + c_0 u$$

Let $W = \left\{ u \in L^2\left((0,T), H_0^1(\Omega)\right) : u_t \in L^2\left((0,T), H^{-1}(\Omega)\right)\right\}$. For $h \in L_T^2$, we say that u is a (weak) solution of the periodic problem

$$\begin{cases} Lu = h & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \ T\text{-periodic} & \end{cases} \tag{1}$$

if u is T-periodic, $u_{|\Omega\times(0,T)} \in W$ and

$$\int_{\Omega\times(0,T)} \left[-u\frac{\partial g}{\partial t} + \langle A\nabla u, \nabla g\rangle + \langle b, \nabla u\rangle g + c_0 u g \right] = \int_{\Omega\times(0,T)} hg$$

for all $g \in C_c^\infty(\Omega \times (0,T))$. For $u \in W$, the inequality $Lu \geq h$ (respectively $\leq$) in $\Omega \times \mathbb{R}$ will be understood in the analogous weak sense.

For $1 \leq r \leq \infty$ let $W_r^{2,1}(\Omega \times (t_0,t_1))$ be the Sobolev space of the functions $u \in L^r(\Omega \times (t_0,t_1))$, $u = u(x_1, ..., x_N, t)$, such that u_t, u_{x_j} and $u_{x_i x_j}$ belong to $L^r(\Omega \times (t_0,t_1))$ for $1 \leq i,j \leq N$, and let $W_{r,T}^{2,1}$ be the space of T-periodic functions such that $u_{|\Omega\times(0,T)} \in W_r^{2,1}(\Omega \times (0,T))$. For $f \in L_T^r$, $r > 1$, we say that u is a strong solution of (1) if $u \in W_{r,T}^{2,1}$ and the equation holds a.e. in the pointwise sense.

Let $f, g : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be two Caratheodory functions, i.e. $(x,t) \to f(x,t,\xi)$ is measurable for all $\xi \geq 0$ and $\xi \to f(x,t,\xi)$ is continuous in $[0,\infty)$ a.e. $(x,t) \in \Omega \times \mathbb{R}$, and the same for g. Assume that $f(.,\xi)$ and $g(.,\xi)$ belong to L_T^r, $r > (N+2)/2$, for all $\xi \geq 0$ and that both are T-periodic in t. Let

H1. There exist $c_f, p_1, p_2, \underline{\xi}, \overline{\xi} > 0$ and $0 \leq a \in L_T^\infty$ such that

$$c_f \xi^{p_1} \leq f(x,t,\xi) \text{ for all } \xi \in \left(0, \underline{\xi}\right] \ a.e.\,(x,t) \in \Omega \times \mathbb{R}, \tag{2}$$

$$f(x,t,\xi) \leq a(x,t)\,\xi^{p_2} \text{ for all } \xi \in \left[\overline{\xi}, \infty\right) \ a.e.\,(x,t) \in \Omega \times \mathbb{R} \tag{3}$$

H2. There exist $b, c_g, q_1, q_2, \underline{\xi}, \overline{\xi} > 0$ with $q_1 > p_2$ if $p_2 \geq 1$ such that

$$c_g \xi^{q_1} \leq g(x,t,\xi) \text{ for all } \xi \in [\overline{\xi}, \infty) \ a.e.\,(x,t) \in \Omega \times \mathbb{R}, \quad (4)$$
$$g(x,t,\xi) \leq b\xi^{q_2} \text{ for all } \xi \in (0, \underline{\xi}] \ a.e.\,(x,t) \in \Omega \times \mathbb{R} \quad (5)$$

Our aim in this paper is to study existence and nonexistence of positive solutions for semilinear periodic parabolic problems of the form

$$\begin{cases} Lu = \lambda f(x,t,u) - g(x,t,u) & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \ T\text{-periodic} \end{cases} \quad (6)$$

where $\lambda > 0$ is a real parameter and f, g satisfy conditions H1 and H2. Let us mention that as a consequence of our proofs the results remain true for the corresponding elliptic problems. For applications we refer to [2], [16].

In order to describe our results and relate them to others in the literature, let us take as an example of the above situation the problem

$$\begin{cases} Lu = \lambda a(x,t)\, h(u)\, u^p - b(x,t)\, u^q := H(x,t,u) & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \ T\text{-periodic} \end{cases} \quad (7)$$

where $0 < a_0 \leq a \in L^\infty_T$, $0 < b_0 \leq b \in L^\infty_T$, $p, q > 0$ and $h : [0, \infty) \to \mathbb{R}$ is a continuous function such that $h(0) \geq 0$ with $h'(0) > 0$ if $h(0) = 0$, and $\sup_{\xi > 0} h(\xi) < \infty$.

When $p = 1 < q$ and $h \equiv 1$, (7) becomes the well-known logistic equation that has been widely studied in recent years. A necessary and sufficient condition for the existence of positive solutions is $\lambda > \lambda_1(a)$, where $\lambda_1(a)$ is the (unique) positive principal eigenvalue of the linear problem with weight

$$\begin{cases} Lu = \lambda a(x,t)\, u & \text{in } \Omega \times \mathbb{R} \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ u \ T\text{-periodic} \end{cases} \quad (8)$$

(see e.g. [14], [17] and the references therein for the elliptic problem and [12], [10] for the periodic parabolic case). However, if for instance

$h(u) = \sin u$, then neither the approach in [12] nor the one in [10] can be applied because $\xi \to \xi^{-1} H(., \xi)$ is no longer decreasing and $\lim_{\xi \to 0^+} \xi^{-1} H(., \xi) = 0$. As a consequence of Theorem 3.1 we shall see that a similar result still holds for (7), namely there exists $\Lambda \geq \lambda_1(a) / \sup_{\xi > 0} h(\xi)$ such that (7) has a positive solution for all $\lambda > \Lambda$ and there is no positive solution if $0 < \lambda < \Lambda$.

If $0 < p, q < 1$ and $h \equiv 1$, in the elliptic case it is also known that there exists some $\Lambda \geq 0$ such that (7) has a positive solution for all $\lambda > \Lambda$ and that there is no positive solution if $0 < \lambda < \Lambda$. In fact, it was proved under additional smoothness assumptions on a and b that if $p \leq q$ then $\Lambda = 0$, and that if $p > q$ then $\Lambda > 0$ (see e.g. [21], [4], [15] and its references). For the periodic parabolic problem, recently the authors have found existence of positive solutions for all λ large enough in [11]. We note however that there it is asked that either $q > 1 - 1/(N+2)$ or b satisfies a quite strong assumption. From Theorem 3.1 below it will follow that a similar result as is in the elliptic case is true for (7) with no restrictions on q and b (and not necessarily $h \equiv 1$) and we shall also have a lower estimate for such a Λ when $p > q$.

If $0 < p < 1 \leq q$ and $h \equiv 1$, existence of positive solutions for (7) was obtained for all $\lambda > 0$ in [10], but again the approach followed there fails if one takes h such that $\lim_{\xi \to 0^+} \xi^{-1} H(., \xi) = 0$ (for the elliptic problem, similar results are given in [4] for $0 < p < q$, $p < 1$, $a \equiv 1$ and $0 < b \in C^\theta\left(\overline{\Omega}\right)$, and in [3] for $0 < p < 1 \leq q$, $a \equiv 1$ and $0 \leq b \in C^\theta\left(\overline{\Omega}\right)$). Theorem 3.1 also extends these results in this case.

Finally, to our knowledge no results are known for (7) when $1 < p < q$, even if $h \equiv 1$, while this elliptic problem has been studied for example in [20], [13] for $a = b = h \equiv 1$ and $(N+2)/(N-2) < p < q$, and recently in [6] for a quasilinear equation that includes the case $1 < p < (N+2)/(N-2)$, $q < 2N/(N-2)$, $a \equiv 1$ and $b \geq 0$ satisfying some additional conditions. Theorem 3.1 shows that similar existence results as the ones quoted above are still valid in this situation, and that there exists a lower estimate for Λ and a positive solution for $\lambda = \Lambda$ in this case.

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2. Preliminaries

We start collecting some known facts about periodic parabolic problems with weight.

Remark 2.1. *i) Let $a \in L_T^r$, $r > (N+2)/2$, and let*

$$P(a) := \int_0^T \operatorname*{esssup}_{x\in\Omega} a(x,t)\,dt.$$

Then $P(a) > 0$ is necessary and sufficient for the existence of a (unique) positive principal eigenvalue $\lambda_1(a)$ for problem (8) (cf. [8], Theorem 3.6). We note that the case $P(a) = +\infty$ is allowed (cf. [8], p. 218).

ii) Let $0 \le \lambda < \lambda_1(a)$ if $\lambda_1(a)$ exists or $\lambda \ge 0$ if $\lambda_1(a)$ does not exist. Then $(L - \lambda a)^{-1} : L_T^r \to C_T$ ($r > (N+2)/2$) is a well defined compact and positive operator (cf. [9], Lemma 2.9). In particular, if $Lu \ge \lambda a u$ (respectively $\le$) then $\lambda \le \lambda_1(a)$ if $\lambda_1(a)$ exists (respectively $\lambda \ge \lambda_1(a)$).

iii) The following comparison principle holds: if $a_1, a_2 \in L_T^r$, $P(a_1) > 0$ and $a_1 \le a_2$ in $\Omega \times \mathbb{R}$, then $\lambda_1(a_1) \ge \lambda_1(a_2)$ and, if in addition $a_1 < a_2$ in a set of positive measure, then $\lambda_1(a_1) > \lambda_1(a_2)$ (cf. [8], Remark 3.7).

The following remark compiles necessary information of some singular periodic parabolic problems.

Remark 2.2. *Let $0 < \alpha < 1/(N+2)$, $0 < \beta < 1$, and consider the problem*

$$\begin{cases} Lv = -v^{-\alpha} + \lambda v^{\beta} & \text{in } \Omega \times \mathbb{R} \\ v = 0 & \text{on } \partial\Omega \times \mathbb{R} \\ v \ T\text{-periodic} \end{cases} \tag{9}$$

Then there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$ (9) has a positive strong solution $v \in W_{r,T}^{2,1}$ for some $r > N+2$. Moreover, $v \in C_T^{1+\theta,(1+\theta)/2}$ and $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega \times \mathbb{R}$, where ν denotes the outward normal unit vector (cf. [7], Theorems 3.1 and 3.3).

3. The theorem

Let

$$P^{\circ} := \text{interior of the positive cone of } C_T^{1+\theta,(1+\theta)/2}$$

THEOREM 3.1. *i) Let f,g satisfying H1 and H2. Then there exists $\Lambda \geq 0$ such that (6) has a (strictly) positive solution $u = u_\lambda \in L_T^\infty$ for all $\lambda > \Lambda$, and if $0 < \lambda < \Lambda$ then there is no positive solution for (6). Moreover, u_λ can be chosen such that*

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{L_T^\infty} = \infty \tag{10}$$

Assume in addition that (3) and (4) hold for all $\xi > 0$.

ii) If $p_2 = 1$, then

$$\Lambda \geq \lambda_1(a) \tag{11}$$

iii) If $p_2 > 1$, then there exists a positive solution $u_\Lambda \in L_T^\infty$ for $\lambda = \Lambda$ and

$$\Lambda \geq \lambda_1(a)^{(q_1-p_2)/(q_1-1)} \left(c_g / \|a\|_{L_T^\infty}\right)^{(p_2-1)/(q_1-1)} \tag{12}$$

iv) If $0 < q_1 < p_2 < 1$, then

$$\Lambda \geq c_g^{(1-p_2)/(1-q_1)} / \left(\|a\|_{L_T^\infty} \left\|L^{-1}\right\|_{L_T^\infty \to L_T^\infty}^{(p_2-q_1)/(1-q_1)}\right) := \widetilde{\Lambda} \tag{13}$$

Also, either if $f(.,\xi), g(.,\xi) \in L_T^r$ for some $r > N+2$ and all $\xi \geq 0$ or if in addition (2) and (5) hold for all $\xi > 0$, then $u_\lambda \in W_{r,T}^{2,1} \cap P^{\circ}$ whenever such u_λ exists.

Proof. In order to prove (i) we start constructing a subsolution for (6). Let α, β, λ_0 be as in Remark 2.2, and let $v = v_{\lambda^*} \in W_{r,T}^{2,1} \cap P^{\circ}$ be a solution of (9) corresponding to some $\lambda^* > \lambda_0$. Let $\underline{\xi} > 0$ be given by H1 and H2 (clearly we may assume that both $\underline{\xi}$ coincide). Choose $k = \underline{\xi} / \|v\|_\infty$ and $\varepsilon = \varepsilon(k) > 0$ such that $b\xi^{q_2} + \lambda^* k^{1-\beta} \xi^\beta \leq k^{1+\alpha} \xi^{-\alpha}$ for all $0 < \xi \leq \varepsilon$. Define $v_k := kv$ and $h(x,t,\xi) := \lambda f(x,t,\xi) - g(x,t,\xi)$, and pick $\lambda \geq \left(\lambda^* k^{1-\beta} \underline{\xi}^\beta + b\underline{\xi}^{q_2}\right) / \left(c_f \varepsilon^{p_1}\right)$. Since $v_k \leq \underline{\xi}$, from (9), (2) and (5) we have that

$$\begin{aligned} Lv_k &= -k^{1+\alpha} v_k^{-\alpha} + \lambda^* k^{1-\beta} v_k^\beta \\ &\leq -b\underline{\xi}^{q_2} \chi_{\{0<v_k<\varepsilon\}} + \lambda^* k^{1-\beta} \underline{\xi}^\beta \chi_{\{\varepsilon \leq v_k\}} \\ &\leq \left(\lambda c_f v_k^{p_1} - b{v_k}^{q_2}\right) \chi_{\{0<v_k<\varepsilon\}} + \left(\lambda c_f \varepsilon^{p_1} - b\underline{\xi}^{q_2}\right) \chi_{\{\varepsilon \leq v_k\}} \\ &\leq h\left(x, t, v_k\right) \end{aligned}$$

and therefore v_k is a subsolution of (6).

On the other side, if $p_2 \geq 1$ we have $p_2 < q_1$ and so recalling (3) and (4) we see that for some constant $K >> 0$ it holds that

$$h\left(x, t, K\right) \leq \lambda \left\|a\right\|_\infty K^{p_2} - c_g K^{q_1} \leq 0 \leq L\left(K\right)$$

Hence, K is a supersolution of (6). Suppose now $0 < p_2 < 1$, and fix $0 < \delta < \lambda_1\left(1\right)$ and $K > \max\left(\overline{\xi}, \left(\lambda \left\|a\right\|_\infty / \delta\right)^{1/(1-p_2)}\right)$, where $\overline{\xi}$ is given by H1. From Remark 2.1 (ii) there exists $0 \leq w \in L_T^\infty$ solution of the Dirichlet periodic problem $Lw = \delta\left(w + K\right)$ in $\Omega \times \mathbb{R}$. Moreover,

$$\begin{aligned} h\left(x, t, w + K\right) &\leq \lambda a\left(x, t\right)\left(w + K\right)^{p_2} \leq \lambda \left\|a\right\|_\infty \left(w + K\right) / K^{1-p_2} \\ &\leq \delta\left(w + K\right) \leq L\left(w + K\right) \end{aligned}$$

and thus $w + K$ is a supersolution of (6). Hence, in any case we can apply [5], Theorem 1, to obtain a solution $0 < u \in L_T^\infty$ of (6).

Let $\Lambda := \inf\left\{\lambda > 0 : \text{there exists } 0 < u_\lambda \in L_T^\infty \text{ solution of (6)}\right\} < \infty$. Let $\lambda > \Lambda$ and let $\lambda > \overline{\lambda} > \Lambda$ such that there exists $u_{\overline{\lambda}} \in L_T^\infty$ solution of (6) for $\lambda = \overline{\lambda}$. Clearly $u_{\overline{\lambda}}$ is a subsolution of (6). Moreover, as above we can choose a supersolution $w \geq \left\|u_{\overline{\lambda}}\right\|_\infty$ and then again Theorem 1 in [5] gives a solution of (6).

Let us prove (10). Let λ_j be an increasing sequence such that $\lambda_j \to \infty$, and let u_{λ_j} be the corresponding positive solutions of (6). An inspection of the above part of the proof shows that we can choose u_{λ_j} such that $\lambda_j \to u_{\lambda_j}$ is increasing and so there exists $\lim_{j\to\infty} \left\|u_{\lambda_j}\right\|_\infty := l \leq \infty$. Suppose $l < \infty$, and let $0 < u_\infty := \lim_{j\to\infty} u_{\lambda_j}$. Dividing (6) by λ_j and going to the limit we find that $f\left(x, t, u_\infty\right) = 0$, which is not possible. Therefore, part (i) of the theorem is proved.

Assume now that (3) and (4) hold for all $\xi > 0$, and let $\lambda > \Lambda$, $0 < u \in L^\infty_T$ be the solution found above. If $p_2 = 1$, (3) and (4) imply $Lu \leq \lambda a u$ and hence Remark 2.1 (ii) gives $\lambda \geq \lambda_1(a)$ and thus (11) follows (note that since (3) holds for all $\xi > 0$, (2) and (3) say that a is not identically zero, i.e. $P(a) > 0$ and so $\lambda_1(a)$ exists).

We prove (iii). Let $\Lambda < \lambda_j$ be a decreasing sequence such that $\lambda_j \to \Lambda$ and let u_{λ_j} be the positive solutions of (6) for $\lambda = \lambda_j$. As before, we can choose u_{λ_j} such that $j \to u_{\lambda_j}$ is decreasing and so $\left\| u_{\lambda_j} \right\|_\infty \leq c$ for some $c > 0$ not depending on j. Moreover, since $\left| h\left(., u_{\lambda_j}\right) \right| \leq \max_{0 \leq \xi \leq \left\| u_{\lambda_j} \right\|_\infty} |h(., \xi)|$, the assumptions on f and g give that $\left\| h\left(x, t, u_{\lambda_j}\right) \right\|_{L^r_T} \leq c$ with c not depending on j $(r > (N+2)/2)$. Thus, from the compactness of $L^{-1} : L^r_T \to C_T$ (cf. Remark 2.1 (ii)) we get some $0 \leq u_\Lambda \in L^\infty_T$ solution of (6) for $\lambda = \Lambda$. In order to show that u_Λ is not identically zero it suffices to prove that $\lim_{j\to\infty} \left\| u_{\lambda_j} \right\|_\infty \neq 0$. Now, suppose $\lim_{j\to\infty} \left\| u_{\lambda_j} \right\|_\infty = 0$, and let $v_j := u_{\lambda_j} / \left\| u_{\lambda_j} \right\|_\infty$. Recalling (3) and (4) we get

$$0 < v_j \leq L^{-1}\left(\lambda_j a v_j {u_{\lambda_j}}^{p_2 - 1} - c_g v_j u_{\lambda_j}^{q_1 - 1} \right)$$

and thus going to the limit the continuity of L^{-1} implies $v_j \to 0$ which is not possible, and so the first assertion of (iii) is proved.

Let $k = \left(c_g / \lambda \left\| a \right\|_\infty\right)^{1/(q_1 - p_2)}$. Since $1 < p_2 < q_1$ we have

$$\begin{aligned} L(ku) &\leq \lambda a k^{1-p_2} (ku)^{p_2} - c_g k^{1-q_1} (ku)^{q_1} \\ &\leq \lambda a k^{1-p_2} (ku)\, \chi_{\{0 < ku \leq 1\}} + (\lambda a k^{1-p_2} - c_g k^{1-q_1})(ku)^{p_2} \chi_{\{ku > 1\}} \\ &\leq \lambda a k^{1-p_2} (ku)\, \chi_{\{0 < ku \leq 1\}} \end{aligned}$$

and so from the last statements in Remark 2.1 (ii) we get $\lambda \geq \lambda_1\left(a k^{1-p_2}\right) = \lambda_1(a)\, k^{p_2 - 1}$ which in turn implies (12).

In order to prove (13) we proceed by contradiction. Suppose there exists a positive solution u for $\lambda = \widetilde{\Lambda}$. Choose $k := \left(\widetilde{\Lambda} \left\| a \right\| \left\| L^{-1} \right\| \right)^{-1/(1-p_2)}$. Recalling (3), (4) and that $0 < p_2 < 1$ a computation shows that $\left\| ku \right\|_\infty \leq 1$. Taking into account this and that $q_1 < p_2$ we find

$$\begin{aligned} L(ku) &\leq \widetilde{\Lambda} a k^{1-p_2} (ku)^{p_2} - c_g k^{1-q_1} (ku)^{q_1} \\ &\leq \left(\|a\| \left\|L^{-1}\right\|\right)^{-1} a (ku)^{p_2} - \left\|L^{-1}\right\|^{-1} (ku)^{q_1} \\ &\leq 0 \end{aligned}$$

Contradiction.

To end the proof, note that any of the last assumptions imply $h(x,t,u) \in L_T^r$ for some $r > N+2$. Since the operator $L^{-1} : L_T^r \to W_{r,T}^{2,1}$ is continuous (see e.g. [19], Section 4) it follows that $u \in W_{r,T}^{2,1}$, and from the Sobolev imbedding theorems (e.g. [18], Lemma 3.3, p. 80) and the strong maximum principle (e.g. [2], Theorem 13.5) we get that $u \in P^{\circ}$. □

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